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TWO RANK SOLUTIONS OF THE ABSTRACT CAUCHY PROBLEM

H. ODETALLAH AND R. KHALIL*

Departement of Mathematics, University of Jordan, Jordan

Abstract. The object of this paper is to discuss the Abstract Cauchy Problem using tensor product technique. We give two rank solution of the problem when the Banach space is a Hilbert space.

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1. Introduction

One of the most important differential equations in Banach spaces, which is involved in many sectors of sciences, is the so-called the Abstract Cauchy Problem, and it has the form:

$$\begin{aligned} Bu'(t) &= Au(t) + f(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (E) \\ u(0) &= x_0 \end{aligned}$$

where u is continuously differentiable vector valued function and f is continuous vector valued function on I ; I is equal to $[0,1]$ or $[0,\infty)$, and both u and f have values in a Banach space X. In Problem (E), A and B are densely defined linear operators on X. If B is not invertible,

*Corresponding author

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then Problem (E) is called degenerate problem, otherwise it is called non-degenerate. If $f = 0$, then Problem (E) is homogenous, otherwise it is called non-homogenous.

Many researchers were and are interested in this Problem and studied it using a variety of methods. In the mid-seventies of the last century, Carroll,R.W. and Showalter,R.E. , released an article about degenerate Abstract Cauchy Problem[4]. In 1996, Thaller, B. and Thaller, S. used the Factorization of degenerate Abstract Cauchy Problem in a Hilbert space[7]. Favini, A. and Yagi, A. ,(1999), produced an interesting material on the degenerate Abstract Cauchy Problem by unifying the methods of semigroups and operational approaches to treat the solvability of such problem[5]. In 2002, the inverse form of Problem (E) was studied by Al Horani, M. , where some conditions were put on operators A and B to convert Problem (E) to a non-degenerate problem[1]. In 2004, Al Horani, M. and Favini, A. discussed the inverse problem when it is of the second order[2]. In 2010, Ziqan, A.M., Al Horani, M. and Khalil, R. used tensor product technique to find a unique solution for the Abstract Cauchy Problem under certain conditions on the operators A and B [8], [9]. Khalil, R. and Abdullah, L. (2010), used the same technique and found an atomic solution for certain degenerate and non-degenerate inverse problems[6].

In this paper we studied the non-homogeneous Abstract Cauchy Problem using the tensor product technique when the non-homogeneous part of the problem is a two rank function. In other words f in Problem (E) is equal to $f_1 \otimes \delta_1 + f_2 \otimes \delta_2$, where f_1, f_1 are real-valued continuous functions on I and δ_1, δ_1 are two orthogonal unit vectors in ℓ^2, the Banach space consisting of the square summable sequences. We did prove the existence of a unique solution for this type of problems when u has the form $u_1 \otimes \delta_1 + u_2 \otimes \delta_2$, where u_1, u_2 are real-valued continuously differentiable functions on I, and with some conditions on the operators A and B. Also, we found an atomic solution for non-degenerate problem of this type of the non-homogeneous Abstract Cauchy Problem.

Throughout the paper, the homogeneous Abstract Cauchy Problem is

$$Bu'(t) = Au(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (E_1)$$

$$u(0) = x_0$$

and the nonhomogeneous Abstract Cauchy Problem is

$$Bu'(t) = Au(t) + f(t) \ldots\ldots\ldots\ldots\ldots\ldots (E_2)$$

$$u(0) = x_0,$$

where A and B are densely defined linear operators on the Banach space X , $u \in C^1(I,X)$, $f \in C(I,X)$ and $x_0 \in X$.

2. Two rank solution of two rank non-homogeneous part of the equation

In this section, we study the problem

$$\begin{aligned} Bu'(t) &= Au(t) + f(t) \qquad \ldots\ldots\ldots\ldots (E_2) \\ u(0) &= x_0 \end{aligned}$$

Where we assume that $u(t) = u_1(t)\delta_1 + u_2(t)\delta_2$, and $f(t) = f_1(t)\delta_1 + f_2(t)\delta_2$ and we assume A and B are densely defined closed operators on ℓ^2.

One of our main results is the following:

Theorem 2.1 In Problem (E_2), let $B = I$, and $u(t) = u_1(t)\delta_1 + u_2(t)\delta_2$, with $u_1(t)$ and $u_2(t)$ are continuously differentiable functions on $[0,\infty)$. Assume further that $f_1(t)$ and $f_2(t)$ are continuous on $[0,\infty)$.Then Problem (E_2) has a unique solution.

Proof : Since $u'(t) = u_1'(t)\delta_1 + u_2'(t)\delta_2$, we get

$$u_1'(t)\delta_1 + u_2'(t)\delta_2 = u_1(t)A\delta_1 + u_2(t)A\delta_2 + f_1(t)\delta_1 + f_2(t)\delta_2 \ldots\ldots\ldots\ldots (2.1)$$

If $[\delta_1,\delta_2]$ is an invariant subspace of A, then the restriction of A to $[\delta_1,\delta_2]$ has a matrix representation $\widehat{A}=\left[a_{ij}\right]_{2\times 2}$, where $a_{ij}=\left\langle A\delta_j,\delta_i\right\rangle, i,j=1,2.$

Taking the inner product of δ_1 and δ_2 to both sides of (1.1), we get

$$\begin{aligned} u_1'(t)\langle\delta_1,\delta_1\rangle+u_2'(t)\langle\delta_2,\delta_1\rangle &= u_1(t)\langle A\delta_1,\delta_1\rangle+u_2(t)\langle A\delta_2,\delta_1\rangle \\ &\quad +f_1(t)\langle\delta_1,\delta_1\rangle+f_2(t)\langle\delta_2,\delta_1\rangle \text{ ....}(2.2) \end{aligned}$$

And

$$\begin{aligned} u_1'(t)\langle\delta_1,\delta_2\rangle+u_2'(t)\langle\delta_2,\delta_2\rangle &= u_1(t)\langle A\delta_1,\delta_2\rangle+u_2(t)\langle A\delta_2,\delta_2\rangle \\ &\quad +f_1(t)\langle\delta_1,\delta_2\rangle+f_2(t)\langle\delta_2,\delta_2\rangle \text{ ....}(2.3) \end{aligned}$$

Since $\{\delta_1,\delta_2\}$ is an orthonormal set, we get from equations (2.2) and (2.3) :

$$u_1'(t)=u_1(t)a_{11}+u_2(t)a_{12}+f_1(t)\text{ ....}(2.4)$$

And

$$u_2'(t)=u_1(t)a_{21}+u_2(t)a_{22}+f_2(t)\text{ ....}(2.5)$$

Now, equations (2.4) and (2.5) represent a non-homogeneous system of two linear differential equations

$$U'(t)=\widehat{A}U(t)+F(t)\text{ ....}(2.6)$$

where $U(t)=\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}$ and $F(t)=\begin{bmatrix}f_1(t)\\ f_2(t)\end{bmatrix}$.

The general solution of system (2.6) U_g is the sum of the homogeneous solution and a particular solution. The details are as follows:

Now, the corresponding homogeneous system of (2.6) is

$$U'(t)=\widehat{A}U(t)\text{ ....}(2.7)$$

For such $\widehat{A}$ we have the following cases:

Case 1: $\widehat{A}$ has distinct real eigenvalues (i.e $\lambda_1 \neq \lambda_2$). Then the general solution of system (2.7) is of the form

$$U_h(t) = c_1 e^{\lambda_1 t} \xi_1 + c_2 e^{\lambda_2 t} \xi_2$$

where ξ_1 and ξ_2 are the corresponding eigenvectors of λ_1 and λ_2, respectively.

Case 2: $\widehat{A}$ has equal eigenvalues (i.e $\lambda_1 = \lambda_2 = \lambda$), then we have the following sub-cases:

Case 2.1: λ has two linearly independent eigenvectors ξ_1 and ξ_2. Then the general solution of system (1.7) is given by

$$U_h(t) = (c_1 \xi_1 + c_2 \xi_2) e^{\lambda t}$$

Case 2.2: λ has a single linearly independent eigenvector ξ. Then the general solution of system (1.7) is given by

$$U_h(t) = (c_1 \xi + c_2 (t\xi + \eta)) e^{\lambda t}$$

where η satisfies equation $\left(\widehat{A} - \lambda I\right) \eta = \xi$ and I is the identity matrix.

Case 3: $\widehat{A}$ has complex conjugate eigenvalues, $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$. Let $\xi_1 + \xi_2 i$ and $\xi_1 - \xi_2 i$ are the corresponding eigenvectors of λ_1 and λ_2, respectively, where ξ_1 and ξ_2 are vectors.

Then the general solution of system (2.7) is given by

$$U_h(t) = e^{at} \left[c_1 (\xi_1 \cos(bt) - \xi_2 \sin(bt)) + c_2 (\xi_1 \sin(bt) + \xi_2 \cos(bt))\right]$$

Now, to get a particular solution, form the matrix $\Psi(t) = (e^{\lambda_1 t} \xi_1 : e^{\lambda_2 t} \xi_2)$ which is known as the fundamental matrix of the system. It is known that the inverse of Ψ exists and a particular solution to system (2.6) is given by the formula:

$$U_p(t) = \Psi(t) \int_0^t \Psi^{-1}(\tau) F(\tau) \, d\tau$$

Furthermore, for any of the above cases the general solution of system (2.6) is of the form

$$U_g(t) = U_h(t) + U_p(t)$$

Where $U_h(t)$ is the general solution of the corresponding homogeneous system and $U_p(t)$ is the particular solution of the system.

By the initial condition $u(0) = x_0$, we have

$$(\xi_1 : \xi_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}(\delta_1 \ \delta_2) = x_0$$

Since ξ_1 and ξ_2 are linearly independent eigenvectors, then the matrix $(\xi_1 : \xi_2)$ is invertible. Multiplying $(\xi_1 : \xi_2)^{-1}$ to both sides, we get

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}(\delta_1 \ \delta_2) = (\xi_1 : \xi_2)^{-1} x_0$$

Taking the inner product of δ_1 and δ_2 to both sides, we get

$$c_i = \left\langle (\xi_1 : \xi_2)^{-1} x_0, \delta_i \right\rangle, \ i = 1, 2.$$

So, the Problem has a unique solution.

Now, if $[\delta_1, \delta_2]$ is not an invariant subspace of A, then $A\delta_1 = a_{11}\delta_1 + a_{12}\delta_2 + a_{13}\theta_1$ and $A\delta_2 = a_{21}\delta_1 + a_{22}\delta_2 + a_{23}\theta_2$, where θ_1, θ_2 are orthogonal with $\{\delta_1, \delta_2\}$, and a_{13}, a_{23} not both are equal to zero.

So, the Problem becomes

$$\begin{aligned} u_1'(t)\,\delta_1 + u_2'(t)\,\delta_2 &= a_{11}u_1(t)\,\delta_1 + a_{12}u_1(t)\,\delta_2 + a_{13}u_1(t)\,\theta_1 \\ &\quad + a_{21}u_2(t)\,\delta_1 + a_{22}u_2(t)\,\delta_2 + a_{23}u_2(t)\,\theta_2 + f_1(t)\,\delta_1 + f_2(t)\,\delta_2 ...... (1.8) \end{aligned}$$

By equating the coefficients of $\delta_1, \delta_2, \theta_1$ and θ_2 in both sides, we have

$$a_{13}u_1(t) = 0 \text{ and } a_{23}u_2(t) = 0$$

So, we have the following cases:

Case (i) : $a_{13} = 0$ and $a_{23} = 0$. This case contradicts the assumption on a_{13} and a_{23}.

Case (ii) : If $u_1(t) = 0$ and $u_2(t) = 0$, then $u(t) = 0$, and hence the Problem has the trivial unique solution.

Case (iii) : If $(a_{13} = 0$ and $u_2(t) = 0)$ or $(a_{23} = 0$ and $u_1(t) = 0)$, then $u(t) = u_i(t)\,\delta_i$, for some $i = 1, 2$.

Then equation (1.6) becomes

$$u_i'(t)\,\delta_i = a_{i1}u_i(t)\,\delta_1 + a_{i2}u_i(t)\,\delta_2 + f_1(t)\,\delta_1 + f_2(t)\,\delta_2 .... (2.9)$$

By taking the inner product of δ_i to both sides of (1.9), we have

$$u_i'(t) = a_{ii}u_i(t) + f_i(t) \ldots\ldots\ldots\ldots (2.10)$$

Equation (2.10) is first order linear differential equation, and has a general solution of the form

$$u_i(t) = e^{a_{ii}t}\left(\int_0^t e^{-a_{ii}\tau} f_i(\tau)\, d\tau + c\right)$$

And by the initial condition $u(0) = x_0$, we have

$$u_i(0) = c$$

Then

$$u(0) = u_i(0)\,\delta_i = c\;\delta_i = x_0$$

Taking the inner product of δ_i to both sides, we have

$$c = \langle x_0, \delta_i \rangle$$

And hence the Problem has a unique solution. $\square$

Theorem 2.2 Consider Problem (E_2). Let $B_2 = B\mid_{[\delta_1,\delta_2]}$be orthogonally diagonalizable linear operator with respect to the orthonormal basis $\{\theta_1, \theta_2\}$ and corresponding eigenvalues λ_1, λ_2 such that $\langle A\theta_j, \delta_i \rangle \neq 0$ for some $i, j \in \{1,2\}$. Then Problem (E_2) has a unique solution.

Proof : Let $D = diag(\lambda_1, \lambda_2)$ be the matrix representation of B_2 with respect to $\{\theta_1, \theta_2\}$.

Now, if $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, then B_2 is invertible and we can use Theorem 2.1, so the problem has a unique solution.

Assume $\lambda_1 \neq 0$ and $\lambda_2 = 0$. Let $u(t) = v_1(t)\,\theta_1 + v_2(t)\,\theta_2$. Then $u'(t) = v_1'(t)\,\theta_1 + v_2'(t)\,\theta_2$.

Hence,

$$v_1'(t)\,B\theta_1 + v_2'(t)\,B\theta_2 = v_1(t)\,A\theta_1 + v_2(t)\,A\theta_2 + f_1(t)\,\delta_1 + f_2(t)\,\delta_2.$$

Since $B\theta_1 = \lambda_1\theta_1$ and $B\theta_2 = 0$, we have

$$\lambda_1 v_1'(t)\,\theta_1 = v_1(t)\,A\theta_1 + v_2(t)\,A\theta_2 + f_1(t)\,\delta_1 + f_2(t)\,\delta_2 \ldots\ldots\ldots\ldots (2.11)$$

Taking the inner product of θ_1 and θ_2 with both sides of (2.11), we get

$$\lambda_1 v_1'(t) = v_1(t)\langle A\theta_1,\theta_1\rangle + v_2(t)\langle A\theta_2,\theta_1\rangle + f_1(t)\langle \delta_1,\theta_1\rangle + f_2(t)\langle \delta_2,\theta_1\rangle \text{ .... } (2.12)$$

and

$$0 = v_1(t)\langle A\theta_1,\theta_2\rangle + v_2(t)\langle A\theta_2,\theta_2\rangle + f_1(t)\langle \delta_1,\theta_2\rangle + f_2(t)\langle \delta_2,\theta_2\rangle \text{ .... } (2.13)$$

Now putting $\alpha_{ji} = \langle A\theta_i,\theta_j\rangle$ and $\beta_{ji} = \langle \delta_i,\theta_j\rangle, i,j = 1,2$, then equation (2.13) gives the following cases:

Case (i) : If $\alpha_{22} \neq 0$, then

$$v_2(t) = \frac{-(\alpha_{21}v_1(t) + \beta_{21}f_1(t) + \beta_{22}f_2(t))}{\alpha_{22}} \text{ .... } (2.14)$$

Substituting (2.14) in (2.12), we get

$$v_1'(t) = K_1 v_1(t) + K_2 f_1(t) + K_3 f_2(t) \text{ .... } (2.15)$$

where $K_1 = \frac{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}{\lambda_1\alpha_{22}}, K_2 = \frac{\alpha_{22}\beta_{11}-\alpha_{12}\beta_{21}}{\lambda_1\alpha_{22}}$ and $K_3 = \frac{\alpha_{22}\beta_{12}-\alpha_{12}\beta_{22}}{\lambda_1\alpha_{22}}$.

Then (2.15) is a first order linear differential equation and it has a general solution of the form

$$v_1(t) = e^{K_1 t}\left(\int_0^t e^{-K_1 t}(K_2 f_1(\tau) + K_3 f_2(\tau))\,d\tau + c\right) \text{ .... } (2.16)$$

Substituting (2.16) in (2.14), we get

$$\begin{aligned} v_2(t) &= -\frac{1}{\alpha_{22}}(\alpha_{21}e^{K_1 t}(\int_0^t e^{-K_1 t}(K_2 f_1(\tau) + K_3 f_2(\tau))\,d\tau + c) \\ &\quad + \beta_{21}f_1(t) + \beta_{22}f_2(t)) \text{ .... } (2.17) \end{aligned}$$

From (2.16), (2.17) and the initial condition $u(0) = x_0$, we can determine constant c uniquely as follows

$$\begin{aligned} u(0) &= x_0 = v_1(0)\theta_1 + v_2(0)\theta_2 \\ &= c\theta_1 - \frac{1}{\alpha_{22}}(\alpha_{21}c + \beta_{21}f_1(0) + \beta_{22}f_2(0))\theta_2 \end{aligned}$$

By taking the inner product of θ_1 with both sides, we get

$$c = \langle x_0,\theta_1\rangle$$

Thus, Problem (E_2) has a unique solution.

Case $(\mathbf{ii})$: If $\alpha_{22} = 0$, then we have the following sub-cases:

Case $(\mathbf{ii}.\mathbf{1})$: If $\alpha_{12} \neq 0$ and $\alpha_{21} \neq 0$, then from equations (2.12) and (2.13)

$$v_1(t) = -\frac{1}{\alpha_{21}} \left(\beta_{21} f_1(t) + \beta_{22} f_2(t)\right) \text{..........} (2.18)$$

And

$$v_2(t) = \frac{1}{\alpha_{12}} \left(\lambda_1 v_1'(t) - \alpha_{11} v_1(t) - \beta_{11} f_1(t) - \beta_{12} f_2(t)\right) \text{..........} (2.19)$$

Substituting equation (2.18) in (2.19), we get a unique solution for Problem (E_2).

Case $(\mathbf{ii}.\mathbf{2})$: If $\alpha_{12} \neq 0$ and $\alpha_{21} = 0$, then in this case we have one equation

$$\lambda_1 v_1'(t) = \alpha_{11} v_1(t) + \alpha_{12} v_2(t) + \beta_{11} f_1(t) + \beta_{12} f_2(t) \text{..........} (2.20)$$

So, we need another equation to find $v_1(t)$ and $v_2(t)$.

Now, by the assumption on A, without loss of generality, we can assume that $\langle A\theta_2, \delta_1 \rangle \neq 0$.

Taking the inner product of δ_1 to both sides of equation (2.11), we get

$$\lambda_1 \beta_{11} v_1'(t) = v_1(t) \langle A\theta_1, \delta_1 \rangle + v_2(t) \langle A\theta_2, \delta_1 \rangle + f_1(t)$$

If $\gamma_{1i} = \langle A\theta_i, \delta_1 \rangle, i = 1, 2$, then the above equation becomes

$$\lambda_1 \beta_{11} v_1'(t) = \gamma_{11} v_1(t) + \gamma_{12} v_2(t) + f_1(t) \text{..........} (2.21)$$

From equations (2.20) and (2.21), we have

$$v_1'(t) = h_1 v_1(t) + h_2 f_1(t) + h_3 f_2(t),$$

where $h_1 = \frac{\gamma_{12}\alpha_{11} - \gamma_{11}\alpha_{12}}{\lambda_1(\gamma_{12} - \alpha_{12}\beta_{11})}, h_2 = \frac{\gamma_{12}\beta_{11} - \alpha_{12}}{\lambda_1(\gamma_{12} - \alpha_{12}\beta_{11})}$ and $h_3 = \frac{\gamma_{12}\beta_{12}}{\lambda_1(\gamma_{12} - \alpha_{12}\beta_{11})}$.

This is a first order linear differential equation, and has a general solution of the form

$$v_1(t) = e^{h_1 t} \left((\int_0^t e^{-h_1 t} \left(h_2 f_1(\tau) + h_3 f_2(\tau)\right) d\tau + c \right) \text{..........} (2.22)$$

By substituting equation (2.22) in (2.21), we determine $v_2(t)$ uniquely.

And again, in equation (2.22), $c = \langle x_0, \theta_1 \rangle$ by the initial condition $u(0) = x_0$.

Case $(\mathbf{ii}.\mathbf{3})$: If $\alpha_{12} = 0$ and $\alpha_{21} \neq 0$, then $v_1(t)$ determine uniquely by equation (2.18). And by substituting equation (2.18) in (2.21), we determine $v_2(t)$ uniquely.

Case $(\mathbf{ii.4})$: If $\alpha_{12} = 0$ and $\alpha_{21} = 0$, then $v_1(t)$ determine uniquely by substituting $\alpha_{12} = 0$ in equation (2.22). By substituting equation (2.22) in (2.21), we determine $v_2(t)$ uniquely.

Hence, the Problem has a unique solution. $\square$

Now let us consider the case where our function u is of rank one while the non-homogenous part of the equation (the right hand of the equation) is of rank two. That is to say: $u = g \otimes x$ is an atom, and the right hand is $f_1 \otimes \delta_1 + f_2 \otimes \delta_2$.

Theorem 2.3 Consider Problem (E_2) with $B = I$. Assume $u(t) = g(t)x$, $g \in C^1(I)$ be such that:

(1) There exists some $x^* \in X^*$ and $h \in C(I)$ such that $\langle g(t)x, x^* \rangle = h(t)$.
(2) $\left(A - \ln\left(\frac{h(1)}{h(0)}\right)I\right)^{-1}$ exists and bounded.
(3) $f_i(0) \neq 0$, $i = 1,2$.

Then Problem (E_2) has a unique solution.

Proof : If we use the tensor product notation, then the Problem becomes

$$g' \otimes x + g \otimes Ax = f_1 \otimes \delta_1 + f_2 \otimes \delta_2 \ldots\ldots\ldots\ldots\ldots (2.23)$$

Since an atom $x \otimes y$ has infinite number of representations $\lambda x \otimes \frac{1}{\lambda} y$, then without loss of generality we can assume that $g(0) = 1$.

Now, equation (3.1) can be written in the form

$$g' \otimes x + g \otimes Ax - f_1 \otimes \delta_1 = f_2 \otimes \delta_2 \ldots\ldots\ldots\ldots\ldots (2.24)$$

Since the sum of three atoms is equal to an atom, then ,[ ], we have the following cases:

Case (i) : If $g' = \lambda g$, then $g(t) = ce^{\lambda t}$ .But since $g(0) = 1$, then $g(t) = e^{\lambda t}$.

By using condition (1), we get $\langle x, x^* \rangle = h(0)$. So

$$h(t) = \langle g(t)x, x^* \rangle = \left\langle e^{\lambda t} x, x^* \right\rangle = e^{\lambda t} \langle x, x^* \rangle = e^{\lambda t} h(0)$$

We conclude from condition (2) ,that $h(0)$ is different from zero. This implies that $\lambda = \ln\left(\frac{h(1)}{h(0)}\right)$. Hence, g is determined.

Now, equation (2.4) becomes

$$e^{\lambda t}\otimes(\lambda x+Ax)-f_1\otimes\delta_1=f_2\otimes\delta_2 \quad\dots\dots\dots\dots\dots\dots (2.25)$$

Again the sum of two atoms is equal to an atom, so, [ ], we have the following subcases:

Case $(i.1)$: If $f_1=\alpha e^{\lambda t}$, then equation (2.25) can be written as

$$e^{\lambda t}\otimes(\lambda x+Ax-\alpha\delta_1)=f_2\otimes\delta_2 \quad\dots\dots\dots\dots\dots\dots (2.26)$$

From equation (2.26), we have equality of two atoms. Consequently, we have

$$e^{\lambda t}=\omega f_2(t) \quad\dots\dots\dots\dots\dots\dots (2.27)$$

and

$$(\lambda x+Ax-\alpha\delta_1)=\frac{1}{\omega}\delta_2 \quad\dots\dots\dots\dots\dots\dots (2.28)$$

Now, since equation (2.27) is true for all t , then using condition (3), we get

$$\omega=\frac{1}{f_2(0)}$$

From equation (2.28), we get

$$(\lambda I+A)x=\alpha\delta_1+\frac{1}{\omega}\delta_2 \quad\dots\dots\dots\dots\dots\dots (2.29)$$

Now, using the value of λ and condition (2), we get

$$x=(\lambda I+A)^{-1}\left(\alpha\delta_1+\frac{1}{\omega}\delta_2\right) \quad\dots\dots\dots\dots\dots\dots (2.30)$$

Now, x will be determined completely if α is determined.

Since $u(t)=e^{\lambda t}x$ and by the initial condition $u(0)=x_0$, we have

$$x_0=x=(\lambda I+A)^{-1}\left(\alpha\delta_1+\frac{1}{\omega}\delta_2\right) \quad\dots\dots\dots\dots\dots\dots (2.31)$$

Taking the inner product of δ_1 to both sides of equation (3.9), we get

$$\alpha=\langle(\lambda I+A)x_0,\delta_1\rangle$$

And hence, x is determined uniquely.

Case $(i.2)$: If $(\lambda x+Ax)=\beta\delta_1$, then we have

$$x=\beta(\lambda I+A)^{-1}\delta_1 \quad\dots\dots\dots\dots\dots\dots (2.32)$$

So, if β is determined, then x will be determined completely. As in Case $(i.1)$, by the initial condition $u(0) = x_0$ and also by taking the inner product of δ_1, we get

$$\beta = \langle (\lambda I + A) x_0, \delta_1 \rangle$$

Thus, x is determined uniquely.

Case (ii) : If $Ax = \mu x$, then equation (2.24) becomes as follows

$$g' \otimes x + \mu g \otimes x - f_1 \otimes \delta_1 = f_2 \otimes \delta_2 ....(2.33)$$

In equation (2.33), if we take the inner product of x^* with both sides and use condition (1), then we get

$$h'(t) + \mu h(t) = f_1(t) \langle \delta_1, x^* \rangle + f_2(t) \langle \delta_2, x^* \rangle ....(2.34)$$

Since equation (3.34) is true for all t,then μ can be determined by the following:

$$\mu = \frac{1}{h(0)} \left(f_1(0) \langle \delta_1, x^* \rangle + f_2(0) \langle \delta_2, x^* \rangle - h'(0) \right)$$

Now, equation (2.33) becomes

$$\left(g' + \mu g \right) \otimes x - f_1 \otimes \delta_1 = f_2 \otimes \delta_2 ....(2.35)$$

We now have the sum of two atoms is equal to an atom. So, we have the following subcases:

Case $(ii.1)$: If $g' + \mu g = -\gamma_1 f_1$, then $g(t) = e^{-\mu t} \left[\int_0^t -\gamma_1 f_1(\tau) e^{\mu \tau} d\tau + c \right]$. And since $g(0) = 1$, then $c = 1$.

Now, g will be determined completely, if μ is determined.

Since $g' + \mu g = -\gamma_1 f_1$, then $g'(t) + \mu g(t) = -\gamma_1 f_1(t)$. Hence

$$(g'(t) + \mu g(t))x = (-\gamma_1 f_1(t))x.....................(2.36)$$

So, if we take the inner product of x^* to both sides of equation (3.36), we get

$$\left\langle g'(t) x, x^* \right\rangle + \langle \mu g(t) x, x^* \rangle = \langle -\gamma_1 f_1(t) x, x^* \rangle ....(2.37)$$

Using condition (1) in equation (2.37), we have

$$h'(t) + \mu h(t) = -\gamma_1 f_1(t) \langle x, x^* \rangle ....(2.38)$$

If we use condition (3), we find that

$$\gamma_1 = -\frac{h'(0)+\mu h(0)}{f_1(0)\,h(0)}$$

So, from equation (2.35), x is determined uniquely in the formula

$$x = \frac{f_1(0)\,\delta_1 + f_2(0)\,\delta_2}{g'(0)+\mu}$$

Case$(ii.2)$: If $x = \gamma_2\delta_1$, then equation (2.35) becomes

$$(\gamma_2(g'+\mu g)-f_1)\otimes\delta_1 = f_2\otimes\delta_2 \text{................}(2.39)$$

Taking the inner product of δ_1 to both sides of equation (2.39), we get

$$\gamma_2(g'+\mu g)-f_1 = 0$$

And then

$$\gamma_2(g'(t)+\mu g(t))-f_1(t) = 0\text{....................}(2.40)$$

Multiplying equation (2.40) by $< x, x^* >$, we get

$$\gamma_2(h'(t)+\mu h(t))-f_1(t)h(0) = 0\text{................}(2.41)$$

And since equation (2.41) is true for all t, then

$$\gamma_2 = \frac{f_1(t)h(0)}{h\prime(0)+\mu h(0)}$$

Then

$$x = \frac{f_1(t)h(0)}{h\prime(0)+\mu h(0)}\delta_1$$

From equation (2.40), g will be determined uniquely by the form

$$g(t) = e^{-\mu t}[\int_0^t \frac{f_1(t)}{\gamma_2}e^{\mu t}dt + c]$$

Where $c = 1$, since $g(0) = 1$.

Now, assume that $x = a_1\delta_1 + a_2\delta_2 + a_3\theta$, where θ is orthogonal with $\{\delta_1, \delta_2\}$.

Then, by the initial condition $u(0) = x_0$, $a_i = \langle x_0, \delta_i\rangle$,and $a_3 = \langle x_0, \theta\rangle$, where $i = 1,2$.

Applying the linear operator A on x, we get

$$Ax = a_1 A\delta_1 + a_2 A\delta_2 + a_3 A\theta$$

So, assume that $\alpha_{ji} = \left\langle A\delta_i, \delta_j\right\rangle$, $\alpha_{3i} = \langle A\delta_i, \theta\rangle$, $\alpha_{j3} = \left\langle A\theta, \delta_j\right\rangle$ and $\alpha_{33} = \langle A\theta, \theta\rangle$ where $i, j = 1,2$.

The Problem becomes

$$g'(t)(a_1\delta_1 + a_2\delta_2 + a_3\theta) + g(t)(a_1 A\delta_1 + a_2 A\delta_2 + a_3 A\theta) = f_1(t)\delta_1 + f_2(t)\delta_2 \ldots\ldots. (2.42)$$

Now, to find g we have the following cases:

Case(1) : If $a_1 \neq 0$, then taking the inner product of δ_1 to both sides of equation (2.42), we have

$$a_1 g'(t) + (a_1\alpha_{11} + a_2\alpha_{12} + a_3\alpha_{13})\, g(t) = f_1(t)$$

then $g(t) = e^{-k_1 t}\left[\int\limits_0^t e^{k_1\tau} f_1(\tau)\, d\tau + c\right]$, where $k_1 = \frac{a_1\alpha_{11} + a_2\alpha_{12} + a_3\alpha_{13}}{a_1}$ and $c = 1$, since $g(0) = 1$.

Case(2) : If $a_1 = 0$, then we have the following sub-cases:

Case(2.1) : If $a_2 \neq 0$, then taking the inner product of δ_2 to both sides of equation (2.42), we have

$$a_2 g'(t) + (a_2\alpha_{22} + a_3\alpha_{23})\, g(t) = f_2(t)$$

then $g(t) = e^{-k_2 t}\left[\int\limits_0^t e^{k_2\tau} f_2(\tau)\, d\tau + c\right]$, where $k_2 = \frac{a_2\alpha_{22} + a_3\alpha_{23}}{a_2}$ and $c = 1$.

Case(2.2) : If $a_2 = 0$, then we have the following sub-cases:

Case$(2.2.1)$: If $a_3 \neq 0$, then taking the inner product of θ to both sides of equation (2.42), we have

$$a_3 g'(t) + a_3\alpha_{33} g(t) = 0$$

then $g(t) = ce^{-\alpha_{33}t}$, where $c = 1$, since $g(0) = 1$.

Case$(2.2.2)$: If $a_3 = 0$, then $x = 0$ and this implies that $u(t) = 0$.

Thus, for all the above cases Problem (E_2) has a unique solution. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests.

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