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ON THE SEMIGROUPS OF QUASI-OPEN TRANSFORMATIONS

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Abstract. In this paper we consider the semigroups of quasi-open transformations. A map f between topological spaces X and Y is quasi-open if for any non-empty open set $U \subset X$, the interior of $f(U)$ in Y is non-empty. We give abstract characterizations of semigroups of continuous quasi-open transformations defined on an open set of Euclidean n-space and semigroups of quasi-open mappings defined on a certain class of topological spaces.

Keywords: Quasi-open map; Lattice-equivalence; T_D-space.

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1. Introduction

Investigation of topological spaces by algebraic methods plays an important role in modern mathematics. The basic method is to investigate topological spaces via groups, rings, semigroups and other structures of mappings. This allows one to recast statements about topological spaces into statements about algebraic structures. Many researchers have focused their efforts on the characterization of topological spaces by semigroups of continuous, open, closed, quasi-open mappings defined on these spaces [2], [3],[5], [6], and [7].

In this paper we investigate semigroups of quasi-open mappings. A map f between topological spaces X and Y is quasi-open if for any non-empty open set $U \subset X$, the interior of $f(U)$ in

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Y is non-empty. If f and g are both continuous and quasi-open, then the function composition is also continuous and quasi-open. Let $CQ(X)$ denote the semigroup of continuous quasi-open mappings from a topological space X into itself with composition of functions as the semigroup operation. It is obvious that if X and Y are homeomorphic then the semigroups $CQ(X)$ and $CQ(Y)$ are isomorphic. We wonder if X and Y are homeomorphic whenever $CQ(X)$ and $CQ(Y)$ are isomorphic. In general, the answer is no. Let X denote any set with more than one element and $\xi \in X$. Consider the topological spaces $Y = (X, \tau_1)$ and $Z = (X, \tau_2)$ where τ_1 is the trivial topology and $\tau_2 = \{\emptyset, \{\xi\}, X\}$. Evidently $CQ(Y)$ and $CQ(Z)$ are isomorphic but Y and Z are not homeomorphic.

In this paper we also investigate semigroups of quasi-open mappings in light of lattice-equivalence. Let $Q(X)$ denote the semigroup of quasi-open maps of a topological space X . If $Q(X)$ and $Q(Y)$ are isomorphic, must X and Y be homeomorphic. Let X denote any set with more than two elements containing the elements η, ξ. Consider the topological spaces $Y = (X, \tau_1)$ and $Z = (X, \tau_2)$ with $\tau_1 = \{\oslash, \{\eta\}, X\}$ and $\tau_2 = \{\oslash, \{\eta\}, X\setminus\{\xi\}, X\}$. Evidently $Q(Y)$ and $Q(Z)$ are isomorphic but Y and Z are not homeomorphic.

The purpose of this paper is to give abstract characterizations of semigroups of continuous quasi-open transformations defined on an open set of Euclidean n-space and semigroups of quasi-open mappings defined on a certain class of topological spaces.

2. A characterization of semigroups of continuous quasi-open mappings

We denote by R^n the n-dimensional Euclidean space with standard topology.

Theorem 2.1. *Let X and Y be open subsets of R^n and R^m respectively, $(n, m > 1)$. The semigroups $CQ(X)$ and $CQ(Y)$ are isomorphic if and only if the spaces X and Y are homeomorphic.*

Proof. It is obvious that if X and Y are homeomorphic then $CQ(X)$ and $CQ(Y)$ are isomorphic. Specifically, if $h : X \to Y$ is a homeomorphism between X and Y, then $f \to hfh^{-1}$ is an isomorphism between $CQ(X)$ and $CQ(Y)$. The necessity of the condition follows from Lemmas 2.2-2.7.

Throughout this paper, φ denotes an isomorphism between semigroups $CQ(X)$ and $CQ(Y)$. Without loss of generality we can assume that X and Y are bounded open sets.

We denote by $CQ_0(X)$ a subset of $CQ(X)$ consisting of all $f \in CQ(X)$ for which $f(X) \subset K_f$ for some compact subset of X. The set $CQ_0(X)$ is an ideal of $CQ(X)$.

Lemma 2.2. *Let $f, g \in CQ(X)$. Then, from $g(X) \subseteq f(X)$ it follows that $\varphi g(Y) \subseteq \overline{\varphi f(Y)}$. In addition, if $\varphi f \in CQ_0(Y)$ then $\varphi g(Y) \subset \overline{\varphi f(Y)}$.*

Proof. Let $\varkappa'(\varphi f) = \tau'(\varphi f)$ for some $\varkappa', \tau' \in CQ(Y)$. Since φ is an isomorphism, there exist $\varkappa, \tau \in CQ(X)$ such that $\varkappa' = \varphi\varkappa$ and $\tau' = \varphi\tau$. Then $(\varphi\varkappa)(\varphi f) = (\varphi\tau)(\varphi f)$ and $\varphi(\varkappa f) = \varphi(\tau f)$. Again, because φ is an isomorphism we get $\varkappa f = \tau f$. From $g(X) \subseteq f(X)$ it follows that for any point $x_1 \in X$ there is a point $x_2 \in X$ such that $g(x_1) = f(x_2)$. Then

$$\varkappa g(x_1) = \varkappa(g(x_1)) = \varkappa(f(x_2)) = \varkappa f(x_2) = \tau f(x_2) = \tau(f(x_2)) = \tau(g(x_1)) = \tau g(x_1),$$

which shows that $\varkappa g = \tau g$. But since φ is an isomorphism we have $\varphi(\varkappa g) = \varphi(\tau g)$ and $\varphi(\varkappa)\varphi(g) = \varphi(\tau)\varphi(g)$. Hence $\varkappa'\varphi(g) = \tau'\varphi(g)$.

Now suppose that the condition $\varphi g(Y) \subseteq \overline{\varphi f(Y)}$ does not hold, i.e. the set $\varphi g(Y) \setminus \overline{\varphi f(Y)}$ is not empty. Let $y' = (\varphi g)y$ be an arbitrary point of $\varphi g(Y) \setminus \overline{\varphi f(Y)}$. Since the set $\varphi g(Y) \setminus \overline{\varphi f(Y)}$ is open there exists a closed n-ball $E \subset \varphi g(Y) \setminus \overline{\varphi f(Y)}$ centered at y'. Let ϕ be any homeomorphism of E, which is constant on the boundary and $\phi(y') \neq y'$. Then the transformation $\gamma : Y \to Y$ defined by

$$\gamma(y) = \begin{cases} y, & \text{if } \; y \in Y \setminus E, \\ \phi(y), & \text{if } \; y \in E, \end{cases}$$

is continuous and quasi-open.

Now let $\varkappa'$ be a homeomorphism of Y. Then for the transformation $\tau' = \varkappa'\gamma$ and for every $y \in Y$ we have

$$\tau'(\varphi f)y = \tau'((\varphi f)y) = \varkappa'\gamma((\varphi f)y) = \varkappa'((\varphi f)y) = (\varkappa'(\varphi f))y.$$

But for the point $y' = (\varphi g)y \in \varphi g(Y) \setminus \overline{\varphi f(Y)}$ we have $\varkappa'(y') \neq \tau'(y')$. Consequently,

$$(\varkappa'(\varphi g))y = \varkappa'((\varphi g)y) = \varkappa'(y') \neq \tau'(y') = \tau'((\varphi g)y) = (\tau'(\varphi g))y.$$

This contradiction proves the first assertion.

Now let $\varphi f \in CQ_0(Y)$. For the second assertion notice that $\overline{\varphi f(Y)}$ is a closed and bounded set but $\varphi g(Y)$ is an open set.

Lemma 2.3. *Let f, g be arbitrary elements of $CQ_0(X)$ such that $f(X) \cap g(X) \neq \emptyset$ and $\varphi f, \varphi g \in CQ_0(Y)$. Then*

$$Int\left[\overline{(\varphi f)Y} \cap \overline{(\varphi g)Y}\right] \neq \emptyset.$$

Proof. Let E be a closed n-ball in $f(X) \cap g(X)$, let E_1 be a closed n-ball containing X and let $\widetilde{\tau}$ denote the homeomorphism from E_1 onto E. Now let τ denote the restriction of this homeomorphism to X. Clearly, $\tau \in CQ_0(X)$ and we have $\tau(X) \subset E \subset f(X)$, $\tau(X) \subset E \subset g(X)$. By Lemma we must have $(\varphi\tau)(Y) \subseteq \overline{(\varphi f)f(Y)}$ and $(\varphi\tau)(Y) \subseteq \overline{(\varphi g)f(Y)}$, i.e., $(\varphi\tau)(Y) \subseteq \overline{(\varphi f)f(Y)} \cap \overline{(\varphi g)f(Y)}$. Therefore $Int\left[\overline{(\varphi f)Y} \cap \overline{(\varphi g)Y}\right] \neq \emptyset$.

Let $x \in X$ and $f_k \in CQ(X)$ for $k = 1, 2, \ldots$. We say that the sequence $\{f_k\}_{k=1}^{\infty}$ of mappings converges to x if the following three conditions are satisfied

(1) $f_k \in CQ_0(X)$, $(\varphi f_k) \in CQ_0(Y)$ for $k = 1, 2, \ldots$,
(2) $\overline{f_{k+1}(X)} \subset f_k(X)$ and $\overline{(\varphi f_{k+1})(Y)} \subset (\varphi f_k)(Y)$ for $k = 1, 2, \ldots$,
(3) $\cap_{k=1}^{\infty} f_k(X) = \{x\}$.

Lemma 2.4. *Let $x \in X$. There exists a sequence $\{f_k\}_{k=1}^{\infty}$ converging to x.*

Proof. Let $x \in X$. Note that the set $CQ_0(Y)$ is an ideal of $CQ(Y)$. Then the set $\varphi^{-1}(CQ_0(Y))$ is an ideal of $CQ(X)$ and hence

$$\left\{ \begin{array}{c} CQ_0(X) \cdot \varphi^{-1}(CQ_0(Y)) \subseteq CQ_0(X), \\ CQ_0(X) \cdot \varphi^{-1}(CQ_0(Y)) \subseteq \varphi^{-1}(CQ_0(Y)). \end{array} \right. \tag{1}$$

First, let us show that there exists $h \in CQ_0(X)$ such that $x \in h(X)$ and $\varphi h \in CQ_0(Y)$. Let $d \in \varphi^{-1}(CQ_0(Y))$, let E_1 be a closed n-ball containing X and let E_2 be a closed n-ball such that $E_2 \subset d(X)$. Further, let $\widetilde{E_1}$ and $\widetilde{E_2}$ be two closed n-balls in X centered at x such that $\widetilde{E_2} \subset \widetilde{E_1}$. There exists a homeomorphism $\widetilde{\varkappa}$ from E_1 onto $\widetilde{E_1}$ such that $\widetilde{\varkappa}(E_2) = \widetilde{E_2}$. Let $\varkappa$ be the restriction of this homeomorphism to X and let $h = \varkappa d$. From (1) it follows that $h \in CQ_0(X)$ and $h \in \varphi^{-1}(CQ_0(Y))$. Hence $\varphi h \in CQ_0(Y)$. Since $E_2 \subset d(X)$ and $\widetilde{\varkappa}(E_2) = \widetilde{E_2}$ we have $x \in h(X)$.

Now, let E_3, E_4 be closed n-balls in $h(X)$ centered at x such that $x \in E_4 \subset E_3$. There exists a homeomorphism $\widetilde{\varkappa}_1$ from E_1 onto E_3 such that $\widetilde{\varkappa}_1(E_2) = E_4$. Let f_1 be the restriction of this homeomorphism to X and let $x_1 = f_1^{-1}(x)$. Clearly $f_1 \in CQ_0(X)$, $x_1 \in E_2$ and $x \in f_1(X)$. Since $f_1(X) \subset E_3 \subset h(X)$, it follows from Lemma that $\varphi f_1(Y) \subseteq \overline{\varphi h(Y)}$ and from $\varphi h \in CQ_0(Y)$ we get $\varphi f_1 \in CQ_0(Y)$.

To the point x_1 there corresponds a mapping $h_1 \in CQ_0(X)$ such that $\varphi h_1 \in CQ_0(Y)$ and $x_1 \in h_1(X)$. Let $\widetilde{E_3}$ be a closed n-ball in $h_1(X)$ centered at x_1. Clearly $x \in f_1\left(\widetilde{E_3}\right)$. Then let us choose closed n-balls E_5, E_6 in $f_1\left(\widetilde{E_3}\right) \cap E_4$ centered at x such that $x \in E_6 \subset E_5$. There exists a homeomorphism $\widetilde{\varkappa}_2$ from E_1 onto E_5 such that $\widetilde{\varkappa}_2(E_2) = E_6$. Let f_2 denote the restriction of this homeomorphism to X and let $x_2 = f_2^{-1}(x)$. Clearly $f_2 \in CQ_0(X)$, $x_2 \in E_2$ and $x \in f_2(X)$. By the definition, we have $f_2(X) \subset E_5 \subset f_1(X)$ but then $\overline{f_2(X)} \subset E_5 \subset f_1(X)$ and by Lemma $\varphi f_2 \in CQ_0(Y)$. Let c_1 denote the element $f_1^{-1} f_2 \in CQ_0(X)$. Clearly $f_1 c_1 = f_2$. Since $c_1(X) \subset f_1^{-1} f_2(X) \subset \widetilde{E_3} \subset h_1(X)$, it follows from Lemma that $\varphi c_1(Y) \subseteq \overline{\varphi h_1(Y)}$. From $\varphi h_1 \in CQ_0(Y)$ we get $\varphi c_1 \in CQ_0(Y)$. Then there exists a compact subset $K_{\varphi c_1}$ in Y such that $\varphi c_1(Y) \subset K_{\varphi c_1}$. Since φ is an isomorphism, we have $(\varphi f_1)(\varphi c_1) = \varphi f_2$. Thus $\overline{\varphi f_2(Y)} = \overline{(\varphi f_1)(\varphi c_1)(Y)} \subseteq \overline{(\varphi f_1)((\varphi c_1)(Y))} \subseteq \overline{(\varphi f_1)\left(K_{\varphi c_1}\right)} = (\varphi f_1)\left(K_{\varphi c_1}\right) \subset \varphi f_1(Y)$ and hence condition 2 holds.

This process yeilds the sequence $\{f_k\}_{k=1}^{\infty}$converging to x. Indeed, the constructed sequence $\{f_k\}_{k=1}^{\infty}$ satisfies the conditions 1 and 2. This sequence satisfies the condition 3 if the sequence $\{r_k\}_{k=1}^{\infty}$ converges to 0, where r_k is the radius of the corresponding closed n-ball E_k.

Lemma 2.5. *Let $x \in X$ and let $\{f_k\}_{k=1}^{\infty}$ be the sequence converging to x. There exists a unique point $\theta x \in Y$ such that the sequence $\{\varphi f_k\}_{k=1}^{\infty}$ converges to θx and the point θx does not depend on the choice of the sequence $\{f_k\}_{k=1}^{\infty}$.*

Proof. The sequence $\{\varphi f_k\}_{k=1}^{\infty}$ satisfies the conditions 1 and 2. Indeed if $(\varphi f_k) \in CQ_0(Y)$ then $\varphi^{-1}(\varphi f_k) = f_k$ and therefore $\varphi^{-1}(\varphi f_k) \in CQ_0(X)$. If $\overline{(\varphi f_{k+1})Y} \subset (\varphi f_k)Y$ then $\overline{(\varphi^{-1}(\varphi f_{k+1}))X} \subset (\varphi^{-1}(\varphi f_k))X$. Let us show that

$$\cap_{k=1}^{\infty}(\varphi f_k)(Y) = \cap_{k=1}^{\infty}\overline{(\varphi f_k)(Y)}. \tag{2}$$

For sake of contradiction, suppose that there exists a point y such that $y \in \cap_{k=1}^{\infty}\overline{(\varphi f_k)Y}$ but $y \notin \cap_{k=1}^{\infty}(\varphi f_k)Y$. Then there exists a natural number m such that $y \in \overline{(\varphi f_m)Y} \setminus (\varphi f_m)Y$. Since

$\overline{(\varphi f_{m+1})Y} \subset (\varphi f_m)Y$ and $y \notin (\varphi f_m)Y$ we must have $y \notin \overline{(\varphi f_{m+1})Y}$ and hence $y \notin \cap_{k=1}^{\infty} \overline{(\varphi f_k)Y}$ contradicting the assumption that $y \in \cap_{k=1}^{\infty} \overline{(\varphi f_k)Y}$. The set $\cap_{k=1}^{\infty} \overline{(\varphi f_k)Y}$ is not empty as the intersection of nested closed sets and therefore the set $\cap_{k=1}^{\infty} (\varphi f_k)Y$ is not empty.

Now let $y \in \cap_{k=1}^{\infty} (\varphi f_k)Y$. Suppose that the set $\cap_{k=1}^{\infty} (\varphi f_k)(Y)$ contains another point y'. Let $\{h'_k\}_{k=1}^{\infty}$ be a sequence which converges to y. Since $y \in (\varphi f_i)Y$ and $y \in h'_j(Y)$ for any i, j, it follows that

$$y \in (\varphi f_i)(Y) \cap h'_j(Y) \tag{3}$$

for all naturals i, j and $\cap_{k=1}^{\infty} \overline{h'_k(Y)} = \cap_{k=1}^{\infty} h'_k(Y) = \{y\}$. Then there exists a natural number m such that $y' \notin \overline{h'_k(Y)}$ whenever $k \geqslant m$. Let h_j denote the mapping $\varphi^{-1}(h'_j)$. Then $\overline{(\varphi^{-1} h'_{j+1})X} \subset (\varphi^{-1} h'_j)X$ and hence $\overline{h_{j+1}(X)} \subset h_j(X)$. Now suppose that $x \notin h_j X$ for some natural j. Then $x \notin \overline{h_{j+1} X}$ and hence there exists a natural number i such that

$$\overline{f_i(X)} \cap \overline{h_{j+1}(X)} = \emptyset. \tag{4}$$

From (3) it follows that $y \in (\varphi f_i)(Y) \cap h'_{j+1}(Y)$ and by Lemma we get $\overline{f_i(X)} \cap \overline{h_{j+1}(X)} \neq \emptyset$ which contradicts (4). Thus for any natural number j the point x belongs to $h_j(X)$. Since the sequence $\{f_k\}_{k=1}^{\infty}$converges to x we must have $f_k(X) \subseteq h_j(X)$ for every natural number $k > k_j$, for some k_j. By Lemma we get $(\varphi f_k)(Y) \subset \overline{h'_j(Y)}$ for $k > k_j$ and therefore $y' \notin (\varphi f_k)(Y)$ for $k > k_j$, $j \geqslant m$. This contradiction proves that the set $\cap_{k=1}^{\infty} (\varphi f_k)(Y)$ consists of one point and $\cap_{k=1}^{\infty} (\varphi f_k)(Y) = \cap_{k=1}^{\infty} \overline{(\varphi f_k)(Y)} = \{y\}$.

Let us denote the point y by $\theta(x)$ and prove that the point $\theta(x)$ does not depend on the choice of the sequence $\{f_k\}_{k=1}^{\infty}$. Let $\{g_k\}_{k=1}^{\infty}$ be another sequence converging to x. Then for any natural number k there exists a natural number i_k such that $f_i(X) \subset g_k(X)$ whenever $i \geqslant i_k$. By Lemma we have $(\varphi f_i)(Y) \subset \overline{(\varphi g_k)(Y)}$. Since the set $\cap_{i=1}^{\infty} (\varphi f_i)(Y)$ consists of one point y, then $y \in (\varphi f_i)(Y)$ for any natural number i, and $y \in \overline{(\varphi g_k)(Y)}$ for any natural number k as well. We can similarly show that

$$\cap_{k=1}^{\infty} (\varphi g_k)(Y) = \cap_{k=1}^{\infty} \overline{(\varphi g_k)(Y)}.$$

Hence $\cap_{i=1}^{\infty} (\varphi f_i)(Y) = \cap_{k=1}^{\infty} (\varphi g_k)(Y) = \cap_{k=1}^{\infty} \overline{(\varphi g_k)(Y)} = \{y\}$, i.e. the point $y = \theta(x)$ does not depend on the choice of the sequence $\{f_k\}_{k=1}^{\infty}$.

Let $\theta : X \to Y$ denote the function which maps a point $x \in X$ to the point $y = \theta(x) \in Y$.

Lemma 2.6. *Let f be an element of $CQ_0(X)$ such that $\varphi f \in CQ_0(Y)$. If $x \in f(X)$ then $\theta(x) \in \overline{\varphi f(Y)}$.*

Proof. Let $x \in f(X)$ and $\{f_k\}_{k=1}^{\infty}$ be a sequence converging to x. Then there exists a natural number i, such that $f_k(X) \subset f(X)$ for all $k \geqslant i$. The sequence $\{f_k\}_{k=i}^{\infty}$ also converges to x and by Lemma it follows that the sequence $\{\varphi f_k\}_{k=i}^{\infty}$converges to y, where $y = \theta(x) \in (\varphi f_i)(Y)$. Since $f_i(X) \subset f(X)$, by Lemma it follows that $(\varphi f_i)(Y) \subset \overline{(\varphi f)(Y)}$ and therefore $y = \theta(x) \in \overline{\varphi f(Y)}$.

Lemma 2.7. *The function $\theta : X \to Y$ is a homeomorphism.*

Proof. The map $\theta : X \to Y$ is surjective. Indeed, let y be any point in Y and $\left\{f'_k\right\}_{k=1}^{\infty}$ be a sequence converging to y. Then the sequence $\left\{\varphi^{-1} f'_k\right\}_{k=1}^{\infty}$converges to some point x in X such that $\theta x = y$.

Let us show that θ is injective. Suppose that $\theta x_1 = \theta x_2 = y'$ for some $x_1, x_2 \in X$, $x_1 \neq x_2$. Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be the sequences converging to x_1 and x_2, respectively. Then the sequences $\{\varphi f_k\}_{k=1}^{\infty}$ and $\{\varphi g_k\}_{k=1}^{\infty}$converge to y', but then the sequences $\left\{\varphi^{-1}(\varphi f_k)\right\}_{k=1}^{\infty} = \{f_k\}_{k=1}^{\infty}$ and $\left\{\varphi^{-1}(\varphi g_k)\right\}_{k=1}^{\infty} = \{g_k\}_{k=1}^{\infty}$ must converge to a unique point. Hence $x_1 = x_2$ contradicting the assumption that $x_1 \neq x_2$.

Let us now show that θ and θ^{-1} are continuous. Let U be any open neighborhood of θx, E be a closed n-ball in Y centered at $\theta(x)$ such that $E \subset U$, and $\{f_k\}_{k=1}^{\infty}$ be a sequence converging to some point x. Then it follows from Lemma that $\cap_{k=1}^{\infty}(\varphi f_k)(Y) = \{\theta x\}$ and $\overline{(\varphi f_{k+1})(Y)} \subset (\varphi f_k)(Y)$. Therefore $\overline{(\varphi f_k)(Y)} \subset E$ for some natural k. Since $x \in f_k(X)$ and the set $f_k(X)$ is an open set, there exists an open neighborhood V of x, such that $V \subset f_k(X)$. Then it follows from Lemma that $\theta(V) \subset \theta(f_k X) \subset \overline{(\varphi f_k)(Y)} \subset E \subset U$. Thus the function θ is continuous. A similar proof shows that θ^{-1} is continuous and thus θ is a homeomorphism.

3. An abstract characterization of semigroups of quasi-open maps

A topological space X is said to be a T_D-space if for every point ξ in X the set $\left\{\overline{\xi}\right\} \setminus \{\xi\}$ is closed [4]. We denote the set $\left\{\overline{\xi}\right\} \setminus \{\xi\}$ by $\{\xi\}'$. Obviously, each T_D-space is T_0-space and each T_1-space is T_D-space.

Lemma 3.1. *Let X be a T_D-space with no isolated points and let $\xi \in X$ and let a, b be arbitrary elements of $Q(X)$. The condition*

$$\forall f, g \in Q(X), fa = ga \to fb = gb \tag{5}$$

is necessary and sufficient for $b(X) \subseteq a(X)$.

Proof. If the condition $b(X) \subseteq a(X)$ is satisfied, then for every $x \in X$ there exists a point $\xi \in X$ such that $b(x) = a(\xi)$. Then

$$fb(x) = f(b(x)) = f(a(\xi)) = fa(\xi) = ga(\xi) = g(a(\xi)) = g(b(x)) = gb(x).$$

So, condition (5) holds.

Now let condition (5) hold for some $a, b \in Q(X)$. Suppose that the set $b(X) \setminus a(X)$ is not empty. For any point $\xi = b(x)$ in $b(X) \setminus a(X)$ there exist $f, g \in Q(x)$, such that $f(\xi) \neq g(\xi)$ but $f(x) = g(x)$ for all $x \in X \setminus \{\xi\}$. Indeed, select a point ξ in X and consider the map $f : X \to X$ defined by

$$f(x) = \begin{cases} \eta_1 & \text{if } x = \xi, \\ x & \text{if } x \neq \xi, \end{cases}$$

and the map $g : X \to X$ defined by

$$g(x) = \begin{cases} \eta_2 & \text{if } x = \xi, \\ x & \text{if } x \neq \xi, \end{cases}$$

where $\eta_1 \neq \eta_2$ are any fixed points in $X \setminus \{\xi\}$. The maps f and g are quasi-open and we have $f(\xi) \neq g(\xi)$ but $f(x) = g(x)$ for all $x \in X \setminus \{\xi\}$. Then for every $x \in X$ the point $a(x)$ is in $X \setminus \{\xi\}$ and therefore $fa(x) = f(a(x)) = g(a(x)) = ga(x)$. But for $\xi = b(x) \in b(X) \setminus a(X)$ we have $fb(x) = f(b(x)) = f(\xi) \neq g(\xi) = g(b(x)) = gb(x)$ which contradicts to (5).

Lemma 3.2. *Let X and Y be T_D-spaces with no isolated points and let $\varphi : Q(X) \to Q(Y)$ be an isomorphism between semigroups $Q(X)$ and $Q(Y)$. If $a(X) \subseteq b(X)$ for some $a, b \in Q(X)$ then $(\varphi a)(Y) \subseteq (\varphi b)(Y)$. Hence if $a(X) = b(X)$ for some $a, b \in Q(X)$ then $(\varphi a)(Y) = (\varphi b)(Y)$.*

Proof. Suppose that $b(X) \subseteq a(X)$. If $f'(\varphi a) = g'(\varphi a)$ for some elements $f', g' \in Q(Y)$ then there exist $f, g \in Q(X)$ such that $f' = \varphi f$ and $g' = \varphi g$. Then $(\varphi f)(\varphi a) = (\varphi g)(\varphi a)$ and since φ is an isomorphism, $\varphi(fa) = \varphi(ga)$ and $fa = ga$. We have $fb = gb$, by Lemma . Again, since φ is an isomorphism, then $(\varphi f)(\varphi b) = (\varphi g)(\varphi b)$ and therefore $f'(\varphi b) = g'(\varphi b)$. Because $f'(\varphi b) = g'(\varphi b)$ is true for every $f', g' \in Q(Y)$ satisfying the condition $f'(\varphi a) = g'(\varphi a)$ it follows from Lemma that $(\varphi b)(Y) \subseteq (\varphi a)(Y)$. In the same way, we could show that if $a(X) \subseteq b(X)$ then $(\varphi a)(Y) \subseteq (\varphi b)(Y)$.

Let X be a T_D-space with no isolated points that has an open base, each element of which is an image of X under a quasi-open mapping and let Λ be a class of all such spaces. For instance, the open subsets of the α-cube I^α, $\alpha \geqslant 1$, the set $\mathbb{R}$ of real numbers with Zariski topology and any topological space X, $|X| \geqslant \aleph_0$, with cofinite topology belong to the class Λ.

Lemma 3.3. *Let $X \in \Lambda$ and let U be any open subset of X. Then there exists a quasi-open mapping $a \in Q(X)$ such that $a(X) = U$.*

Proof. Let $X \in \Lambda$ and $\Im$ is an open base of X. Suppose that U is an open subset of X and $i : U \to X$ is the inclusion map, which is open map. Let $V_1 \in \Im$ and $V_1 \subset U$, then there exists a quasi-open mapping f from X onto V_1. Consider the restriction of f to $X \setminus \overline{U}$. Since restriction of a quasi-open map to an open set is quasi-open, this map is quasi-open. Denote by g the extension of this mapping to $X \setminus U$ obtained by assigning all boundary points of U to any fixed point in U. The mapping $a : X \to U$ defined by

$$a(x) = \begin{cases} i(x), & \text{if} \quad x \in U, \\ g(x), & \text{if} \ x \in X \setminus U, \end{cases}$$

is a quasi-open map and $a(X) = U$.

Let X be a topological space. The family $O(X)$ of all open sets of X is a complete distributive lattice if set inclusion is taken as the ordering. By the duality principle for ordered sets, two topological spaces X and Y are homeomorphic if and only if lattices $O(X)$ and $O(Y)$ are isomorphic [4].

Lemma 3.4. *Let $X, Y \in \Lambda$. If the semigroups $Q(X)$ and $Q(Y)$ are isomorphic then the lattices $O(X)$ and $O(Y)$ are lattice-isomorphic.*

Proof. Let U be any open subset of X. By Lemma there exists a quasi-open function $a \in Q(X)$ such that $a(X) = U$. Since the semigroups $Q(X)$ and $Q(Y)$ are isomorphic there exists a quasi-open function $a' \in Q(Y)$ such that $\varphi a = a'$. Let $a'(Y) = U'$. We define a map θ from $O(X)$ to $O(Y)$ by assinging to each open set $U \subset X$ the set $U' \subset Y$. The map θ does not depend on the choice of $a \in Q(X)$. Indeed , if $a(X) = U$ and $b(X) = V$ then Lemma says that $(\varphi a)(Y) = (\varphi b)(Y) = U'$. Let U and V be any two different open subsets of X. By Lemma there exist two quasi-open functions $a, b \in Q(X)$ such that $a(X) = U$ and $b(X) = U$. Since the semigroups $Q(X)$ and $Q(Y)$ are isomorphic it follows from Lemma that $(\varphi a)Y \neq (\varphi b)Y$. Hence θ is bijective. Now suppose that U' is an arbitrary open set in Y. Since the semigroups $Q(X)$ and $Q(Y)$ are isomorphic it follows from Lemma that there exists an open set $U \subset X$ such that $\theta(U) = U'$. Again it follows from Lemma that if $U \subseteq V$ then $\theta(U) \subseteq \theta(V)$. From Theorem 2.1 of [4] it follows that the topological spaces X and Y are homeomorphic.

Theorem 3.5. *Let $X, Y \in \Lambda$. The semigroups $Q(X)$ and $Q(Y)$ are isomorphic if and only if the spaces X and Y are homeomorphic.*

Proof. It is obvious that if X and Y are homeomorphic then $Q(X)$ and $Q(Y)$ are isomorphic. Specifically, if h is a homeomorphism from X onto Y, then $f \to hfh^{-1}$ is an isomorphism from $Q(X)$ onto $Q(Y)$. The proof of the necessary condition follows from Lemmas -.

Conflict of Interests

The author declares that there is no conflict of interests.

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