Available online at http://scik.org
J. Semigroup Theory Appl. 2015, 2015:6
ISSN: 2051-2937

ABOUT JENSEN'S INEQUALITY AND CAUCHY'S TYPE MEANS FOR POSITIVE C_0-SEMIGROUPS

GUL I HINA ASLAM*, MATLOOB ANWAR

School of Natural Sciences, National University of Sciences and Technology, Islamabad, Pakistan

Abstract. A Jensen's type inequality is obtained for a semigroup of positive linear operators and a superquadratic mapping defined on a Banach lattice algebra. The corresponding mean value theorems conduct the authors to find a new set of Cauchy's type means.

Keywords: Jensen's type inequalities; Positive semigroup of operators; Cauchy's type means; Superquadratic mappings.

2010 AMS Subject Classification: 47D03, 46B42, 43A35, 43A17.

1. Introduction and preliminaries

A consequential theory of Cauchy type means has been developed [4, 5, 6, 7, 8, 9], which is both substantial and elegant. In this paper we shall define new means on the C_0-semigroup of bounded linear positive operators, defined on a Banach lattice algebra. The intention to generalize the concept of Cauchy's type means for operator-semigroups, is not very unaccustomed. As recently in [10], a new theory of power means is introduced on a C_0-group of continuous linear operators and Cauchy's type mean are obtained.

*Corresponding author

Received April 2, 2015

The notion of Banach lattice was introduced to get a common abstract setting. Within this framework, one can talk about the ordering of elements. Therefore, the phenomena related to positivity can be generalized. It had mostly been studied in various types to spaces of real-valued functions, e.g. the space $C(K)$ of continuous functions over a compact topological space K, the Lebesque space $L^1(\mu)$ or even more generally the space $L^p(\mu)$ constructed over measure space $(X,\sum,\mu)$ for $1 \leq p \leq \infty$. We shall use without further explanation the terms order relation (ordering), ordered set, supremum, infimum.

Before moving on, to define Banach lattice, we shall firstly go through the definition of vector lattice.

Definition 1.1. A (real) vector space V endowed with an ordering $\geq$, such that it satisfies

O_1**::** $v \leq w$ implies $v+u \leq w+u$ for all $u,v,w \in V$,

O_2**::** $v \geq 0$ implies $\lambda v \geq 0$ for al $v \in V$ and $\lambda \geq 0$,

is known as an *ordered vector space* (see [11]).

It can be easily noted that, O_1 expresses the translation invariance and thus implies that the ordering of an ordered vector space V can be completely determined by the positive part $V_+ = \{v \in V : v \geq 0\}$ of V. In other words, $v \leq w$ if and only if $w - v \in V_+$.

Moreover, the other property O_2, shows that the positive part of V is a convex set and a cone with vertex 0 (mostly called the *positive cone* of V).

If for any two elements $v,w \in V$, a supremum $sup(v,w)$ and thus an infumum $inf(v,w)$ can be defined, an ordered vector space V turns into a *vector lattice*. It is trivially understood that the existence of supremum of any two elements in an ordered vector space implies the existence of supremum of finite number of elements in V. Moreover, $v \geq w$ implies $-v \leq -w$, so the existence of finite infima therefore implied.

Below are few importantly defined quantities;

$$sup(v,-v) = |v|, \quad \textbf{(absolute value of v)}$$
$$sup(v,0) = v^+, \quad \textbf{(positive part of v)}$$
$$sup(-v,0) = v^-. \quad \textbf{(negative part of v)}$$

Some compatibility axiom between norm and order is required to move from vector lattice to a Banach lattice. Which is given in the following short way:

$$|v| \leq |w| \quad implies \quad \|v\| \leq \|w\|. \tag{1}$$

The norm defined on a vector lattice is called a lattice norm.

Now, we are in position to define a Banach lattice in a formal way.

Definition 1.2.

- A *Banach lattice* is a Banach space V endowed with an ordering $\leq$, such that $(V, \leq)$ is a vector lattice with a lattice norm defined on it.
- A Banach lattice with the property that, $u, v \in V_+$, implies $uv \in V_+$, is called *Banach lattice algebra*. If the multiplicative identity element $e \in V$, it ultimately turns to unital Banach lattice algebra.

A linear mapping T from an ordered Banach space V into itself is *positive* (denoted by: $T \geq 0$) if $T(v) \in V_+$, for all $v \in V_+$. The set of all positive linear mappings forms a convex cone in the space $L(V)$ of all linear mappings from V into itself, defining the natural ordering of $L(V)$. The absolute value of T, if it exists, is given by

$$|T|(v) = sup\{T(u) : |u| \leq v\}, \quad (v \in V_+).$$

Thus $T : V \to V$ is positive if and only if $|T(v)| \leq T(|v|)$ holds for any $v \in V$.

Lemma 1.1. [[11], P-249] *A bounded linear operator T on a Banach lattice V is a positive contraction if and only if $\|(Tv)^+\| \leq \|v^+\|$ for all $v \in V$.* □

An operator A on V satisfies the positive minimum principle if for all $v \in D(A)_+ = D(A) \cap V_+$, $\phi \in V'_+$

$$\langle v, \phi \rangle = 0 \quad implies \quad \langle Av, \phi \rangle \geq 0. \tag{2}$$

Definition 1.3. A (one parameter) C_0-semigroup (or strongly continuous semigroup) of operators on a Banach space X is a family $\{Z(t)\}_{t \geq 0} \subset B(X)$ such that

(i): $Z(s)Z(t) = Z(s+t)$ for all $s, t \in \mathbb{R}^+$.

(ii): Z(0)=I, the identity operator on X.

(iii): for each fixed $f \in X, Z(t)f \to f$(with respect to the norm on X) as $t \to 0^+$,

where $B(X)$ denotes the space of all bounded linear operators defined on a Banach space X.

Definition 1.4. The (infinitesimal) generator of $\{Z(t)\}_{t\geq 0}$ is the densely defined closed linear operator $A : X \supseteq D(A) \to R(A) \subseteq X$ such that

$$D(A) = \{f : f \in X, lim_{t\to 0^+} A_t f \; exists \, in X\}$$

$$Af = lim_{t\to 0^+} A_t f \;\; (f \in D(A)),$$

where, for $t > 0$,

$$A_t f = \frac{[Z(t) - I]f}{t} \;\; (f \in X).$$

□

Let $\{Z(t)\}_{t\geq 0}$ be the strongly continuous positive semigroup, defined on a Banach lattice V. The positivity of the semigroup is equivalent to

$$|Z(t)f| \leq Z(t)|f|, \quad t \geq 0, \quad f \in V,$$

where for positive contraction semigroups $\{Z(t)\}_{t\geq 0}$, defined on a Banach lattice V we have;

$$\|(Z(t)f)^+\| \leq \|f^+\|, \quad for \, all \, f \in V.$$

The literature presented in [11], guarantees the existence of the strongly continuous positive semigroups and positive contraction semigroups on Banach lattice V with some conditions imposed on the generator of the strongly continuous positive semigroup and the very important amongst them is, that it must always satisfy (2).

For X be a unital Banach algebra with identity element e. We shall call the strongly continuous semigroup $\{Z(t)\}_{t\geq 0}$ defined on X, a *normalized semigroup*, whenever it satisfies

$$Z(t)(e) = e, \quad for \, all \quad t > 0. \tag{3}$$

The notion of normalized semigroup is inspired from normalized functionals [13].

Example 1.1. *Let* $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$*, and* $X = C(\Gamma)$*. The rotation semigroup* $\{Z(t)\}_{t\geq 0}$ *is defined as,* $Z(t)f(z) = f(e^{it} \cdot z)$*,* $f \in X$*. The identity element* $E \in X$*, s.t. for all* $z \in \Gamma$*,* $E(z) = z$*.*

Then $Z(t)E(z) = E(e^{it} \cdot z) = e^{it} \cdot z$. Or we can say that any complex number $z = e^{ix}$ is mapped to $e^{i(x+t)}$. $Z(t)$ satisfies (3) only when t is a multiple of 2π. It can easily be verified that $Z' = \{Z(2\pi t)\}_{t\geq 0}$ is a subgroup of $Z = \{Z(t)\}_{t\geq 0}$, as $Z(2\pi t)Z(2\pi s) = Z(2\pi(t+s))$. Therefore $Z' = \{Z(2\pi t)\}_{t\geq 0}$ is a normalized semigroup. □

For real and continuous functions φ, χ on a closed interval $K := [k_1, k_2]$, such that φ, χ are differentiable in the interior of K and $\chi' \neq 0$, throughout the interior of K. A very well know Cauchy mean value theorem guarantees the existence of of a number $\zeta \in (k_1, k_2)$, such that

$$\frac{\varphi'(\zeta)}{\chi'(\zeta)} = \frac{\varphi(k_1) - \varphi(k_2)}{\chi_(k_1) - \chi(k_2)}.$$

Now, if the function $\frac{\varphi'}{\chi'}$ is invertible, then the number ζ is unique and

$$\zeta := \Big(\frac{\varphi'}{\chi'}\Big)^{-1}\Big(\frac{\varphi(k_1) - \varphi(k_2)}{\chi_(k_1) - \chi(k_2)}\Big).$$

The number ζ is called *Cauchy's mean value* of numbers k_1, k_2. It is possible to define such a mean for several variables, in terms of divided difference. Which is given by

$$\zeta := \Big(\frac{\varphi^{n-1}}{\chi^{n-1}}\Big)^{-1}\Big(\frac{[k_1, k_2, ..., k_n]\varphi}{[k_1, k_2, ..., k_n]\chi}\Big).$$

This mean value was first defined and examined by Leach and Sholander [**?**]. The integral representation of Cauchy mean is given by

$$\zeta := \Big(\frac{\varphi^{n-1}}{\chi^{n-1}}\Big)^{-1}\Big(\frac{\int_{E_{n-1}} \varphi^{n-1}(k.u)du}{\int_{E_{n-1}} \chi^{n-1}(k.u)du}\Big)',$$

where $E_{n-1} := \{(u_1, u_2, ..., u_n) : u_i \geq 0, 1 \leq i \leq n,\ , \sum_{i=1}^{n-1} u_i \leq 1\}$, is (n-1) dimensional simplex, $u = (u_1, u_2, ..., u_n), u_n = 1 - \sum_{i=1}^{n-1} u_i, du = du_1 du_2 ... du_n$ and $k.u = \sum_{i=1}^{n} u_i k_i$.

A mean which can be expressed in the similar form as of Cauchy mean, is called *Cauchy type mean*. The purpose of our work is to introduce new means of Cauchy type defined on aC_0-semigroup of positive operators.

2. Jensen's type inequality and corresponding means

The Jensen type inequality for superquadratic function on isotonic linear functionals, is given in [[13], Theorem 10]. In [1], the corresponding Cauchy type means are defined.

In the present paper, we shall firstly prove the Jensen's type inequality for semigroup of positive linear operators defined on a Banach lattice algebra. Result will be followed by some generalized mean value theorems, bringing in a new set of Cauchy type means.

Definition 2.1. Let V be a Banach lattice algebra. A mapping $\phi : V_+ \to V$ is superquadratic, provided that for all $v \geq 0$ there exists a constant vector $C(v)$ such that

$$\phi(u) - \phi(v) - \phi(|u-v|) \geq C(v)(u-v) \tag{4}$$

for all $u \geq 0$. We say that the mapping ϕ is *subquadratic* if $-\phi$ is superquadratic.

Theorem 2.1. *Let $\{Z(t)\}_{t\geq 0}$ be a strongly continuous positive semigroup of operators defined on Banach lattice algebra V. Then for $g \in V_+$ and the continuous superquadratic mapping $\phi : V_+ \to V$, we have;*

(5)

$$\phi\left([Z(t)g]^{-1}[Z(t)(gf)]\right) \leq \frac{Z(t)[g\phi(f)] - Z(t)\left[g\phi\left(\left|f - [Z(t)g]^{-1}[Z(t)(gf)]\right|\right)\right]}{[Z(t)g]}, \quad f \in V_+.$$

If ϕ is subquadratic then a reversed inequality in (5) holds.

Proof. Since the mapping ϕ is superquadratic, inequality (4) holds for all $u, v \geq 0$. As $f, g \geq 0$ and the operator $Z(t)$ is positive for all $t \geq 0$, we have $[Z(t)g]^{-1}[Z(t)(gf)] \geq 0$. Setting $u = f$ and $v = [Z(t)g]^{-1}[Z(t)(gf)]$ in (4), we obtain;

$$\begin{aligned} \phi(f) \ \ &\geq \ \ \phi\left([Z(t)g]^{-1}[Z(t)(gf)]\right) + C\left[[Z(t)g]^{-1}[Z(t)(gf)]\right]\left[f - [Z(t)g]^{-1}[Z(t)(gf)]\right] \\ &\quad + \phi\left(\left|f - [Z(t)g]^{-1}[Z(t)(gf)]\right|\right), \end{aligned}$$

for all $t \geq 0$. Multiplying the above inequality by $g \in V_+$, we get

$$\begin{aligned} g\phi(f) \ \ &\geq \ \ g\phi\left([Z(t)g]^{-1}[Z(t)(gf)]\right) + C\left[[Z(t)g]^{-1}[Z(t)(gf)]\right]\left[gf - g[Z(t)g]^{-1}[Z(t)(gf)]\right] \\ &\quad + g\phi\left(\left|f - [Z(t)f]^{-1}[Z(t)(gf)]\right|\right). \end{aligned}$$

By applying the operator $Z(t)$ on both sides, we get for all $t \geq 0$;

$$\begin{aligned} Z(t)[g\phi(f)] \geq\ & Z(t)[g]\phi\Big([Z(t)g]^{-1}[Z(t)(gf)]\Big) \\ & +C\Big[[Z(t)g]^{-1}[Z(t)(gf)]\Big]\Big[Z(t)(gf)-Z(t)[g][Z(t)g]^{-1}[Z(t)(gf)]\Big] \\ & +Z(t)\Big[g\phi\Big(\Big|f-[Z(t)f]^{-1}[Z(t)(gf)]\Big|\Big)\Big]. \end{aligned}$$

The assertion (5) follows directly. □

Throughout the remaining article, V shall denote the (real) unital Banach lattice algebra with identity element e, until and unless stated otherwise.

Theorem 2.2. *Let $\{Z(t)\}_{t\geq 0}$ be a normalized strongly continuous positive semigroup of operators defined on V; then for a continuous superquadratic operator $\phi : V_+ \to V$, we have*

$$\phi[Z(t)f] \leq Z(t)[\phi(f)] - Z(t)[\phi(|f-Z(t)f|)], \quad f \in V_+. \tag{6}$$

If the mapping ϕ is subquadratic, then the inequality above is reversed.

Proof. Since $\{Z(t)\}_{t\geq 0}$ is a normalized semigroup it must satisfy (3). By taking $g \equiv e$ in Theorem (2.1), we obtain (6). □

Definition 2.2. Let $\{Z(t)\}_{t\geq 0}$ be a strongly continuous normalized positive semigroup of operators defined on V; then for a continuous operator $\phi : V_+ \to V$, we define an other operator $\Lambda_\phi : V_+ \to V$;

$$\Lambda_\phi := Z(t)[\phi(f)] - \phi[Z(t)f] - Z(t)[\phi(|f-Z(t)f|)], \quad f \in V_+. \tag{7}$$

If ϕ is continuous superquadratic mapping then, $\Lambda_\phi \geq 0$.

Below we give an operator analogue of [[2], Lemma 3.1].

Lemma 2.1. *Suppose $\phi : V_+ \to V$ is continuously differentiable and $\phi(0) \leq 0$. If ϕ' is superadditive or $f \to \frac{\phi'(f)}{f}$, $\quad f \in V_+$, is increasing, then ϕ is superquadratic.*

Lemma 2.2. *Let $\phi \in C^2[V_+]$ and $u, U \in V$ be such that*

$$u \leq \left(\frac{\phi'(f)}{f}\right)' = \frac{f\phi''(f)-\phi'(f)}{f^2} \leq U, \quad \forall f > 0. \tag{8}$$

Consider the operators $\phi_1, \phi_2 : V_+ \to V$ defined as:

$$\phi_1(f) = \frac{Uf^3}{3} - \phi(f), \quad \phi_2 = \phi(f) - \frac{uf^3}{3}.$$

Then the mappings $f \to \frac{\phi_1'(f)}{f}$ and $f \to \frac{\phi_2'(f)}{f}$ are increasing. If also $\phi_i(0) = 0, i = 1, 2$, then these are superquadratic mappings.

Proof. By using the inequality (6), it can be easily seen that the mappings $f \to \frac{\phi_1'(f)}{f}$ and $f \to \frac{\phi_2'(f)}{f}$ are increasing. Moreover, if $\phi_i(0) = 0, i = 1, 2$, Lemma (2.1) implies these to be superquadratic. □

Theorem 2.3. *Let $\{Z(t)\}_{t\geq 0}$ be a positive normalized C_0-semigroup of operators defined on V and $\frac{\phi'}{f} \in C^1(V_+)$ and $\phi(0) = 0$, then the following inequality holds*

$$\Lambda_\phi = \frac{\xi\phi''(\xi) - \phi'(\xi)}{3\xi^2}\{Z(t)[f^3] - [Z(t)f]^3 - Z(t)(|f - Z(t)f|^3)\}, \quad f \in V_+. \tag{9}$$

Proof. Suppose that $u = min_{f\in V_+}(\frac{\phi'(f)}{f})'$ and $U = max_{f\in V_+}(\frac{\phi'(f)}{f})$ exists. Taking ϕ_1 instead of ϕ in (6), we get for $f \in V_+$;

$$Z(t)[\phi(f)] - \phi[Z(t)f] - Z(t)[\phi(|f - Z(t)f|)] \leq \frac{U}{3}\Big\{Z(t)[f^3] - [Z(t)f]^3 - Z(t)[|f - Z(t)f|]\Big\}.$$

Similarly, by taking ϕ_2 instead of ϕ in (6), we get for $f \in V_+$;

$$Z(t)[\phi(f)] - \phi[Z(t)f] - Z(t)[\phi(|f - Z(t)f|)] \geq \frac{u}{3}\Big\{Z(t)[f^3] - [Z(t)f]^3 - Z(t)[|f - Z(t)f|]\Big\}.$$

Since, $\phi = f^3$ is superquadratic and $Z(t) \in \{Z(t)\}_{t\geq 0}$ is the positive operator, therefore

$$Z(t)[f^3] - [Z(t)f]^3 - Z(t)[|f - Z(t)f|] \geq 0, \quad f \in V_+.$$

By combining the above two inequalities and using (8), we obtain that, there exists $\xi \in V_+$, such that the assertion (9) holds.

Theorem 2.4. *Let $\{Z(t)\}_{t\geq 0}$ be a positive normalized C_0-semigroup of operators defined on V and $\frac{\phi'}{f}, \frac{\psi'}{f} \in C^1(V_+)$ such that, $\phi(0) = \psi(0) = 0$, we have*

$$\frac{\Lambda_\phi}{\Lambda_\psi} = \frac{\xi\phi''(\xi) - \phi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = K(\xi), \quad \xi \in V_+, \tag{10}$$

provided the denominators do not vanish. If K is invertible, we have the following new mean;

$$\xi = K^{-1}\Big(\frac{\Lambda_\phi}{\Lambda_\psi}\Big), \quad \Lambda_\psi \neq 0, \tag{11}$$

Proof. Lets consider a function $\Omega = c_1\phi - c_2\psi$, where

$$c_1 = \Lambda_\psi, \quad c_2 = \Lambda_\phi.$$

Then for $f \in V_+$;

$$\frac{\Omega'}{f} = c_1\frac{\phi'}{f} - c_2\frac{\psi'}{f} \in C^1(V_+).$$

One may calculate that $\Lambda_\Omega = 0$ and using Lemma (2.2) with $\phi = \Omega$ we obtain;

$$[c_1(\xi\phi''(\xi) - \phi'(\xi)) - c_2(\xi\psi''(\xi) - \psi'(\xi))]\Big\{Z(t)[f^3] - [Z(t)f]^3 - Z(t)[|f - Z(t)f|]\Big\} = 0, f \in V_+.$$

Since $\phi = f^3$ is superquadratic and $\{Z(t)\}_{t\geq 0}$ is semigroup of positive operators, therefore we may conclude that

$$\frac{c_2}{c_1} = \frac{\xi\phi''(\xi) - \phi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = \frac{\Lambda_\phi}{\Lambda_\psi}, \quad \xi \in V_+,$$

providing the denominator do not vanish. This completes the proof. □

We shall denote the set of all invertible strictly monotone continuous operators, defined from V to itself, by $G_M(V)$.

Definition 2.3. For a positive normalized C_0-semigroup $\{Z(t)\}_{t\geq 0}$, defined on a Banach lattice V and $F \in G_M(V)$, we define the generalized mean:

$$M_F(Z, f, t) := F^{-1}\{Z(t)F(f)\}, \quad f \in X. \tag{12}$$

For the sake of simplicity, the set of all elements of G_M, whose second order derivative (in Gateaux's sense) exits, shall be denoted by $C^2G_M(V)$.

Theorem 2.5. *Let $\{Z(t)\}_{t\geq 0}$ be a positive normalized C_0-semigroup defined on V and $H, F, K \in C^2G_M(V)$. Let for $f \in V_+, \frac{H\circ F^{-1}(f)}{f}, \frac{K\circ F^{-1}(f)}{f} \in C^1(V)$ with $H \circ F^{-1}(0) = 0 = K \circ F^{-1}(0)$, then for $f \in V_+$ and $t \geq 0$;*

$$\frac{H(M_H(Z,f,t)) - H(M_F(Z,f,t)) - H(M_H(F^{-1}|F[Z(\tau)f] - FM_F(Z,f,t)|, f, t))}{K(M_H(Z,f,t)) - K(M_F(Z,f,t)) - K(M_K(F^{-1}|F[Z(\tau)f] - FM_F(Z,f,t)|, f, t))}$$

$$= \frac{F(\eta)\{H''(\eta)F'(\eta) - H'(\eta)F''(\eta) - H'(\eta)[F'(\eta)]^2\}}{F(\eta)\{K''(\eta)F'(\eta) - K'(\eta)F''(\eta) - K'(\eta)[F'(\eta)]^2\}}, \tag{13}$$

holds for some $\eta \in V_+$, provided the denominator do not vanish.

Proof. By choosing the operators ϕ and ψ in Theorem 2.4, such that

$$\phi = H \circ F^{-1}, \quad \psi = K \circ F^{-1} \quad and \quad Z(t)f = F[Z(t)f], \quad f \in V_+,$$

where $H, F, K \in C^2 G_M(V)$. We find that there exists $\xi \in V_+$, such that

$$\begin{aligned} &\frac{H(M_H(Z,f,t)) - H(M_F(Z,f,t)) - H(M_H(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|, f,t))}{K(M_H(Z,f,t)) - K(M_F(Z,f,t)) - K(M_K(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|, f,t))} \\ = \; &\frac{\xi\{H''(F^{-1}\xi)F'(F^{-1}\xi) - H'(F^{-1}\xi)F''(F^{-1}\xi) - H'(F^{-1}\xi)[F'(F^{-1}\xi)]^2\}}{\xi\Big\{K''(F^{-1}\xi)F'(F^{-1}\xi) - K'(F^{-1}\xi)F''(F^{-1}\xi) - K'(F^{-1}\xi)[F'(F^{-1}\xi)]^2\Big\}}. \end{aligned}$$

Therefore, by setting $F^{-1}(\xi) = \eta$, we find that there exists $\eta \in X$, such that the assertion (13) follows directly. □

The above theorem accredit us to define new means. Set

$$L(\eta) = \frac{F(\eta)\{H''(\eta)F'(\eta) - H'(\eta)F''(\eta) - H'(\eta)[F'(\eta)]^2\}}{F(\eta)\{K''(\eta)F'(\eta) - K'(\eta)F''(\eta) - K'(\eta)[F'(\eta)]^2\}},$$

and when $F \in G(V)$;

$$\eta = L^{-1}\Big(\frac{H(M_H(Z,f,t)) - H(M_F(Z,f,t)) - H(M_H(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|, f,t))}{K(M_H(Z,f,t)) - K(M_F(Z,f,t)) - K(M_K(F^{-1}|F[Z(t)f] - FMF(Z,f,t)|, f,t))}\Big)$$

Remark 2.1. For $(V, \|.\|)$ a Banach lattice algebra, it follows from Theorem 2.5 that

$$m \leq \Big\|\frac{H(M_H(Z,f,t)) - H(M_F(Z,f,t)) - H(M_H(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|, f,t))}{K(M_H(Z,f,t)) - K(M_F(Z,f,t)) - K(M_K(F^{-1}|F[Z(t)f] - FM_F(Z,f,t)|, f,t))}\Big\| \leq M,$$

Where m and M are respectively, the minimum and maximum values of

$$\Big\|\frac{F(\eta)\{H''(\eta)F'(\eta) - H'(\eta)F''(\eta) - H'(\eta)[F'(\eta)]^2\}}{F(\eta)\{K''(\eta)F'(\eta) - K'(\eta)F''(\eta) - K'(\eta)[F'(\eta)]^2\}}\Big\|, \quad \eta \in V.$$

□

Definition 2.4. [12] Let V be a Banach algebra with unit e. For $f \in V$, we define a function $log(f)$ from V to V;

$$log(f) = -\sum_{n=1}^{\infty} \frac{(e-f)^n}{n} = -(e-f) - \frac{(e-f)^2}{2} - \frac{(e-f)^3}{3} - \ldots$$

for $||(e-x)|| \leq 1$. □

In correspondence with the usual definition of generalized power means for isotonic functionals [1], we shall define the generalized power means for semigroup of operators, as follows.

Definition 2.5. Let X be a Banach space and $\{Z(t)\}_{t\in\mathbb{R}}$ the C_0-semigroup of linear operators on X. For $f \in X$ and $t \in \mathbb{R}_+$, the genralized power mean is defined as;

$$M_{G_r}(Z,f,t) = \begin{cases} \Big(Z(t)[f^r]\Big)^{1/r}, & r \neq 0, \\ \\ exp[Z(t)[log(f)]], & r = 0. \end{cases} \tag{14}$$

Now, we shall prove an important result which is going to lead us, to define the Cauchy's type means on C_0-semigroup of operators.

Corollary 2.1. *Let all the conditions of Theorem 2.5 are satisfied. For $r,s,l \in \mathbb{R}_+$ such that $r \neq l; l \neq 2s$, we have*

$$\frac{M^r_{G_r}(Z,f,t) - M^r_{G_s}(Z,f,t) - M^r_{G_r}(|[Z(\tau)f]^s - M^s_{G_s}(Z,f,t)|^{\frac{1}{s}},f,t)}{M^l_{G_l}(Z,f,t) - M^l_{G_s}(Z,f,t) - M^l_{G_l}(|[Z(\tau)f]^s - M^s_{G_s}(Z,f,t)|^{\frac{1}{s}},f,t)} = \frac{r(r-2s)}{l(l-2s)}\eta^{r-l} \tag{15}$$

The assertion (15) holds for some η, provided that the denominators do not vanish.

Proof. For $r,s,l \in \mathbb{R}_+$ and $f \in V_+$, if we set

$$H(f) = f^r, \quad F(f) = f^s, \quad K(f) = f^l$$

in Theorem (2.5), the assertion in (15) follows directly.

Ultimately, we shall define means of the Cauchy's type on C_0-semigroup of positive linear operators defined on Banach lattice algebra V.

Definition 2.6. Let $r,s,l \in \mathbb{R}_+$ and $\{Z(t)\}_{t\geq 0} \subset B(V)$ be a normalized C_0-semigroup of positive linear operators on a unital Banach lattice algebra V. Then

(16)

$$\mathfrak{M}^{l,s}_{G_r}(Z,f,t) = \Big(\frac{l(l-2s)}{r(r-2s)}\frac{M^r_{G_r}(Z,f,t) - M^r_{G_s}(Z,f,t) - M^r_{G_r}(|[Z(\tau)f]^s - M^s_{G_s}(Z,f,t)|^{\frac{1}{s}},f,t)}{M^l_{G_l}(Z,f,t) - M^l_{G_s}(Z,f,t) - M^l_{G_l}(|[Z(\tau)f]^s - M^s_{G_s}(Z,f,t)|^{\frac{1}{s}},f,t)}\Big)^{\frac{1}{r-l}}.$$

is a mean of the Cauchy's type. This definition is true for all $r \neq l \neq s \neq 0$ and other cases can be taken as limiting cases, as in [8].

3. Conclusion

Firstly, we have proved a Jensen's type inequality for a normalized semigroup of positive linear operators and a superquadratic mapping, defined on a Banach lattice algebra. A systematic procedure has been used to prove the corresponding mean value theorems, which lead us to a new set of means. These means are Cauchy's type means for the mentioned operators. By following the similar procedure, many functional inequalities can be generalized for the operator semigroups and corresponding means can be obtained.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] S. Abramovich, G. Farid, J. Pečarić, More about Jensen's inequality and Cauchy's means for superquadratic functions, J. Math. Inequal., 7 (2013), 11-24.

[2] S. Abramovich, G. Jameson, G. Sinnamon, Refining Jensens inequality, Bull, Math. Soc. Sci. Math. Roumanie 47 (2004), 3-14.

[3] L. Ali Khan, Mean value theorem in topological vector spaces, C. R. Math. Rep. Acad. Sci. Canada, 19 (1997), 24-27.

[4] M. Anwar, Josip Pečarić, Cauchys means of Levinson type, J. Inequal. Pure Appl. Math. 9 (2008), 1-8.

[5] M. Anwar and Josip Pečarić, M. Rodic Lipanović, Cauchy-type means for positive linear funtionals, Tamakang J. Math. 42 (2011), 511-530.

[6] M. Anwar and Josip Pečarić, Cauchy's means for signed measures, Bull. Malays. Math. Sci. Soc. 34 (2011), 31-38.

[7] M. Anwar and Josip Pečarić, Cauchy means of Mercer's type, Utilitas Mathematica, 84 (2011), 201-208.

[8] M. Anwar and Josip Pečarić, New Cauchy's Means, J. Inequal. Appl. 2008 (2008), Article ID 163202.

[9] M. Anwar, J. Jakšetić, J. Pečarić and Atiq Ur Rehman, Exponential convexity, positive semi-definite matrices and fundamental inequalities, J. Math. Inequal. 4 (2010), 171-189.

[10] G.I. Hina Aslam, Matloob Anwar, Cauchy type means on one-parameter C_0-group of operators, J. Math. Inequal. 9 (2015), 631-639.

[11] R. Nagel (ed.), t One-parameter Semigroups of Positive Operators, Lect. Notes in Math. Springer-Verlag, (1986).

[12] A. H. Salas, The exponential and logarithmic functions on commutative Banach algebras, Int. J. Math. Anal. 4 (2010), 2079 - 2088.

[13] Senka Banić, Sanja Varošanec, Functional inequalities for superquadratic functions, J. Pure Appl. Math. 43 (2008), 717-729.