Available online at http://scik.org
J. Semigroup Theory Appl. 2015, 2015:3
ISSN: 2051-2937

DERIVATIONS ON QS- ALGEBRAS

SAMY M. MOSTAFA*, R. A. K. OMAR AND MOSTAFA A. HASSAN
Department of mathematics -Faculty of Education -Ain Shams University Roxy, Cairo, Egypt

Abstract. In this paper, we introduce the notions of (ℓ, r) $((r, \ell))$-derivations of a QS-algebras, (r, ℓ) $((\ell, r))$-*t*-derivations of a QS-algebras, *t*- *bi*-derivations of a QS-algebras and we investigate several interesting basic properties.

Keywords: QS-algebras; $(\ell, r)((r, \ell))$-derivations of a QS-algebras; (r, ℓ) $((\ell, r))$-*t*-derivations of a QS-algebras; *t*- *bi*- derivations of a QS-algebras.

2010 AMS Subject Classification: 03G25, 06F35.

1. Introduction

In 1966, Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [10,11,16]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Neggers et al [8] introduced a notions, called Q-algebras, which is a generalization of BCH / BCI / BCK-algebras and generalized some theorems discussed in BCI-algebras. Moreover, Ahn and Kim [15] introduced the notions of QS-algebras which is a proper subclass of Q-algebras. Kondo [13] proved that, each theorem of QS-algebras is provable in the theory of Abelian groups and conversely each theorem of Abelian groups is provable in the theory of QS-algebras. Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. Several authors [2,6,7,13,14] have studied derivations in rings and near rings. Jun and Xin [17] applied the notions of derivations in ring and near-ring theory to *BCI*-algebras, and they also introduced a new concept called a regular

*Corresponding author
Received March 5, 2015

derivations in *BCI* -algebras. They investigated some of its properties, defined a *d* -derivations ideal and gave conditions for an ideal to be *d*-derivations. Later, Abujabal and Al-Shehri [5], defined a left derivations in *BCI*-algebras and investigated a regular left derivations. Zhan and Liu [9] studied f-derivations in BCI-algebras and proved some results. Muhiuddin and Al-roqi [3,4] introduced the notions of (α,β)-derivations in a BCI-algebras and investigated related properties. They provided a condition for a (α,β)- derivations to be regular. They also introduced the concepts of a $d_{(\alpha,\beta)}$- invariant (α,β)-derivations and α-ideal, and then they investigated their relations. Furthermore, they obtained some results on regular (α,β) -derivations. Moreover, they studied the notions of *t*-derivations on BCI-algebras [4] and obtain some of its related properties. Further, they characterized the notions of *p*-semisimple BCI-algebras X by using the notions of *t*-derivations. In this paper we introduce the notions of $(\ell,r)((r,\ell))$-derivations of a QS-algebras, (r,ℓ) $((\ell,r))$-*t*-derivations of a QS-algebras, *t*- *bi*-derivations of a QS-algebras and investigate some related properties.

2. Preliminaries

In this section, we recall some basic definitions and results that are needed for our work.

Definition 2.1[15] A QS-algebra $(X,*,0)$ is a non-empty set X with a constant 0 and a binary operation $*$ such that for all $x, y, z \in X$ satisfying the following axioms:

(QS-1) $(x*y)*z=(x*z)*y$.

(QS-2) $x*0=x$.

(QS-3) $x*x=0$.

(QS-4) $(x*y)*(x*z)=z*y$.

Definition 2.2 [15] Let $(X,*,0)$ be a QS-algebra, we can define a binary relation $\leq$ on X as, $x\leq y$ if and only if $x*y=0$, this makes X as a partially ordered set.

Proposition 2.3[15] Let $(X,*,0)$ be a QS-algebra. Then the following hold: $\forall\ x, y, z \in X$.

1. $x \le y \ \ implies \ z * y \le z * x.$
2. $x \le y \ \ and \ \ y \le z \ \ imply \ x \le z.$
3. $x * y \le z \ \ implies \ x * z \le y.$
4. $(x * z) * (y * z) \le x * y.$
5. $x \le y \ \ implies \ x * z \le y * z.$
6. $0 * (0 * (0 * x)) = 0 * x.$

Lemma 2.4[12] Let (X , $*$,0) be a QS-algebra. If $x * y = z, \ then \ x * z = y \ \ \forall x, y, z \in X.$

Lemma 2.5[12] Let (X , $*$,0) be a QS-algebra. $0 * (x * y) = y * x \ \ \forall x, y \in X.$

Corollary 2.6[12] Let (X , $*$,0) be a QS-algebra. $0 * (0 * x) = x \ \ \forall x \in X.$

Lemma 2.7 [12] Let (X , $*$,0) be a QS- algebra. $x * (0 * y) = y * (0 * x) \ \ \forall x, y \in X.$

Proposition 2.8 Let (X , $*$,0) be a QS-algebra. Then the following hold: $\forall \ x, y, z \in X$ **.**

1. $x * (x * y) = y$.
2. $x * (x * (x * y)) = x * y$ **.**
3. $(x * (x * y)) * y = 0$.
4. $(x * z) * (y * z) = x * y$.
5. $(x * y) * x = 0 * y$.
6. $x * 0 = 0 \Rightarrow x = 0$.
7. $0 * (x * y) = (0 * x) * (0 * y)$.
8. $x * y = 0, y * x = 0 \Rightarrow x = y$**.**

Proof. 1. $x * (x * y) = \overbrace{(x * 0) * (x * y)}^{from \ Def \ 2.1. \ (QS-2)} = \overbrace{y * 0}^{from \ Def \ 2.1. \ (QS-4)} = y.$

2. $x * (x * (x * y)) = \overbrace{x * y}^{from \ \text{Proposition} \ 2.8. \ 1}$.

3. $(x * (x * y)) * y = \overbrace{y * y}^{from \ \text{Proposition} \ 2.8. \ 1} = 0.$

4. $(x*z)*(y*z) \leq x*y$ clear from Proposition 2.3. 4

$$(x*y)*((x*z)*(y*z)) = \overbrace{(x*y)*((0*(z*x))*(0*(z*y)))}^{from\ Lemma\ 2.5.} = \overbrace{(x*y)*((z*y)*(z*x))}^{from\ Def\ 2.1.\ (QS-4)} =$$

$$\overbrace{(x*y)*(x*y)}^{from\ Def\ 2.1.\ (QS-4)} = 0,\ then\ x*y \leq (x*z)*(y*z).$$

$Hence\ (x*z)*(y*z) = x*y.$

5. $(x*y)*x = \overbrace{(x*x)*y}^{from\ Def\ 2.1.\ (QS-1)} = 0*y.$

6. $If\ x*0 = 0,\ then\ \overbrace{x=0}^{from\ Def\ 2.1.\ (QS-2)}$.

7. $0*(x*y) = \overbrace{(x*x)*(x*y)}^{from\ Def\ 2.1.\ (QS-3)} = \overbrace{y*x}^{from\ Def\ 2.1.\ (QS-4)} = \overbrace{(0*x)*(0*y)}^{from\ Def\ 2.1.\ (QS-4)}.$

8. $x*y = 0 \Rightarrow x \leq y\ and\ y*x = 0 \Rightarrow y \leq x,\ then\ x = y$.

Example 2.9 [12] Let $X = \{0,1,2\}$ be a set in which the operation $*$ is defined as follows:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then (X , $*$, 0) is a QS-algebra.

Definition 2.10 Let (X , $*$,0) be a QS-algebra and S be a non-empty subset of X , then S is called subalgebra of X if $x*y \in S\ \ \forall x, y \in S$.

Definition 2.11 (X , $*$,0) is a QS-algebra , $x, y \in X$ we denote $x \wedge y = y*(y*x)$.

3. Derivations of QS-algebras

Definition 3.1 Let (X , $*$,0) be a QS-algebra . A map $d : X \to X$ is called a left- right derivation (briefly (l,r)-derivation) of X if d($x*y) = (d(x)*y) \wedge (x*d(y))$ $\forall\ x,y \in X$.
Similarly, a map d: $X \to X$ is called a right- left derivation (briefly (r,l) -derivation) of X if d($x*y) = (x*d(y)) \wedge (d(x)*y)$ $\forall\ x,y \in X$ .A map $d : X \to X$ is called a derivation of X if d is both a (l,r)-derivation and a (r,l) -derivation of X .

Example 3.2 Let $X = \{0,1,2\}$ be a QS-algebra, in which the operation $*$ is defined as follows:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	0	0

Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 0 & if \ \ x = 0 \\ 1 & if \ \ x = 1 \\ 2 & if \ \ x = 2 \end{cases}$$

Then it is clear that d is a derivation of X .

Definition 3.3 Let (X , $*$,0) be a QS-algebra and $d : X \to X$ be a map of a QS-algebra X , then d is called regular if d (0)=0.

Proposition 3.4 Let (X , $*$,0) be a QS-algebra

1. If d is a (l,r)-derivation of X , then $d(x) = d(x) \wedge x$ $\forall x \in X$.
2. If d is a (r,l) -derivation of X , then

   d is regular $\Leftrightarrow d(x) = x \wedge d(x)$ $\forall x \in X$.

Pr*oof.* 1. Let d be a (l,r)-derivation of X .Then

$$d(x) = d(x*0) = (d(x)*0) \wedge (x*d(0)) = d(x) \wedge (x*d(0)) = (x*d(0))*((x*d(0))*d(x))$$

$$= \overbrace{(x*d(0))*((x*d(x))*d(0))}^{from\ Def\ 2.1.(QS-1)} = \overbrace{x*(x*d(x))}^{from\ \ Pro\ \ 2.8.\ \ 4} = d(x) \wedge x.$$

2. Let d be regular (r,l)-derivation of X. Then

$d(x)=d(x*0)=(x*d(0))\wedge(d(x)*0)=(x*0)\wedge d(x)=x\wedge d(x)$.

Conversely, let d be a (r,l)-derivation of X and $d(x)=x\wedge d(x)$ $\forall\, x\in X$, then we get

$d(0)=0\wedge d(0)=d(0)*(d(0)*0)=d(0)*d(0)=0$. Hence d is regular.

Lemma 3.5 Let (X ,$*$,0) be a QS-algebra and d be a (l,r)-derivation of X. Then the following hold $\forall\, x,y\in X$.

1. $d(x*y)=d(x)*y$.
2. $d(0)=d(x)*x$ *and if* d *is regular, then* $d(x)\le x$.

Proof. Clear.

Lemma 3.6 Let (X ,$*$,0) be a QS- algebra and d be a (r,l)-derivation of X. Then

1. $d(x*y)=x*d(y)$ $\forall\, x,y\in X$.
2. $d(0)=x*d(x)$ *and if* d *is regular, then* $x\le d(x)$.

Proof. Clear.

Theorem 3.7 Let (X ,$*$,0) be a QS-algebra and d be a regular (r,l)-derivation of X. Then the following hold: $\forall\, x,y\in X$.

1. $d(x)=x$.
2. $d(x)*y=x*d(y)$.
3. $d(x*y)=d(x)*y=x*d(y)=d(x)*d(y)$.
4. $Ker(d)=\{x\in X : d(x)=0\}$ is a subalgebra of X.

Proof. 1. Since d is a regular (r,l)-derivation of X, we have

$$d(x)=d(x*0)=\overbrace{x*d(0)}^{\text{from Theorem 3.6. 1}}=x*0=x.$$

2. Since d is a regular (r,l)-derivation of X, then by Theorem 3.7. 1,we have

$d(x)=x$ $\forall x\in X$. Then $d(x)*y=x*y=x*d(y)$.

3. Since d is a regular (r,l)-derivation of X, then by Theorem 3.7. 1, we have $d(x)=x\ \forall x\in X$. Then $d(x*y)=d(x)*y=x*d(y)=d(x)*d(y)=x*y$.
4. Since d is a regular, $d(0)=0,$ then $0\in Ker(d),$ which implies that $Ker(d)$ is non-empty set .Let $x,y\in Ker(d),$ then $d(x)=0,\ d(y)=0,$ hence we have $d(x*y)=x*y=d(x)*d(y)=0*0=0,$ therefore $(\boldsymbol{x*y})\in \boldsymbol{Ker(d)}$ and $Ker(d)$ is a subalgebra of X.

Lemma 3.8 Let (X ,$*$,0) be a QS-algebra and d be a derivation on X. If $x\le y\ \forall x,y\in X$.*Then* $d(x)=d(y)$.

Pr*oof*.We have

$$x\le y \Leftrightarrow x*y=0\ ,then\ d(x)=\overbrace{d(x*0)}^{from\ Def\ 2.1.\ (QS-2)}=d(x*(x*y))=\overbrace{d(y)}^{from\ Propostion\ 2.8.\ 1}.$$

4. t-Derivations on QS -Algebras

Definition 4.1 Let (X ,$*$,0) be a QS-algebra .Then for any $t\in X$, we define a self map $d_t : X\to X$ by $d_t(x)=x*t\ \forall\ x\in X$.

Definition 4.2 Let (X ,$*$,0) be a QS -algebra .Then for any $t\in X$, A self map $d_t : X\to X$ is called a t-(l,r)-derivation of X if it satisfies the condition $d_t(x*y)=(d_t(x)*y)\wedge(x*d_t(y))\ \forall\ x,y\in X$. Similarly for any $t\in X$, A self map $d_t : X\to X$ is called a $t-(r,l)$-derivation of X if it satisfies the condition $d_t(x*y)=(x*d_t(y))\wedge(d_t(x)*y)\ \forall\ x,y\in X$. And for any $t\in X$, A self map $d_t : X\to X$ is called a $t-$derivation of X if d_t is both a $t-(l,r)$-derivation and a $t-(r,l)$-derivation of X .

Example 4.3 Let $X=\{0,1,2\}$ be a QS -algebra in which the operation $*$ is defined as follows:

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

Define a map $d_t : X \to X$ by

$$d_t(x) = \begin{cases} x & \forall x \in X & if\ t = 0 \\ 0 & \forall x \in X & if\ t = 1,2 \end{cases}$$

Then it is clear that d_t is a derivation of X .

Definition 4.4 Let (X ,$*$,0) be a QS -algebra and $d_t : X \to X$ be a map of a QS -algebra X , then d_t is called t-regular if $d_t(0) = 0$.

Proposition 4.5 Let (X ,$*$,0) be a QS -algebra.

1. If d_t is a t-(l,r)-derivation of X , then $d_t(x) = d_t(x) \wedge x \quad \forall x \in X$.
2. If d_t is a t-(r,l)-derivation of X , then

   d_t is regular $\Leftrightarrow d_t(x) = x \wedge d_t(x)\ \forall x \in X$ **.**

Proof. 1. Let d_t be a t-(l,r)-derivation of X .Then

$$d_t(x) = d_t(x*0) = (d_t(x)*0) \wedge (x * d_t(0)) = d_t(x) \wedge (x * d_t(0)) = (x * d_t(0)) * ((x * d_t(0)) * d_t(x))$$

$$= \overbrace{(x * d_t(0)) * ((x * d_t(x)) * d_t(0))}^{from\ Def\ 2.1.(QS-1)} = \overbrace{x * (x * d_t(x))}^{from\ Lemma 2.2.\ 2.} = d_t(x) \wedge x.$$

2. Let d_t be regular t-(r,l)-derivation of X . Then

   $d_t(x) = d_t(x*0) = (x * d_t(0)) \wedge (d_t(x)*0) = (x*0) \wedge d_t(x) = x \wedge d_t(x)$.

   Conversely, let d_t be a $t-(r,l)$-derivation of X and satisfied $d_t(x) = x \wedge d_t(x) \quad \forall\ x \in X$, then we get $d_t(0) = 0 \wedge d_t(0) = d_t(0) * (d_t(0)*0) = d_t(0) * d_t(0) = 0$.

Theorem 4.6 Let (X ,$*$,0) be a QS-algebra and d_t be a t-(l,r)-derivation of X . Then the following hold : $\forall\ x, y \in X$.

1. $d_t(x*y) = d_t(x)*y$.

2. $d_t(0) = d_t(x) * x$.

3. If $x \le y$, then $d_t(x) \le d_t(y)$.

Proof. 1. $d_t(x*y) = (d_t(x)*y) \wedge (x*d_t(y)) =$

$$(x*d_t(y))*((x*d_t(y))*(d_t(x)*y)) = \overbrace{d_t(x)*y}^{from\ \text{Proposition}\ 2.8.\ 1}.$$

2. $$d_t(0) = d_t(x*x) = \overbrace{d_t(x)*x}^{from\ Theorem\ 4.6.\ 1}.$$

3. Let $x \le y$, then $d_t(x)*d_t(y) = (x*t)*(y*t) = \overbrace{(x*y)}^{from\ \text{Pro}\ 2.8.\ 4} = 0$ .Thus $d_t(x) \le d_t(y)$.

Lemma 4.7 Let (X ,$*$,0) be a QS -algebra and d_t be a *t*-(r,l)-derivation of X . Then

$d_t(x*y) = x*d_t(y) \quad \forall\ x,y \in X$.

Proof. Clear.

Theorem 4.8 Let (X ,$*$,0) be a QS -algebra and d_t be a regular *t*-(r,l)-derivation of X . Then the following hold $\forall\ x,y \in X$.

1. $d_t(x) = x$.
2. $d_t(x)*y = x*d_t(y)$.
3. $d_t(x*y) = d_t(x)*y = x*d_t(y) = d_t(x)*d_t(y)$.
4. $Ker(d_t) = \{x \in X : d_t(x) = 0\}$ is a subalgebra of X .

Proof. 1. Since d_t is a regular *t*-(r,l)-derivation of X , $\forall\ x,y \in X$, we have

$$d_t(x) = d_t(x*0) = \overbrace{x*d_t(0)}^{from\ Lemma\ 4.7.} = x*0 = x.$$

2. Since d_t is a regular *t*-(r,l)-derivation of X, then by Theorem 4.8. 1, we have

$d_t(x) = x\ \forall x \in X$. Then $d_t(x)*y = x*y = x*d_t(y)$.

3. Since d_t is a regular *t*-(r,l)-derivation of X, then by Theorem 4.8. 1 $d_t(x) = x\ \forall x \in X$,

hence we have $d_t(x*y) = d_t(x)*y = x*d_t(y) = d_t(x)*d_t(y) = x*y$.

4. Since d_t is a regular, $d_t(0) = 0$, then $0 \in Ker(d_t)$, hence we have

$Ker(d_t)$ is a non-empty set.

Let $x, y \in Ker(d_t)$, then $d_t(x) = 0$, $d_t(y) = 0$, hence we have

$d_t(x * y) = x * y = d_t(x) * d_t(y) = 0 * 0 = 0$, therefore $(x * y) \in Ker(d_t)$.

Then $Ker(d_t)$ is a subalgebra of X .

Lemma 4.9 Let (X , $*$,0) be a QS -algebra and d_t be a derivation on X . If $x \le y\ \forall x, y \in X$.*Then* $d_t(x) = d_t(y)$.

Proof. We know

$$x \le y \Leftrightarrow x * y = 0 \text{ , then } d_t(x) = \overbrace{d_t(x * 0)}^{\text{from Def 2.1. } (QS-2)} = d_t(x * (x * y)) = \overbrace{d_t(y)}^{\text{from Propostion 2.8. 1}} .$$

5. Generalized t-Derivations of QS -Algebras

Definition 5.1 Let X be a QS- algebra. A mapping $D_t : X \times X \to X$ is called a generalized *t*-$(l, r)-$derivation if there exists an *t*-$(l, r)-$derivation $d_t : X \to X$ such that $D_t(x * y) = (D_t(x) * y) \wedge (x * d_t(y))\ \forall x, y \in X$. Similarly a mapping $D_t : X \to X$ is called a generalized *t*-$(r, l)-$derivation if there exists an *t*-$(r, l)-$derivation $d_t : X \to X$ such that $D_t(x * y) = (x * D_t(y)) \wedge (d_t(x) * y)\ \forall x, y \in X$.

Moreover if D_t is both a generalized *t*-$(l, r)-$and $(r, l)-$derivation, we say that D_t is a generalized *t*-derivation.

Example 5.2 Let $X = \{0,1,2,3\}$ be a QS -algebra in which the operation $*$ is defined as follows:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define a map $d_t : X \to X$ and a map $D_t : X \times X \to X$ by

$d_t(x) = x * t$ and $D_t(x) = t * x$ $\forall \ x \in X$

Then it is clear that D_t is a generalized t-derivation of X .

Definition 5.3 Let X be a QS-algebra and $D_t : X \to X$ be a map of a QS-algebra X , then D_t is called $t-$regular if $D_t(0)$=0.

Proposition 5.4 Let D_t be a self-map of a QS-algebra X .Then

1. if D_t is a generalized t-$(l,r)-$derivation of X ,then $D_t(x) = D_t(x) \wedge x \ \forall x \in X$
2. if D_t is a generalized t-$(r,l)-$derivation of X ,then

   D_t is t-regular $\Leftrightarrow D_t(x) = x \wedge d_t(x) \ \forall x \in X$.

Proof. 1. if D_t is a generalized t-$(r,l)-$derivation of X , then there exists an t-derivation d_t such that $D_t(x*y) = (D_t(x)*y) \wedge (x*d_t(y)) \ \forall x, y \in X$ .Hence, we get

$$D_t(x) = D_t(x*0) = (D_t(x)*0) \wedge (x*d_t(0)) = \overbrace{D_t(x) \wedge (x*d(0))}^{from\ Def\ 2.1\ (QS-2)} =$$

$$(x*d(0))*((x*d(0))*D_t(x)) = \overbrace{(x*d(0))*((x*D_t(x))*d(0))}^{from\ Def\ 2.1\ (QS-1)} = \overbrace{x*(x*D_t(x))}^{from\ Proposition\ 2.8.\ 4} = D_t(x) \wedge x.$$

2. if D_t is a generalized t-$(r,l)-$derivation of X , then there exists an t-$(r,l)-$derivation d_t such

   that $D_t(x*y) = (x*D_t(y)) \wedge (d_t(x)*y) \ \forall x, y \in X$ .Hence, we get

$$D_t(x) = D_t(x*0) = (x*D_t(0)) \wedge (d_t(x)*0) = (x*0) \wedge d_t(x) = x \wedge d_t(x).$$

Proposition 5.5 Let X be a QS-algebra and D_t is a generalized t-$(l,r)-$derivation of X , then the following hold $\forall x, y \in X$:

1. $D_t(x*y) = d_t(x)*y$.
2. $D_t(0) = D_t(x)*x$.
3. $D_t(x*D_t(x)) = 0$.

Proof. Clear.

Proposition 5.6 Let X be a QS-algebra and D_t is a generalized t-$(r,l)-$derivation of X, then the following hold $\forall x, y \in X$:

1. $D_t(x) = d_t(x)$.
2. $D_t(x * y) = x * d_t(y)$.
3. $D_t(D_t(x) * x) = 0$.

Proof. Clear.

6. On t-Bi-Derivations of QS –Algebras

Definition 6.1 Let X, Y be QS - algebras. We define an operation $*$ on the Cartesian product $X \times Y$ of $X\, and\, Y$ as follows $(x_1, y_1) * (x_2, y_2) = (x_1 * x_2, y_1 * y_2)$ $\forall\ (x_i, y_i) \in X \times Y,\ i = 1,2$. Then it is clear ($X \times Y$, $*$,(0,0)) a QS -algebra, and it is called the product of X, Y.

Lemma 6.2 If ($X,*,0)$ is a QS -algebra, then ($X \times Y$, $*$,0) is a QS –algebra.

Proof. Clear.

Definition6.3 Let X be a QS - algebra and $d_t : X \to X$ be a mapping. A mapping $D_t : X \times X \to X$ is defined by $D_t(x, y) = (x * y) * t$.

Definition 6.4 Let ($X,*,0)$ is a QS-algebra and $D_t : X \times X \to X$ be a mapping. If D_t satisfies the identity $D_t(x * y, z) = (D_t(x, z) * y) \wedge (x * D_t(y, z))$ for all $x, y, z \in X$, then D_t is called *t*-$left-right bi$- derivation (briefly t-$(l,r)-bi$- derivation). Similarly if D_t satisfies the identity $D_t(x * y, z) = (x * D_t(y, z)) \wedge (D_t(x, z) * y)$ for all $x, y, z \in X$, then D_t is called t-$right-left bi$-derivation (briefly t-$(r,l)-bi$- derivation). Moreover if D_t is both an (r, ℓ) and (ℓ, r) t- bi-derivation, it is called that D_t is t- bi- derivation.

Example 6.5 Let $X=\{0,1,2,3\}$ be a QS -algebra in which the operation $*$ is defined as follows:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define a map $D_t : X\times X\to X$ by

$D_t(x,y)=t*(x*y) \qquad \forall\ x,y,z,t\in X$

Then it is clear that D_t is t- bi- derivation of X .

Definition 6.6 Let X be a QS-algebra and $D_t : X\times X\to X$ be a mapping .If $D_t(0,z)=0$, $\forall z\in X$, D_t is called component wise regular. In particular if $D_t(0,0)=d_t(0)=0,$ D_t is called d_t-regular.

Proposition 6.7 Let X be a QS-algebra and $D_t : X\times X\to X$ be a mapping .Then

1. If D_t is a t-$(l,r)-bi$- derivation, then $D_t(x,z)=D_t(x,z)\wedge x \quad \forall x,z\in X$
2. If D_t is a t-$(r,l)-bi$- derivation, then

D_t is component wise regular $\Leftrightarrow D_t(x,z)=x\wedge D_t(x,z)\ \forall x,z\in X$.

Proof. 1. Let D_t be a t-$(l,r)-bi$- derivation. Then $\forall x,z\in X$

$$D_t(x,z)=D_t(x*0,z)=(D_t(x,z)*0)\wedge(x*D_t(0,z))$$
$$=\overbrace{D_t(x,z)\wedge(x*D_t(0,z))}^{from\ Def\ 2.1\ (QS-2)}$$
$$=(x*D_t(0,z))*((x*D_t(0,z))*(D_t(x,z))$$
$$=\overbrace{(x*D_t(0,z))*((x*D_t(x,z))*D_t(0,z))}^{from\ Def\ 2.1.\ (QS-1)}=\overbrace{x*(x*D_t(x,z))}^{from\ Proposition\ 2.3.\ 4}=D_t(x,z)\wedge x.$$

2. Let D_t be component wise regular *t*-$(r,l)-bi$- derivation.

Then $D_t(x,z) = D_t(x*0,z) = (x*D_t(0,z)) \wedge (D_t(x,z)*0) =$

$$(x*0) \wedge (D_t(x,z)*0) = \overbrace{x \wedge D_t(x,z)}^{from\ Def\ 2.1.\ (QS-2)} .$$

Conversely, let D_t be a *t*-$(r,l)-bi$- derivation and $D_t(x,z) = x \wedge D_t(x,z) \quad \forall x,z \in X$. Then we get

$$D_t(0,z) = 0 \wedge D_t(0,z) = D_t(0,z)*(D_t(0,z)*0) = D_t(0,z)*D_t(0,z) = 0.$$

Theorem 6.8 Let X be a QS- algebra and $D_t : X \times X \to X$ be a *t*-$(l,r)-bi$- derivation. Then

1. $D_t(x*y,z) = x*D_t(y,z) \quad \forall x,y,z \in X$.
2. $x*D_t(x,z) = y*D_t(y,z) \quad \forall x,y,z \in X$.

Pr*oof*. 1. Let D_t be a *t*-$(l,r)-bi$- derivation. Then $\forall x,y,z \in X$

$$D_t(x*y,z) = (x*D_t(y,z)) \wedge (y*D_t(x,z)) = (y*D_t(x,z))*((y*D_t(x,z))*(x*D_t(y,z)))$$

$$= \overbrace{x*D_t(y,z)}^{from\ \mathrm{Proposition}\ 2.3.\ 1} .$$

2. Let D_t be a *t*-$(l,r)-bi$- derivation. Then $\forall x,z \in X$

$$D_t(0,z) = \overbrace{D_t(x*x,z)}^{from\ Def\ 2.1.\ (QS-3)} = (x*D_t(x,z)) \wedge (x*D_t(x,z))$$

$$= (x*D_t(x,z))*((x*D_t(x,z)*(x*D_t(x,z)) = (x*D_t(x,z))*0 = \overbrace{x*D_t(x,z).}^{from\ Def\ 2.1.\ (QS-2)}$$

Thus, we can write $D_t(0,z) = x*D_t(x,z) = y*D_t(y,z) \ \forall y \in X$.

Lemma 6.9 Let X be a QS-algebra and $D_t : X \times X \to X$ be a component wise regular *t*-$(l,r)-bi$- derivation . Then $D_t(x,z) = x \quad \forall x,z \in X$.

Pr*oof*. Since D_t is a component wise regular, then $D_t(0,z) = 0$, $\forall z \in X$ .Then

$$D_t(x,z) = \overbrace{D_t(x*0,z)}^{from\ Def\ 2.1.\ (QS-2)} = (x*D_t(0,z)) \wedge (0*D_t(x,z)) = (x*0) \wedge (0*D_t(x,z))$$

$$= x \wedge (0*D_t(x,z)) = (0*D_t(x,z))*((0*D_t(x,z))*x) = \overbrace{x}^{from\ \mathrm{Proposition}\ 2.8.\ 1} .$$

Proposition 6.10 Let X be a QS-algebra and $D_t : X \times X \to X$ be a *t*-$(l,r)-bi$- derivation. If there exist $a \in X$ such that $D_t(x,z) * a = 0 \quad \forall x,z \in X$, then $D_t(x*a,z) = 0$.

Proof. Since D_t is a *t*-$(l,r)-bi$- derivation, we get

$$D_t(x*a,z) = (D_t(x,z)*a) \wedge (x*D_t(a,z)) = 0 \wedge (x*D_t(a,z))$$

$$= (x*D_t(a,z))*((x*D_t(a,z))*0) = \overbrace{(x*D_t(a,z))*(x*D_t(a,z))}^{from\ Def\ 2.1.\ (QS-2)} = \overbrace{0}^{from\ Def\ 2.1.\ (QS-3)}.$$

Proposition 6.11 Let X be a QS-algebra and $D_t : X \times X \to X$ be a *t*-$(r,l)-bi$-derivation. If there exist $a \in X$ such that $a*D_t(x,z) = 0 \quad \forall x,z \in X$, then $D_t(a*x,z) = 0$.

Proof. Since D_t is a *t*-$(r,l)-bi$- derivation, we get

$$D_t(a*x,z) = (a*D_t(x,z)) \wedge (D_t(a,z)*x) = 0 \wedge (D_t(a,z)*x)$$

$$= (D_t(a,z)*x)*((D_t(a,z)*x)*0) = \overbrace{(D_t(a,z)*x)*(D_t(a,z)*x)}^{from\ Def\ 2.1.\ (QS-2)} = \overbrace{0}^{from\ Def\ 2.1.\ (Q-3)}.$$

7. Conclusion

Derivation is a very interesting and important area of research in the theory of algebraic structures in mathematics. In the present paper, The notion of (ℓ,r) $((r,\ell))$-derivations of a QS-algebras, $(\ell,r)((r,\ell))$ *t*-derivations of a QS-algebras, *t*- *bi*- derivations of a QS-algebras are introduced and investigated, also some useful properties of these types derivations in QS-algebras. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as BCH-algebras, Hilbert algebras, BF-algebras, J-algebras, WS-algebras, CI-algebras, SU-algebras, BCL-algebras, BP-algebras and BO-algebras , PU- algebras and so forth. The main purpose of our future work is to investigate the fuzzy derivations ideals in QS-algebras, which may have a lot of applications in different branches of theoretical physics and computer science.

Conflict of Interests

The author declares that there is no conflict of interests.

REFERENCES

[1] A. M. Kamal, “σ-derivations on prime near-rings,” Tamkang Journal of Mathematics, 32(2001), no. 2, 89–93.

[2] Hvala, “Generalized derivations in rings,” Communications in Algebra, 26(1998), no. 4, 1147–1166.

[3] G. Muhiuddin and Abdullah M. Al-Roqi On (α, β)-Derivations in BCI-Algebras. Discrete Dynamics in Nature and Society 2012(2012), Article ID 403209.

[4] G. Muhiuddin and Abdullah M. Al-roqi, On *t*-Derivations of BCI-Algebras, Abstract and Applied Analysis 2012(2012), Article ID 872784, 12 pages.

[5] H. A. S. Abujabal and N. O. Al-Shehri, On left derivations of BCI-algebras, Soochow Journal of Mathematics, 33(2007), no. 3, 435–444.

[6] H. E. Bell and L.-C. Kappe, “Rings in which derivations satisfy certain algebraic conditions,” Acta Mathematica Hungarica, 53(1989), no. 3-4, 339–346.

[7] H. E. Bell and G. Mason, “On derivations in near-rings, near-rings and near-fields,” North-Holland Mathematics Studies, 137(1987), 31–35.

[8] J. Neggers, S. S. Ahn and H. S. Kim, On Q-algebras, Int. J. Math. Math. Sci. 27(12) (2001), 749- 757.

[9] J. Zhan and Y. L. Liu, “On f-derivations of BCI-algebras,” International Journal of Mathematics and Mathematical Sciences, 11(2005), 1675–1684.

[10] K.Iseki: An algebra related with a propositional calculi, *Proc. Japan Acad. Ser A Math. Sci.*, 42 (1966), 26-29.

[11] K Iseki and Tanaka S: An introduction to theory of BCK-algebras, *Math. Japo.*, 23 (1978) 1-26.

[12] M. Kindo, On the class of QS-algebras, International Journal of Mathematics and Mathematical Sciences, 49(2004), 2630-2631.

[13] M. Breˇsar and J. Vukman, “On left derivations and related mappings,” Proceedings of the American Mathematical Society, 110(1990), no. 1, 7–16.

[14] M. Breˇsar, “On the distance of the composition of two derivations to the generalized derivations,” Glasgow Mathematical Journal, 33(1991), no. 1, pp. 89–93.

[15] S.S. Ahn and H.S. Kim, ‘‘On QS-algebras,’’ Journal of the Chungcheong Mathematical Society, 12(1999), 34-35.

[16] Y.Imai and K.Iseki: On axiom systems of Propositional calculi, *XIV, Proc. Japan Acad. Ser A, Math Sci.*, 42(1966),19-22.

[17] Y. B. Jun and X. L. Xin, “On derivations of BCI-algebras,” Information Sciences, 159(2004), no. 3-4, 167–176.

[18] Appendix

Algorithm for QS-algebras.

Input (X :set, $*$:binary operation)

Output (“ X is a QS-algebra or not”)

Begin

If $X=\phi$ then go to (1.);

End If

If $0\notin X$ then go to (1.);

End If

Stop: =false;

$i:=1$;

While $i\leq|X|$ and not (Stop) do

If $x_i * x_i \neq 0$, $x_i * 0 \neq x_i$ then

Stop: = true;

End If

$j:=1$, k:=1

While $j,k\leq|X|$ and not (Stop) do

If) $(x_i * y_j)*z_k \neq (x_i * z_k)*y_j$, $(x_i * y_j)*(x_i * z_k)\neq z_k * y_j$ then

Stop: = true;

EndIf

End While

End While

If Stop then

(1.) Output (“ X is not a QS-algebra”)

Else

Output (“ X is a QS-algebra”)

End If

End