Available online at http://scik.org
J. Semigroup Theory Appl. 2015, 2015:7
ISSN: 2051-2937

CONFORMABLE FRACTIONAL SEMIGROUPS OF OPERATORS

THABET ABDELJAWAD[1,*], MOHAMMED AL HORANI[2,3], ROSHDI KHALIL[2]

[1]Department of Mathematics and General Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia

[2]Department of Mathematics, The University of Jordan, Amman, Jordan

[3]Department of Mathematics, Faculty of Science, University of Hail, Saudi Arabia

Abstract. Let X be a Banach space, and $T : [0,\infty) \to \mathscr{L}(X,X)$, the bounded linear operators on X. A family $\{T(t)\}_{t\geq 0} \subseteq \mathscr{L}(X,X)$ is called a one-parameter semigroup if $T(s+t) = T(s)T(t)$, and $T(0) = I$, the identity operator on X. The infinitesimal generator of the semigroup is the derivative of the semigroup at $t = 0$. The object of this paper is to introduce a (conformable) fractional semigroup of operators whose generator will be the fractional derivative of the semigroup at $t = 0$. The basic properties of such semigroups will be studied.

Keywords: Fractional semigroups; Fractional abstract Cauchy problem; Operator.

2010 AMS Subject Classification: 26A33.

1. Introduction and preliminaries

Let X be a Banach space, and $\mathscr{L}(X,X)$ be the space of all bounded linear operators on X. A family $\{T(t)\}_{t\geq 0} \subseteq \mathscr{L}(X,X)$ is called a one-parameter semigroup if:

$(i)\, T(0) = I$, the identity operator on X.

$(ii)\, T(s+t) = T(s)T(t)$ for all $s,\ t \geq 0$.

*Corresponding author

Received August 1, 2015

If, in addition, for each fixed $x \in X$, $T(t)x \longrightarrow x$ as $t \longrightarrow 0^{+}$, then the semigroup is called c_0-semigroup or strongly continuous semigroup.

Semigroups of operators proved to be a very fruitful tool to solve differential equations. One of the classical vector valued differential equations is the so called the abstract Cauchy problem, precisely,

$$u(t) = Au(t), \quad t > 0,$$

$$u(0) = x,$$

where $A : D(A) \subseteq X \longrightarrow X$ a linear operator of an appropriate type, $x \in X$ is given and $u : [0,\infty) \longrightarrow X$ is the unknown function. We refer to [5] and [7] for basic theory of semigroups of operators and the abstract Cauchy problem. For the inverse form of the abstract Cauchy problem, we refer to [9], see also [2].

Fractional semigroups are related to the problem of fractional powers of operators initiated first by Bochner, see [4]. Balakrishnan, see [3], studied the problem of fractional powers of closed operators and the semigroups generated by them. The fractional Cauchy problem associated with a Feller semigroup was studied by Popescu, see [8].

In the literature, there are many definitions of fractional derivative. To mention some:

(i) Riemann - Liouville Definition. For $\alpha \in [n-1,n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} \, dx.$$

(ii) Caputo Definition. For $\alpha \in [n-1,n)$, the α derivative of f is

$$D_a^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} \, dx.$$

However, the following are the setbacks of one definition or the other:

(i) The Riemann-Liouville derivative does not satisfy $D_a^{\alpha}(1) = 0$ ($D_a^{\alpha}(1) = 0$ for the Caputo derivative), if α is not a natural number.

(ii) All fractional derivatives do not satisfy the known formula of the derivative of the product of two functions:

$$D_a^\alpha(fg) = fD_a^\alpha(g) + gD_a^\alpha(f).$$

(iii) All fractional derivatives do not satisfy the known formula of the derivative of the quotient of two functions:

$$D_a^\alpha(f/g) = \frac{gD_a^\alpha(f) - fD_a^\alpha(g)}{g^2}.$$

(iv) All fractional derivatives do not satisfy the chain rule:

$$D_a^\alpha(f \circ g)(t) = f^{(\alpha)}\big(g(t)\big)\, g^{(\alpha)}(t).$$

(v) All fractional derivatives do not satisfy: $D^\alpha D^\beta f = D^{\alpha+\beta} f$, in general.

(vi) All fractional derivatives, specially Caputo definition, assumes that the function f is differentiable.

In [6], the authors gave a new definition of fractional derivative which is a natural extension to the usual first derivative as follows:

Given a function $f : [0,\infty) \longrightarrow \mathbb{R}$. Then for all $t > 0, \quad \alpha \in (0,1)$, let

$$T_\alpha(f)(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

T_α is called **the conformable fractional derivative of f of order α.**

Let $f^{(\alpha)}(t)$ stands for $T_\alpha(f)(t)$. Hence $f^{(\alpha)}(t) = \lim_{\varepsilon\to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$.

If f is $\alpha-$differentiable in some $(0,b)$, $b > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, **then let**

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

The conformable derivative satisfies all the classical properties of derivative.

Further, according to this derivative, the following statements are true, see [6].

1. $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$,

2. $T_\alpha(\sin \frac{1}{\alpha} t^\alpha) = \cos \frac{1}{\alpha} t^\alpha$,

3. $T_\alpha(\cos\frac{1}{\alpha}t^\alpha) = -\sin\frac{1}{\alpha}t^\alpha$,

4. $T_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$.

The $\alpha-$**fractional integral of a function** f **starting from** $a \geq 0$, see [6], is :

$$I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}}\,dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1)$.
For more about higher conformable fractional integrals and derivatives in left and right senses and other basic concepts we refer to [1].

The object of this paper is two folds: To introduce the fractional semigroups of operators associated with the conformable fractional derivative, then as an application we study the fractional abstract Cauchy problem according to the conformable fractional derivative which was introduced in [6]. Indeed, we prove that fractional $\alpha-$ semigroup, is the classical solution of fractional abstract Cauchy problem. Throughout this paper, $\alpha \in (0,1]$.

2. The basic definition

Definition 2.1. Let $\alpha \in (0,a]$ for any $a > 0$. For a Banach space X, A family $\{T(t)\}_{t\geq 0} \subseteq \mathscr{L}(X,X)$ is called a fractional $\alpha-$semigroup (or $\alpha-$semigroup) of operators if:

(i) $T(0) = I$,

(ii) $T(s+t)^{\frac{1}{\alpha}} = T(s^{\frac{1}{\alpha}})T(t^{\frac{1}{\alpha}})$ for all $s,t \in [0,\infty)$.

Clearly, if $\alpha = 1$, then $1-$semigroups are just the usual semigroups.

Example 2.1. Let A be a bounded linear operator on X. Define $T(t) = e^{2\sqrt{t}A}$. Then $\{T(t)\}_{t\geq 0}$ is a $\frac{1}{2}-$semigroup. Indeed:

(i) $T(0) = e^{0A} = I$.

(ii) $T(s+t)^2 = e^{2\sqrt{(s+t)^2}A} = e^{2(s+t)A} = e^{2sA}e^{2tA} = T(s^2)T(t^2)$.

For the definition of conformable fractional exponential matrix and power series expansions see [1].

Example 2.2. Let $X = C[0,\infty)$, the space of real valued continuous functions on $[0,\infty)$. Define $(T(t)f)(s) = f(s+2\sqrt{t})$. Then one can easily show that T is a $\frac{1}{2}-$ semigroup of operators.

Definition 2.2. An $\alpha-$semigroup $T(t)$ is called a c_0-semigroup if, for each fixed $x \in X$, $T(t)x \longrightarrow x \quad \text{as } t \longrightarrow 0^+$.

The conformable $\alpha-$derivative of $T(t)$ at $t=0$ is called the $\alpha-$ **infinitesimal generator** of the fractional $\alpha-$semigroup $T(t)$, with domain equals

$$\left\{ x \in X : \lim_{t\to 0^+} T^{(\alpha)}(t)x \ exists \right\}.$$

We will write A for such generator.

Theorem 2.1. Let $\{T(t)\}_{t\geq 0} \subseteq \mathscr{L}(X,X)$ be a $c_0-\alpha-$semigroup with infinitesimal generator A, $0<\alpha\leq 1$. If $T(t)$ is continuously $\alpha-$differentiable and $x \in D(A)$, then

$$T^{\alpha}(t)x = AT(t)x = T(t)Ax.$$

Proof. Let us begin with

$$\begin{aligned} T^{(\alpha)}(t)x &= \lim_{\varepsilon\to 0} \frac{T(t+\varepsilon t^{1-\alpha})x - T(t)x}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{T(t^{\alpha} + (t+\varepsilon t^{1-\alpha})^{\alpha} - t^{\alpha})^{\frac{1}{\alpha}}x - T(t)x}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{T(t^{\alpha} + ((t+\varepsilon t^{1-\alpha})^{\alpha} - t^{\alpha}))^{\frac{1}{\alpha}}x - T(t)x}{\varepsilon}. \end{aligned}$$

Since $T(t)$ is an $\alpha-$semigroup of operators, then $T(a+b)^{\frac{1}{\alpha}} = T\left(a^{\frac{1}{\alpha}}\right)T\left(b^{\frac{1}{\alpha}}\right)$. Hence

$$\begin{aligned} T^{(\alpha)}(t)x &= \lim_{\varepsilon\to 0} \frac{T(t^{\alpha})^{\frac{1}{\alpha}}T((t+\varepsilon t^{1-\alpha})^{\alpha} - t^{\alpha}))^{\frac{1}{\alpha}}x - T(t)x}{\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{T(t)[T((t+\varepsilon t^{1-\alpha})^{\alpha} - t^{\alpha}))^{\frac{1}{\alpha}}x - T(0)x]}{\varepsilon}. \end{aligned}$$

Now, using the Mean Value Theorem for conformable fractional derivative, see [6], we get

$$\frac{T(t)[T((t+\varepsilon t^{1-\alpha})^{\alpha} - t^{\alpha}))^{\frac{1}{\alpha}}x - T(0)x]}{\varepsilon} = T(t)T^{(\alpha)}(c)x\frac{[(t+\varepsilon t^{1-\alpha})^{\alpha} - t^{\alpha}]}{\alpha\varepsilon}$$

for some $0 < c < (t+\varepsilon t^{1-\alpha})^{\alpha} - t^{\alpha}$.

If $\varepsilon \to 0$, then $c \to 0$, and $\lim_{\varepsilon\to 0} T^{(\alpha)}(c) = T^{(\alpha)}(0) = A$. Consequently,

$$T^{(\alpha)}(t)x = T(t)Ax \lim_{\varepsilon\to 0} \frac{[(t+\varepsilon t^{1-\alpha})^{\alpha} - t^{\alpha}]}{\alpha\varepsilon}.$$

Using L'Hopital's Rule, we get

$$\lim_{\varepsilon\to 0} \frac{[(t+\varepsilon t^{1-\alpha})^{\alpha} - t^{\alpha}]}{\alpha\varepsilon} = 1. \quad \textit{Hence}\ T^{(\alpha)}(t)x = T(t)Ax.$$

Similarly, one can show that $T(t)x \in D(A)$ and $T^{(\alpha)}(t)x = AT(t)x$.

This ends the proof.

Let $X = C[0,\infty)$ be the space of continuous real-valued functions such that $\lim_{x\to\infty} f(x)$ is finite, with the sup norm. Define $T : [0,\infty) \to \mathscr{L}(X,X)$, by

$$\big(T(t)f\big)(s) = f(s + \frac{1}{\alpha}t^{\alpha}).$$

Claim: T is an $\alpha-$semigroup.

Indeed:

$$\begin{aligned}
\big(T(t+k)^{\frac{1}{\alpha}} f\big)(s) &= f\big(s + \frac{1}{\alpha}[(t+k)^{\frac{1}{\alpha}}]^{\alpha}\big) \\
&= f(s + \frac{1}{\alpha}t + \frac{1}{\alpha}k) \\
&= \big(T(t^{\frac{1}{\alpha}})T(k^{\frac{1}{\alpha}})f\big)(s).
\end{aligned}$$

It is almost immediate that $T(0) = I$ and $T(t)f \in X$ whenever $f \in X$ and that

$$\|T(t)f\|_{\infty} \le \|f\|_{\infty}, \quad t \ge 0,$$

so that $T(t) \in \mathscr{L}(X,X)$. Since the operator $T(t)$ is a translation operator corresponding to moving the graph of f $\frac{1}{\alpha}t^{\alpha}$ units to the left and chopping off the part to the left of the origin, it is known from the literature that $T(t)f$ is right-continuous at 0. So $T(t)$ is an $\alpha-$semigroup.

Theorem 2.2. The infinitesimal generator of the above semigroup is

$$Af(s) = f'(s),$$

$$D(A) = \{f \in X : f' \textit{ exists in } X\}.$$

Proof.

$$\begin{aligned} T^{(\alpha)}(t)f(s) &= t^{1-\alpha}T'(t)f(s) \\ &= t^{1-\alpha}\lim_{\varepsilon\to 0}\frac{T(t+\varepsilon)f(s)-T(t)f(s)}{\varepsilon} \\ &= t^{1-\alpha}\lim_{\varepsilon\to 0}\frac{f(s+\frac{1}{\alpha}(t+\varepsilon)^{\alpha}-f(s+\frac{1}{\alpha}t^{\alpha})}{\varepsilon} \\ &= t^{1-\alpha}\lim_{\varepsilon\to 0}\frac{f(s+\frac{1}{\alpha}(t+\varepsilon)^{\alpha})-f(s+\frac{1}{\alpha}t^{\alpha})}{\varepsilon}.\frac{f(s+t+\varepsilon)-f(s+t)}{f(s+t+\varepsilon)-f(s+t)} \\ &= t^{1-\alpha}\lim_{\varepsilon\to 0}\frac{f(s+\frac{1}{\alpha}(t+\varepsilon)^{\alpha})-f(s+\frac{1}{\alpha}t^{\alpha})}{f(s+t+\varepsilon)-f(s+t)}.\frac{f(s+t+\varepsilon)-f(s+t)}{\varepsilon}. \end{aligned}$$

Now

$$\lim_{\varepsilon\to 0}\frac{f(s+t+\varepsilon)-f(s+t)}{\varepsilon}=f'(s+t),\ \lim_{\varepsilon\to 0}\frac{f(s+\frac{1}{\alpha}(t+\varepsilon)^{\alpha}-f(s+\frac{1}{\alpha}t^{\alpha})}{f(s+t+\varepsilon)-f(s+t)}=\frac{0}{0}.$$

Use L'Hopital's rule (with respect to ε) to get

$$\lim_{\varepsilon\to 0}\frac{f(s+\frac{1}{\alpha}(t+\varepsilon)^{\alpha})-f(s+\frac{1}{\alpha}t^{\alpha})}{f(s+t+\varepsilon)-f(s+t)}=(t)^{\alpha-1}f'(s+\frac{1}{\alpha}(t)^{\alpha})/f'(s+t).$$

Thus the product gives

$$T^{(\alpha)}(t)f(s)=f'(s+\frac{1}{\alpha}t^{\alpha}).$$

Now take the limit as $t\to 0$ to get

$$T^{(\alpha)}(0)f(s)=f'(s).$$

Hence $Af=f'$. This completes the proof.

Let us show how our theory can be applied to obtain information about solutions of certain problems. In particular, we want to use the fractional semigroups approach for solving the so-called $\alpha-$ abstract Cauchy problem.

Definition 2.2. Let X be a Banach space, $A:D(A)\subseteq X\longrightarrow X$ a linear operator and $u_0\in X$. A function $u:[0,\infty)\longrightarrow X$ is a solution of the $\alpha-$ abstract Cauchy problem

(1) $$u^{(\alpha)}(t)=Au(t),\quad t>0,$$

(2) $$u(0)=u_0$$

if:

(i) u is continuous on $[0,\infty)$,

(ii) u is continuously $\alpha-$differentiable on $(0,\infty)$,

(iii) $u(t)\in D(A)$ for $t>0$,

(iv) u satisfies (1)-(2).

Theorem 2.3. Let X be a Banach space and A the infinitesimal generator of a $c_0-\alpha-$semigroup $\{T(t)\}_{t\geq 0}\subseteq \mathscr{L}(X,X)$. If $u_0\in D(A)$, then problem (1)-(2) has one and only one solution u, namely,

$$u(t)=T(t)u_0.$$

Proof. Clearly $u(t)=T(t)x$ is a solution of problem (1)-(2). For uniqueness, let u be a solution of (1)-(2). Then

$$\begin{aligned}
[T(t-s)u(s)]^{(\alpha)} &= T(t-s)u^{(\alpha)}(s)-AT(t-s)u(s)\\
&= T(t-s)u^{(\alpha)}(s)-T(t-s)Au(s)\\
&= T(t-s)[u^{(\alpha)}(s)-Au(s)]\\
&= 0.
\end{aligned}$$

Applying I_α^0, in s, we have

$$T(t-t)u(t)-T(t)u_0=0\Rightarrow u(t)=T(t)u_0.$$

This completes the proof.

Remark 2.1. Let $X=C[0,\infty)$ be the space of continuous real-valued functions such that $\lim\limits_{x\to\infty} f(x)$ is finite, with the sup norm. Define the operator A by

$$Af(s)=f'(s),$$
$$D(A)=\{f\in X : f' \; exists \; in \; X\}.$$

Then A is a generator of the above $\alpha-$semigroup of translation. If $u_0\in D(A)$, then problem (1)-(2) has the unique solution $u(t)=T(t)u_0$, where $T(t)$ is an $\alpha-$semigroup generated by A.

Now it is readily seen that if g is continuously differentiable on $[0,\infty)$, then

$$u(x,t) = g\left(x + \frac{1}{\alpha}t^{\alpha}\right)$$

is the unique solution of the problem

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial u}{\partial x}, \quad x > 0, \ t > 0,$$
$$u(x,0) = g(x), \quad x > 0.$$

REFERENCES

[1] T. Abdeljawad, On Conformable Fractional Calculus, J. Comput. Appl. Math. 279 (2015), 57-66.

[2] M. Al Horani, Projection Method for Solving Degenerate First-Order Identification Problem, J. Math. Anal. Appl. 364 (2010), 204-208.

[3] A. V.Balakrishnan, Fractional Powers Of Closed Operators And The Semigroups Generated By Them, Pacific J. Math. 10 (1960), 419-439.

[4] S. Bochner, Diffusion Equation And Stochastic Processes, Proc. Nat. Acad. Sciences, U.S.A., 35 (1949), 368-370.

[5] K.J. Engel, R. Nagel, One Parameter Semigroups For Linear Evolution Equations, Graduate Texts in Math., Springer-Verlag, 2000.

[6] R. Khalil, M. Al Horani, A.Yousef, M. Sababheh, A new definition Of fractional derivative, J. Comput. Appl. Math. 264 (2014) 65–70.

[7] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differntial Equations, Springer-Verlag, 1983.

[8] E. Popescu, On The Fractional Cauchy Problem Associated With a Feller Semigroup, Romanian Academy, Mathematical Reports, 12 (2010), 181–188.

[9] I.Prilepko, G. Orlovsky, A.Vasin, Methods For Solving Inverse Problems In Mathematical Physics, Marcel Dekker. Inc. New York, 2002.