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CANCELLATIVE SEMIGROUPS ADMITTING CONJUGATES

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Abstract. We prove conjugate analogs of Mal'cev-Neumann-Taylor-Levi's theorems. In other words, we characterize semigroups embeddable in nilpotent groups of class n by means of conjugacy laws involving semigroup laws described by Mal'cev, Neumann and Taylor. Moreover we prove that a cancellative semigroup admitting conjugates is embeddable in a nilpotent group of class 2 if and only if it satisfies the conjugacy law $x^{y^z} = x^y$. Also adapting Ore's techniques we describe an exact procedure for embedding a cancellative semigroup admitting conjugates into a group.

Keywords: Cancellative semigroup; Conjugate; Embedding; Nilpotent.

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1. Introduction

While every subsemigroup of a group is cancellative, a famous theorem of A.I. Mal'cev [4] shows that not every cancellative semigroup is embeddable in a group. In fact, Mal'cev gave

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an infinite set of necessary and sufficient conditions for the possibility of imbedding a cancellation semigroup into a group and proved that no finite set of such conditions would suffice. This calls for finite sets of sufficient conditions that would guarantee the group embeddabilty. Patterned after the classical quotient construction, Oystein Ore [2] discovered the ”principle of common left multiple” to imbed a non-commutative domain into a division ring. Using this as a backdrop, Mal’cev, B.H. Neumann and Taylor developed semigroup equivalents of nilpotent groups of class n and proved that cancellative semigroups of nilpotent class n are embeddable in groups of the same nilpotency class. In this paper, we study some equational classes of enriched semigroups - i.e. semigroups admitting conjugates - and prove that all the valid group theory implications do carry over to the equational theory of semigroups admitting conjugates.

2. Preliminaries

Definition 2.1. Let S be a semigroup. If for all x,y and z in S, $xy = xz$ implies $y = z$, and also $yx = zx$ implies $y = z$, then we say S is a cancellative semigroup.

Definition 2.2. Let S be a semigroup, and $x, y \in S$. If there exist an element $z \in S$ such that $xy = yz$, then z is called conjugate of x by y and it is denoted by x^y. If for all elements x and y in S, x^y exists we say S admits conjugates.

Therefore for any x and y in a cancellative semigroup S that admits conjugates, x^y is unique and

$$xy = yx^y. \qquad (\circledast)$$

Lemma 2.1. *Let S be a cancellative semigroup which admits conjugates and let x, y and z be in S, then:*

(1) $x^x = x$,

(2) $(x^y)^z = x^{yz}$,

(3) $(xy)^z = x^z y^z$.

Proof. (1) : Since $xx = xx^x$ by $(\circledast)$, so by left cancellation we get $x^x = x$.

(2) : Using $(\circledast)$:

$$(yz)x^{yz} = x(yz) = (xy)z = (yx^y)z = y(x^y z) = y\big(z(x^y)^z\big) = (yz)\,(x^y)^z\,,$$

and by left cancellation we get $(x^y)^z = x^{yz}$.

(3) : Using $(\circledast)$:

$$z(xy)^z = (xy)z = x(yz) = x(zy^z) = (xz)y^z = (zx^z)y^z = z(x^z y^z)\,,$$

and by left cancellation we get $(xy)^z = x^z y^z$.

Definition 2.3. Let S be a cancellative semigroup which admits conjugates. For any elements a, b, c and d in S we define

$$a \diagdown b = \{(x,y) |\, ay = xb^y,\ \ x,y \in S\}\,.$$

The set of all $a \diagdown b$ is denoted by $\overline{S}$, i.e.

$$\overline{S} = \{a \diagdown b \mid\ a,b \in S\},$$

and in $\overline{S}$ we define binary operation $*$ as

$$(a \diagdown b) * (c \diagdown d) = ac \diagdown db^c\,.$$

3. Embedding into groups

First we prove an analog of the classical Ore's condition [2].

Theorem 3.1. *Let S be a cancellative semigroup which admits conjugates. For any a, b, x,y,u and v in S, if*

$$ay = xb^y, \qquad cy = xd^y, \qquad av = ub^v\,,$$

then $cv = ud^v$.

Proof. First multiplying $ay = xb^y$ by v we get $(xb^y)v = (ay)v$ now using Lemma 2.1 and $(\circledast)$ we get

$$(ay)v = a(yv) = a(vy^v) = (av)y^v$$

$$
\begin{aligned}
&= (ub^v)y^v && \text{by hypothesis of the Theorem}\\
&= u(b^v y^v)\\
&= u(by)^v && \text{by Lemma 2.1}\\
&= u(yb^y)^v && \text{by } (\circledast)\\
&= u\big(y^v (b^y)^v\big) && \text{by Lemma 2.1}\\
&= (uy^v)(b^y)^v .
\end{aligned}
$$

Also

$$(xb^y)v = x(b^y v) = x\big(v(b^y)^v\big) = (xv)(b^y)^v .$$

Therefore we have $(xv)(b^y)^v = (uy^v)(b^y)^v$ and by right cancellation we get

$$xv = uy^v. \tag{1}$$

On the other hand multiplying $cy = xd^y$ by v we get $(xd^y)v = (cy)v$ now using Lemma and $(\circledast)$ we get

$$
\begin{aligned}
(xd^y)v = x(d^y v) = x\big(v(d^y)^v\big) &= (xv)(d^y)^v\\
&= (uy^v)(d^y)^v && \text{by } (1)\\
&= u\big(y^v (d^y)^v\big)\\
&= u\big(y^v (d^{yv})\big) && \text{by Lemma 2.1}\\
&= u\big(y^v (d^{vy^v})\big) && \text{by } (\circledast)\\
&= u\big(y^v (d^v)^{y^v}\big) && \text{by Lemma 2.1}\\
&= u(d^v y^v) && \text{by } (\circledast)\\
&= (ud^v)y^v .
\end{aligned}
$$

Also

$$(cy)v = c(yv) = c(vy^v) = (cv)y^v .$$

Therefore we have $(ud^v)y^v = (cv)y^v$ and by right cancellation we get $cv = ud^v$.

Lemma 3.1. *Let S be a cancellative semigroup which admits conjugates and let a, b, c and d be in S then*

(1) *If $(a \searrow b) \cap (c \searrow d) \neq \varnothing$, then $a \searrow b = c \searrow d$,*

(2) $a \searrow a = b \searrow b$.

Proof. To prove part 1 : First we note that since $ab = ab^b$ so $(a,b) \in a \searrow b$ that is $a \searrow b \neq \varnothing$. Now if $(a \searrow b) \cap (c \searrow d) \neq \varnothing$, let assume $(x_1, y_1) \in (a \searrow b) \cap (c \searrow d)$ then, $ay_1 = x_1 b^{y_1}$ and $cy_1 = x_1 d^{y_1}$. Now for any (u,v) in $a \searrow b$ since $av = ub^v$ by Theorem 3.1 must $cv = ud^v$, which means $(u,v) \in c \searrow d$ so $a \searrow b \subseteq c \searrow d$. Similarly for any (u,v) in $c \searrow d$ since $cv = ud^v$ by Theorem 3.1 must $av = ub^v$, so $c \searrow d \subseteq a \searrow b$ and therefore $a \searrow b = c \searrow d$.

To prove part 2 : Since $ax = xa^x$ and also $bx = xb^x$ so $(x,x) \in a \searrow a \cap b \searrow b$ and therefore by part 1 must $a \searrow a = b \searrow b$.

Lemma 3.2. *Let S be a cancellative semigroup which admits conjugates and let a, b, u and v be in S then*

(1) $au \searrow bu = a \searrow b$,

(2) $ua \searrow bu^a = a \searrow b$,

(3) $au \searrow u = av \searrow v$.

Proof. (1) : For any (x,y) in $a \searrow b$ we have $ay = xb^y$ now since

$$(au)y = a(uy) = a(yu^y) = (ay)u^y = (xb^y)u^y = x(b^y u^y) = x(bu)^y ,$$

so $(x,y) \in au \searrow bu$ that is $a \searrow b \subseteq au \searrow bu$ and by Lemma 3.1 $au \searrow bu = a \searrow b$.

(2) : For any (x,y) in $a \searrow b$ we have $ay = xb^y$ now since

$$(ua)y = u(ay) = (ay)u^{ay} = (xb^y)u^{ay} = x(b^y u^{ay}) = x(bu^a)^y ,$$

so $(x,y) \in ua \searrow bu^a$ that is $a \searrow b \subseteq ua \searrow bu^a$ and by Lemma 3.1 $ua \searrow bu^a = a \searrow b$.

(3) : First we note that $(au,u) \in (au \searrow u)$ because $(au)u = (au)u^u$; now since $(av)u = a(vu) = a(uv^u) = (au)v^u$ so $(au,u) \in (av \searrow v)$ that is $au \searrow u \cap av \searrow v \neq \varnothing$ and by Lemma 3.1 $au \searrow u = av \searrow v$.

Theorem 3.2. *Let S be a cancellative semigroup which admits conjugates. Then $(\overline{S}, *)$ is a group.*

Proof. **(i) Closure** : For all $a \diagdown b, c \diagdown d \in \overline{S}$, $(a \diagdown b) * (c \diagdown d) = ac \diagdown db^c$ is a well defined element of $\overline{S}$.

(ii) Associativity : For all $a \diagdown b, c \diagdown d, f \diagdown g \in \overline{S}$,

$$\begin{aligned}\big((a \diagdown b) * (c \diagdown d)\big) * (f \diagdown g) &= (ac \diagdown db^c) * (f \diagdown g)\\ &= (ac)f \diagdown g(db^c)^f\\ &= a(cf) \diagdown (gd^f)b^{cf} \qquad \text{by Lemma 2.1}\\ &= (a \diagdown b) * (cf \diagdown gd^f)\\ &= (a \diagdown b) * \big((c \diagdown d) * (f \diagdown g)\big).\end{aligned}$$

(iii) Identity : Since by of Lemma 3.1, for any u and v in S, $u \diagdown u = v \diagdown v$, let $E = u \diagdown u = v \diagdown v$ then E is the identity element because by Lemma 3.2

$$(a \diagdown b) * E = (a \diagdown b) * (u \diagdown u) = au \diagdown ub^u = au \diagdown bu = a \diagdown b;$$

also

$$E * (a \diagdown b) = (u \diagdown u) * (a \diagdown b) = ua \diagdown bu^a = a \diagdown b.$$

(iv) Inverses : For each $a \diagdown b \in \overline{S}$ there exist $b \diagdown a$ in $\overline{S}$ such that

$$(a \diagdown b) * (b \diagdown a) = ab \diagdown ab^b = ab \diagdown ab = E;$$

also

$$(b \diagdown a) * (a \diagdown b) = ba \diagdown ba^a = ba \diagdown ba = E,$$

so $(a \diagdown b)^{-1} = b \diagdown a$. Therefore $(\overline{S}, *)$ is a group.

Theorem 3.3. *If S is a cancellative semigroup which admits conjugates, then it can be embedded into a group.*

Proof. Define $\phi : S \to \overline{S}$ by $\phi(a) = au \diagdown u$ where $u \in S$; then ϕ is a well defined function because by by Lemma 3.2, $au \diagdown u = av \diagdown v$. ϕ is a one to one function because if $\phi(a) = \phi(b)$ then $au \diagdown u = bu \diagdown u$ which means for any $(x, y) \in au \diagdown u = bu \diagdown u$, $(au)y = xu^y$ and $(bu)y = xu^y$ so we have $(au)y = (bu)y$ and by right cancellation we get $a = b$.

In order to show that ϕ is a homomorphism that is for $a,b \in S$, $\phi(ab) = \phi(a) * \phi(b)$ since $\phi(ab) = (ab)u \setminus u = abu \setminus u$ and

$$\phi(a) * \phi(b) = (au) \setminus u) * (bu) \setminus u) = ((au)(bu) \setminus uu^{bu}) = aubu \setminus u^b u\,,$$

we need to show that $abu \setminus u = aubu \setminus u^b u$ .But for any (x,y) in $abu \setminus u$ we have $abuy = xu^y$ now since

$$\begin{aligned}
(aubu)y &= a(u(buy)) \\
&= a((buy)u^{buy}) && \text{by } (\circledast) \\
&= (abuy)u^{buy} \\
&= (xu^y)u^{buy} && \text{since } (x,y) \in (abu \setminus u) \\
&= x(u^y u^{(bu)y}) \\
&= x(uu^{bu})^y && \text{by Lemma 2.1} \\
&= x(u^b u)^y && \text{by } (\circledast)\,,
\end{aligned}$$

so $(x,y) \in (aubu \setminus u^b u)$ that is $(abu \setminus u) \subseteq (aubu \setminus u^b u)$ and by Lemma 3.1 $abu \setminus u = aubu \setminus u^b u$.

3. Conjugacy laws

In 1942, Levi [1] proved that a group satisfies the commutator law $[[x,y],z] = [x,[y,z]]$ if and only if the group is of nilpotent of class at most 2 (see [3] for a modern proof by A. G. Kurosh). By a classical result of Mal'cev [4](also, independently by Neumann and Taylor [5]), a cancellation semigroup satisfies the semigroup law $xyzyx = yxzxy$ if and only it is a subsemigroup of a **group** of nilpotent class at most 2. Accordingly, we call a semigroup to be of nilpotent of class 2 if it satisfies the law $xyzyx = yxzxy$. In this section we prove an analog of Levi's theorem for conjugates by characterizing semigroups embeddable in groups of nilpotent of class 2 by means of a single conjugacy law. But first we mention a trivial equivalency just for the record:

Theorem 4.1. *Let S be a cancellative semigroup which admits conjugates, then the following laws are equivalent in S:*

(1) $xy = yx$ *(commutativity),*

(2) $x^y = x$ *(conjugacy),*

(3) $x^{(y^z)} = (x^y)^z$ *(associativity of conjugates).*

Proof. It is clear that $1 \Rightarrow 2$ and $2 \Rightarrow 3$. Now $3 \Rightarrow 1$ because if $x^{(y^z)} = (x^y)^z$ is a law then for $y = x$ we get $x^{x^z} = (x^x)^z = x^z$. Multiplying both sides by x^z gives $x^z x^{x^z} = x^z x^z$. Using $(\circledast)$ for the left side we get $xx^z = x^z x^z$; right cancellation gives $x = x^z$ therefore $xz = zx^z = zx$.

Theorem 4.2. *Let S be a cancellative semigroup which admits conjugates, then S is nilpotent of class 2 if and only if it satisfies the conjugacy law $x^{y^z} = x^y$.*

Proof. By Mal'cev [4] and or by Neumann and Taylor [5] it is enough to prove that $xyzyx = yxzxy \Leftrightarrow x^{y^z} = x^y$. If $x^{y^z} = x^y$ is a law in S, then first we note that using $(\circledast)$ twice, gives $yx = xy^x = y^x x^{y^x} = y^x x^y$. Now

$$
\begin{aligned}
yxzxy &= (xy^x)z(yx^y) && \text{by } (\circledast)\\
&= (xy^x)(zy)x^y\\
&= (xy^x)x^y(zy)^{x^y} && \text{by } (\circledast)\\
&= x(y^x x^y)(zy)^x && \text{by assumption}\\
&= x(yx)(zy)^x && \text{by above note}\\
&= xy\big(x(zy)^x\big)\\
&= xy\big((zy)x\big) && \text{by } (\circledast).\\
&= xyzyx.
\end{aligned}
$$

Conversely if $xyzyx = yxzxy$ is a law in S, then first we note that using $(\circledast)$ and canceling y we get

$$x^y zyx = xzxy. \tag{2}$$

In (2) replacing y by y^z we get $x^{y^z}zy^zx = xzxy^z$ which means

$$x^{y^z}yzx = xzxy^z\,.$$

Now multiplying both sides by uzy we get

$$x^{y^z}yzxuzy \;=\; xzxy^zuzy\,. \tag{3}$$

But

$$\begin{aligned} x^{y^z}yzxuzy &= x^{y^z}yz(xu)zy \\ &= x^{y^z}zy(xu)yz && \text{by assumption} \\ &= x^{y^z}(zyxuyz)\,. \end{aligned}$$

Also

$$\begin{aligned} xzxy^zuzy &= xzx(y^zuzy) \\ &= xzx(yuyz) && \text{by } (2) \\ &= (xzxy)uyz \\ &= (x^yzyx)uyz && \text{by } (2) \\ &= x^y(zyxuyz)\,. \end{aligned}$$

Therefore by substitution in (3) we get $x^{y^z}(zyxuyz) = x^y(zyxuyz)$ and right cancellation gives $x^{y^z} = x^y$.

Corollary 4.1. *Let S be a cancellative semigroup which admits conjugates, then S is nilpotent of class 2 if and only if it satisfies the conjugacy law $x^{yz} = x^{zy}$.*

Proof. Using the above theorem it is enough to prove that $x^{yz} = x^{zy}$ is a law if and only if $x^{y^z} = x^y$ is a law in S. If S satisfies the law $x^{yz} = x^{zy}$ then since $x^{yz} = x^{zy^z} = (x^z)^{y^z}$ so $(x^z)^{y^z} = (x^z)^y$. Let $t = x^z$ then $t^{y^z} = t^y$.

Conversely if S satisfies the law $x^{y^z} = x^y$ then

$$\begin{aligned} x^{yz} &= (x^z)^{y^z} \\ &= (x^z)^y && \text{by using the law for } x^z, y \text{ and } z \end{aligned}$$

$= x^{zy}$.

Following [4] and [5], we define a semigroup S to be nilpotent of class 3 if it satisfies the law $(xyzyx)u(yxzxy) = (yxzxy)u(xyzyx)$; inductively we say S is of nilpotent class n if it satisfies the law $fug = guf$ where the law $f = g$ defines semigroups of nilpotent class $n-1$ and u is a new variable not occurring in the terms f or g.

Theorem 4.3. *Let S be a cancellative semigroup which admits conjugates, then S is nilpotent of class n if and only if it satisfies the $(n+1)$-variable conjugacy law $x^f = x^g$ where x is a variable not occurring in the terms f or g.*

Proof. Assume that S satisfies the law $x^f = x^g$. Let $x = ft$ where t is a new variable, then since $x^f = (ft)^f = ft^f = tf$ so must $tf = (ft)^g$. Therefore $fug = (fu)g = g(fu)^g = g(uf) = guf$ which means S is nilpotent of class n.

Conversely assume that S is nilpotent of class n that is $fug = guf$ is a law. Then by the very definition of conjugates, we have $xf = fx^f$. Premultiplying both sides of this equation by gy, where y is a new variable, we get $gyxf = gyfx^f$. Using the nilpotent identity $fug = guf$ twice, we obtain $fyxg = fygx^f$. Left canceling the common term fy we get $xg = gx^f$. But $xg = gx^g$, therefore $gx^g = gx^f$. Finally left canceling the common term g, we obtain the desired conjugacy law $x^f = x^g$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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