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## ON THE ONE HALF OF AN ARF NUMERICAL SEMIGROUP

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**Abstract.** In this study, we will characterize the Arf numerical semigroup that is computed in a quotient of an Arf numerical semigroup by an positive integer. We will also obtain the one half of this special Arf numerical semigroup and give some results about this numerical semigroup.

**Keywords:** Numerical semigroup, Arf numerical semigroups, Frobenius number, one half numerical semigroup.

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### 1. Introduction

Let  $\mathbb{N}$  be the set non-negative integers. A numerical semigroup  $S$  is a subset of  $\mathbb{N}$  which contains the zero, is closed under addition and generates  $\mathbb{Z}$  as a group where  $\mathbb{Z}$  denotes the set of the integer. From this definition, we can deduce that  $S$  admits a unique minimal system of generators  $\{n_1 < \dots < n_p\}$ , if we obtain the set  $S = \left\{ \sum_{i=1}^p a_i n_i : a_1, \dots, a_p \in \mathbb{N} \right\}$  and no proper subset of  $\{n_1 < \dots < n_p\}$  is a system of generators of  $S$  [1].

$\mathbb{N} \setminus S$  is finite, and the largest integer not belonging to  $S$  is known as the Frobenius number of  $S$ , usually denoted by  $F(S)$ . We define  $n(S) = \#\left(\{0, 1, \dots, F(S)\} \cap S\right)$ , where  $\#(A)$  denotes the cardinality of any set  $A$ . It is also well known that

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$S = \langle n_1, n_2, n_3, \dots, n_p \rangle = \{0, s_1, s_2, s_3, \dots, s_n, F(S)+1, \rightarrow \dots\}$ , where " $\rightarrow$ " means that every integer greater than  $F(S)+1$  belongs to  $S$ ,  $n = n(S)$  and  $s_i < s_{i+1}$  for  $i = 1, 2, \dots, n$ . [3].

Let  $S$  be a numerical semigroups and  $p$  be a positive integer. Then  $\frac{S}{p} = \{x \in \mathbb{N} : px \in S\}$  is also a numerical semigroup, called as the quotient of  $S$  by  $p$ . We say that  $\frac{S}{2}$  is the half of the numerical semigroup  $S$  [2].

A numerical semigroup  $S$  is an Arf numerical semigroup if for every  $x, y, z \in S$  such that  $x \geq y \geq z$ , we have that  $x + y - z \in S$  [4].

In this paper, our principal aim is to characterize the Arf numerical semigroup that is one half of an Arf numerical semigroup and give relations between  $S$  and  $\frac{S}{2}$ .

## 2. Main Results

Our first goal in this section is to show that quotient of an Arf numerical semigroup by an positive integer is Arf numerical semigroup. In addition, we will obtain Arf numerical semigroup  $\frac{S}{2}$  and find the numbers  $F\left(\frac{S}{2}\right)$  and  $n\left(\frac{S}{2}\right)$ . Lastly, we will give also the relations between  $S$  and  $\frac{S}{2}$ .

**Theorem 2.1.** Let  $S$  be an Arf numerical semigroup and  $p$  be a positive integer. Then  $\frac{S}{p}$  is an Arf numerical semigroup.

**Proof.** We choose all members  $x, y, z$  in  $\frac{S}{p} \setminus \{0\}$  such that  $x \geq y \geq z$ . Then it is clear that  $px, py, pz \in S \setminus \{0\}$ . Thus we obtain that  $px + py - pz \in S$ , since  $S$  is an Arf numerical semigroup. So,  $p(x + y - z) \in S$  and  $x + y - z \in \frac{S}{p}$ . It follows that  $\frac{S}{p}$  is an Arf numerical semigroup.

**Example 2.2.** The numerical semigroup  $S = \{0, 7, 14, 21, 27, \rightarrow \dots\}$  is an Arf numerical semigroup.

So, for  $p = 5$  we obtain the set  $\frac{S}{5} = \{0, 6, \rightarrow \dots\}$  which is an Arf numerical semigroup.

**Theorem 2.3.** Let  $a \in \mathbb{Z}^+$  and  $a \geq 2$ . If  $(a_n)$  is a sequence with general term

$$a_n = \begin{cases} 1, & \text{if } n = 1 \\ a^{n-2}, & \text{if } n \neq 1 \end{cases}, \quad \text{that is } (a_n) = (1, a^0, a^1, \dots, a^{n-2}, \dots), \quad \text{then}$$

$S = \{0, x_1, x_1 + x_2, \dots, x_1 + \dots + x_n \rightarrow \dots\} = \{0, a^{n-2}t, (a^{n-2} + a^{n-3})t, \dots, (a^{n-2} + \dots + a^0 + 1)t, \rightarrow \dots\}$  is an Arf numerical semigroup where  $x_n = t$ ,  $x_{n-1} = a^0t$ ,  $x_{n-2} = a^1t, \dots, x_1 = a^{n-2}t$  for  $t \in \mathbb{N}$  [5].

**Theorem 2.4.** If the numerical semigroup  $S$  is given as in Theorem 2.3 then the one half of  $S$

$$\text{is } \frac{S}{2} = \left\{ 0, \frac{s_1}{2}, \frac{s_2}{2}, \dots, \frac{s_{r-1}}{2}, \frac{s_r}{2}, \rightarrow \dots \right\}.$$

**Proof.** Let  $a, t, n \in \mathbb{Z}^+$ ,  $a \geq 2$  and  $t$  be positive even integer. Suppose

$$A = \left\{ 0, \frac{s_1}{2}, \frac{s_2}{2}, \dots, \frac{s_{r-1}}{2}, \frac{s_r}{2}, \rightarrow \dots \right\}. \text{ If } x \in A \text{ then we write } x = \frac{s_i}{2} \text{ for } i = 1, 2, \dots, r \text{ or } x = \frac{s_r}{2} + k \text{ for}$$

$k = 1, 2, \dots$ . Therefore we find the equalities  $2x = 2 \frac{s_i}{2} = s_i$  or  $2x = 2 \left( \frac{s_r}{2} + k \right) = s_r + 2k$ . So we

obtain  $2x \in S$ . Finally, we have  $x \in \frac{S}{2}$ .

Conversely, if  $x \in \frac{S}{2}$ , then we have  $2x \in S$ . Then we can write  $x = \frac{s_i}{2} \in \frac{S}{2}$  for  $s_i \in S$  and for

$i = 1, 2, \dots, r$ . There is the at least one  $k \in \mathbb{N}$  such that  $2x = s_r + 2k \in S$  for  $2x > s_r$ . Thus we

obtain  $x = \frac{s_i}{2} \in A$ .

**Example 2.5.** If we choose  $a = 7$ ,  $n = 7$  and  $t = 2$  then we obtain

$$x_7 = 2, x_6 = 2, x_5 = 14, x_4 = 98, x_3 = 686, x_2 = 4802, x_1 = 33614.$$

So  $S = \{0, 33614, 38416, 39102, 39200, 39214, 39216, 39218, \rightarrow \dots\}$  is an Arf numerical

semigroup and we can write  $\frac{S}{2} = \{0, 16807, 19208, 19551, 19600, 19607, 19608, 19609, \rightarrow \dots\}$

from Theorem 2.4.

We will take  $a, t, n \in \mathbb{Z}^+$  where  $a, t, n \geq 2$  in the following Theorems, Corollaries and proofs.

**Corollary 2.6.** The one half of the numerical semigroup

$S = \{0, a^{n-2}t, (a^{n-2} + a^{n-3})t, \dots, (a^{n-2} + \dots + a^0 + 1)t, \rightarrow \dots\} = \{0, s_1, s_2, \dots, s_r, \rightarrow \dots\}$  is as the follows:

- i. If  $a$  is even integer and  $t$  is odd integer, then  $\frac{S}{2} = \left\{0, \frac{s_1}{2}, \frac{s_2}{2}, \dots, \frac{s_{r-2}}{2}, \frac{s_r}{2}, \rightarrow \dots\right\}$
- ii. If  $a, t$  are odd integers and  $n$  is even integer, then  $\frac{S}{2} = \left\{0, \frac{s_2}{2}, \frac{s_4}{2}, \dots, \frac{s_{r-2}}{2}, \frac{s_r}{2}, \rightarrow \dots\right\}$
- iii. If  $a, n, t$  are odd integers, then  $\frac{S}{2} = \left\{0, \frac{s_2}{2}, \frac{s_4}{2}, \dots, \frac{s_{r-1}}{2}, \frac{s_{r+1}}{2}, \rightarrow \dots\right\}$ .

We will avoid proving Corollary 2.6. because it is trivial.

**Proposition 2.7.** If  $S$  is an Arf numerical semigroup such that

$S = \{0, a^{n-2}t, (a^{n-2} + a^{n-3})t, \dots, (a^{n-2} + \dots + a^0 + 1)t, \rightarrow \dots\}$  then we find  $n(S) = n$  and

$$F(S) = (a^{n-2} + \dots + a^0 + 1)t - 1 \quad [5].$$

**Theorem 2.8.** The Frobenius number of the one half of numerical semigroup

$S = \{0, a^{n-2}t, (a^{n-2} + a^{n-3})t, \dots, (a^{n-2} + \dots + a^0 + 1)t, \rightarrow \dots\} = \{0, s_1, s_2, \dots, s_r, \rightarrow \dots\}$  is

$$F\left(\frac{S}{2}\right) = \begin{cases} \frac{F(S)}{2} & ; \text{if } a, n, t \text{ are positive odd integers} \\ \frac{F(S) - 1}{2} & ; \text{otherwise} \end{cases}$$

**Proof.** Since  $a, n$  and  $t$  are odd positive integers and from Corollary 2.6., we get

$$\frac{S}{2} = \left\{0, \frac{s_2}{2}, \frac{s_4}{2}, \dots, \frac{s_{r-1}}{2}, \frac{s_{r+1}}{2}, \rightarrow \dots\right\}. \text{ Thus we find } F\left(\frac{S}{2}\right) = \frac{s_{r+1}}{2} - 1 = \frac{s_r + 1}{2} - 1 = \frac{s_r - 1}{2} = \frac{F(S)}{2}.$$

For the otherwise, the one half of original semigroup is written as one of the

$$\frac{S}{2} = \left\{0, \frac{s_1}{2}, \frac{s_2}{2}, \dots, \frac{s_{r-1}}{2}, \frac{s_r}{2}, \rightarrow \dots\right\}, \quad \frac{S}{2} = \left\{0, \frac{s_1}{2}, \frac{s_2}{2}, \dots, \frac{s_{r-2}}{2}, \frac{s_r}{2}, \rightarrow \dots\right\} \text{ or}$$

$$\frac{S}{2} = \left\{0, \frac{s_2}{2}, \frac{s_4}{2}, \dots, \frac{s_{r-2}}{2}, \frac{s_r}{2}, \rightarrow \dots\right\}. \text{ For all three cases, we obtain } F\left(\frac{S}{2}\right) = \frac{s_r}{2} - 1. \text{ Finally, we have}$$

$$F\left(\frac{S}{2}\right) = \frac{s_r}{2} - 1 = \frac{s_r - 2}{2} = \frac{(s_r - 1) - 1}{2} = \frac{F(S) - 1}{2}.$$

**Theorem 2.9.** The following equalities are true for numerical semigroup

$$S = \{0, a^{n-2}t, (a^{n-2} + a^{n-3})t, \dots, (a^{n-2} + \dots + a^0 + 1)t, \rightarrow \dots\} = \{0, s_1, s_2, \dots, s_r, \rightarrow \dots\} .$$

a) If  $t$  is even integer, then  $n\left(\frac{S}{2}\right) = n(S)$

b) If  $a$  is even integer and  $t$  is odd integer, then  $n\left(\frac{S}{2}\right) = n(S) - 1$

c) If  $a, t$  are odd integers and  $n$  is even integer, then  $n\left(\frac{S}{2}\right) = \frac{n(S)}{2}$

d) If  $a, n, t$  are odd integers, then  $n\left(\frac{S}{2}\right) = \frac{n(S) + 1}{2}$ .

**Proof.** Using Theorem 2.4. and Corollary 2.6., for the one half of numerical semigroup  $S = \{0, a^{n-2}t, (a^{n-2} + a^{n-3})t, \dots, (a^{n-2} + \dots + a^0 + 1)t, \rightarrow \dots\} = \{0, s_1, s_2, \dots, s_r, \rightarrow \dots\}$ , the following cases hold.

a) If  $t$  is even integer, then  $\frac{S}{2} = \left\{0, \frac{s_1}{2}, \frac{s_2}{2}, \dots, \frac{s_{r-1}}{2}, \frac{s_r}{2}, \rightarrow \dots\right\}$  and

$$n\left(\frac{S}{2}\right) = \#\left(\left\{0, 1, \dots, F\left(\frac{S}{2}\right)\right\} \cap \frac{S}{2}\right) = \#\left(\left\{0, \frac{s_1}{2}, \frac{s_2}{2}, \dots, \frac{s_{r-1}}{2}\right\}\right) = [(r-1) - 0] + 1 = r = n(S)$$

b) If  $a$  is even integer and  $t$  is odd integer, then  $\frac{S}{2} = \left\{0, \frac{s_1}{2}, \frac{s_2}{2}, \dots, \frac{s_{r-2}}{2}, \frac{s_r}{2}, \rightarrow \dots\right\}$  and

$$n\left(\frac{S}{2}\right) = \#\left(\left\{0, 1, \dots, F\left(\frac{S}{2}\right)\right\} \cap \frac{S}{2}\right) = \#\left(\left\{0, \frac{s_1}{2}, \frac{s_2}{2}, \dots, \frac{s_{r-2}}{2}\right\}\right) = [(r-2) - 0] + 1 = n(S) - 1$$

c) If  $a, t$  are odd integers and  $n$  is even integer, then  $\frac{S}{2} = \left\{0, \frac{s_2}{2}, \frac{s_4}{2}, \dots, \frac{s_{r-2}}{2}, \frac{s_r}{2}, \rightarrow \dots\right\}$  and

$$n\left(\frac{S}{2}\right) = \#\left(\left\{0, 1, \dots, F\left(\frac{S}{2}\right)\right\} \cap \frac{S}{2}\right) = \#\left(\left\{0, \frac{s_2}{2}, \frac{s_4}{2}, \dots, \frac{s_{r-2}}{2}\right\}\right) = \left[\frac{(r-2) - 0}{2}\right] + 1 = \frac{n(S)}{2}$$

d) If  $a, n, t$  are odd integers, then  $\frac{S}{2} = \left\{0, \frac{s_2}{2}, \frac{s_4}{2}, \dots, \frac{s_{r-1}}{2}, \frac{s_{r+1}}{2}, \rightarrow \dots\right\}$  and

$$n\left(\frac{S}{2}\right) = \#\left(\left\{0, 1, \dots, F\left(\frac{S}{2}\right)\right\} \cap \frac{S}{2}\right) = \#\left(\left\{0, \frac{s_2}{2}, \frac{s_4}{2}, \dots, \frac{s_{r-1}}{2}\right\}\right) = \left[\frac{(r-1) - 0}{2}\right] + 1 = \frac{n(S) + 1}{2}$$

**Example 2.10.** If we choose  $a = 3, n = 7$  and  $t = 5$  then we obtain the numerical semigroup

$S = \{0, 1215, 1620, 1755, 1800, 1815, 1820, 1825, \rightarrow \dots\}$  . So we find  $F(S) = 1824$  and  $n(S) = 7$  .

The one half of numerical semigroup  $S$  is  $\frac{S}{2} = \{0, 810, 900, 910, 913, \rightarrow \dots\}$  . Moreover

$$F\left(\frac{S}{2}\right) = 912 = \frac{F(S)}{2} \text{ and } n\left(\frac{S}{2}\right) = 4 = \frac{n(S)+1}{2} .$$

### Conflict of Interests

The author declares that there is no conflict of interests.

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