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IDEMPOTENTS, BANDS AND GREEN'S RELATIONS IN TERNARY SEMIGROUPS

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Abstract. This paper is for one part a generalization of some results obtained by Miyuki Yamada [20] in the case of binary semigroups to ternary semigroups. We prove analogous of almost all the results previously cited. We prove in particular that the set of the idempotents in regular ternary semigroup is a band (that is, a semigroup). In a second part we continue our investigations started in [13; 14] on these semigroups, as on the structure of the set E(S) of idempotents of the ternary semigroup S. The particular case of ternary inverse semigroup has been studied and a relationship between the existence of idempotents and the inverse elements has been caracterized. The documents [5]; [9] and [10] have been intensively used. We asked two questions and the answer for the second one will be the subject of a forcoming paper. We use many references in our work the most important are those used as bibliography.

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1. Preliminaries

Definition 1.1. A nonempty set S is called a ternary semigroup if there exists a ternary operation; $\circ : S \times S \times S \longrightarrow S$, written as $\circ(a,b,c) \longmapsto a \circ b \circ c$ satisfying the following identity for any $a,b,c,d,e \in S$, $(a \circ b \circ c) \circ d \circ e = a \circ (b \circ c \circ d) \circ e = a \circ b(c \circ d \circ e) \circ c$.

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In all the paper, when no ambiguity can be made; the element $\circ(a,b,c)$ will be simply denoted by abc.

Definition 1.2. An element $1 \in S$ is called a unity if $\forall x,y \in S, 1\circ x\circ y = x\circ 1\circ y = x\circ y\circ 1$ and $1\circ 1\circ x = x$.

In the sequel our ternary semigroups are supposed to have a unity which we always denote by 1.

Definition 1.3. An element $a \in S$ is an inverse of an element $b \in S$ if $aba = a$ and $bab = b$. An element is then said to be regular if it has at least one inverse. An element $b \in S$ is a weak inverse of an element a if $aba = a$.

Definition 1.4.

(1) A ternary semigroup S is called regular ternary semigroup if every element of S has at least an inverse.
(2) A ternary semigroup S is called inversive ternary semigroup if every element of S has a unique inverse.

Definition 1.5. Let S be a ternary semigroup. An element a of S is said to be Von Neumann regular if it has at least one weak inverse or equivalently, $a = axa$ for some $x \in S$, and S is called a Von Neumann regular semigroup if every element of S is Von Neumann regular.

Definition 1.6. Let S be a ternary semigroup. An element a of S is said to be an idempotent if $a.a.a = a$.

Remark 1.1. It is clear that an idempotent element is invertible and has itself as an inverse.

Definition 1.8. A ternary semigroup S is said to be

(1) commutative if $a.b.c = \tau(a.b.c)$ for any transposition $\tau \in S_3$.
(2) cyclicly commutative if $abc = C(abc)$ for the two cycles of order 3 in S_3.

Remark 1.2. If S is commutative, then S is cyclicly commutative. The converse is false.

Remark 1.3. Any associative binary operation ”.” on a set S defines a ternary operation γ by

$$\gamma(a,b,c) = (a.b).c.$$

Definition 1.11. Let S be a ternary semigroup. If there exists an element $0 \in S$ such that $0.x.y = x.0.y = x.y.0 = 0$ for all $x, y \in S, S$ is said to have a zero element and 0 is called the zero of the ternary semigroup S.

Let A, B, C be three subsets of S. Then by $A.B.C$ or simply ABC, we mean the set:

$$ABC = \{a_i.b_i.c_i \text{ with } a_i \in A, b_i \in B, c_i \in C\}.$$

Definition 1.12. A subset H of S is called a ternary subsemigroup if

$$a.b.c \in H,\ \forall a,b,c, \in H and 1 \in H.$$

The following definitions in the case of ternary operations analogous to the definitions in the case of binary semigroups.

Definition 1.13.

(1) A Von Neumann regular ternary semigroup is called strictly regular if the set of its idempotents is a subsemigroup.
(2) An orthodox ternary semigroup is a strictly regular ternary semigroup such that there exists an element $a \in S$ with $x.a.x = x\ \forall x \in S$.
(3) A band is a ternary semigroup where any element is an idempotent.

Definition 1.14.

(1) A semilattice is a commutative band.
(2) A left (resp. right) zero band S is a band satisfying the equation $x.x.y = x\ (resp \circ yxx = x)\ \forall x, y \in S$.
(3) A rectangular band S is a band which satisfies $x.y.x = x\ \forall x, y \in S$.
(4) A normal band S is a band satisfying $x.(y.z.t).x = x.(\sigma(yzt))x\ \forall x, y, z, t \in S$ and $\sigma \in S_3$.
(5) A regular band S is a band satisfying $x.(y.x.z).x = x.y.z\ \forall x, y, z \in S$.

Example 1.1. Let I, J be any two non-empty sets, the cartesian product endowed with the following ternary operation:

$$\forall (a,b),(c,d),(e,f) \in I \times J; (a,b).(c,d).(e,f) = (a,b)$$

is a rectangular band because

(1) (a) $(a,b).(c,d).[(e,f).(i,j).(k,l)] = (a,b).(c,d).(e,l) = (a,l)$,
  (b) $(a,b).[(c,d).(e,f).(i,j)].(k,l) = (a,b).(c,i).(k,l) = (a,l)$,
  (c) $[(a,b).(c,d).(e,f)].(i,j)].(k,l) = (a,f).(c,i).(k,l) = (a,l)$

(2) $(a,b).(a,b).(a,b) = (a,b)$,

(3) $(a,b).(c,d).(a,b) = (a,b)$.

2. Regular, inverse semigroups and idempotents

Proposition 2.1. Let e be an idempotent of a strictly regular ternary semigroup S. Then, every inverse of e is an idempotent.

Proof. Let x be an inverse of e then $x.e.x = x$ and $e.x.e = e$ so

$$x = x.e.x = x.(e.e.e).x = x.e.(e.e.x) = x.(e.e.e).(e.e.x) = (x.e.e).e.(e.e.x).$$

But

$$(x.e.e).(x.e.e).(x.e.e) = x.e.[e.(x.e.e).(x.e.e)] = x.e.[(e.x.e).e.(x.e.e)] = x.e.[e.e.(x.e.e)] =$$

$$x.e.[e.(e.x.e).e] = x.e.[e.e.e] = x.e.e$$

and

$$(e.e.x).(e.e.x).(e.e.x) = e.[e.x.(e.e.x)].(e.e.x) = e.[(e.x.e).e.x](e.e.x) = e.[e.e.x].(e.e.x) =$$

$$e.e.[(e.x.e).e.x] = e.e.[e.e.x] = (e.e.e).e.x = e.e.x$$

so $e.e.x$ and $x.e.e$ are idempotents. The set $E(S)$ is a subsemigroup then $(x.e.e).e.(e.e.x)$ is an element of $E(S)$ and so is x.

Definition 2 .1. An element a of a ternary semigroup S commutes with $b \in S$ if $a.b.x = b.a.x$ and $x.a.b = x.b.a$ for any $x \in S$.

Lemma 2 .1. If a^* is an inverse of an element $a \in S$ and an idempotent e commutes with a then $a^*.a.e$ and $e.a^*$ are idempotent.

Proof. $(a^*.a.e).(a^*.a.e).(a^*.a.e) = (a^*.e.a).(a^*.a.e).(a^*.a.e) = a^*.e.[a.(a^*.a.e).(a^*.a.e)] = a^*.[(a.a^*.a).e.(a^*.a.e)] = a^*.e.[a.e.(a^*.a.e)] = a^*.e.[e.(a.a^*.a).e] = a^*.e.[e.a.e] = a^*.a.(e.e.e) = a^*.a.e$ since e is an idempotent.

In other hand, $(e.a.a^*).(e.a.a^*).(e.a.a^*) = (e.a.a^*).(a.e.a^*).(e.a.a^*) = e.(a.a^*a)[e.a^*.(e.a.a^*)] = e.a.[e.a^*.(e.a.a^*)] = e.e.[a.a^*(a.e.a^*)] = e.e.[(a.a^*a).e.a^*] = e.e.(a.e.a^*) = (e.e.e)a.a^* = e.a.a^*$.

In the sequel $a.b.c$ will be just denoted abc.

Theorem 2.3. Let S be a strictly regular ternary semigroup. Let $e, f \in E(S)$ such that $efe = e$ and $fef = f$. Then, if $a, c \in S$ commute with any idempotent; any inverse x of aec is also an inverse of afc.

Proof. Let a^*, c^* be two inverses of a, c respectively.

$aec = (aec)x(aec) = [ae(cc^*c)]x[(a.a^*a)ec] \implies$
$a^*(aec)c^* = a^*[[ae(cc^*c)]x[(a.a^*a)ec]]c^* = a^*[ae(cc^*c)][x[(a.a^*a)ec]c^*] =$
$(a^*ae)(cc^*c)[x[(a.a^*a)ec]c^*] = [(a^*ae)cc^*]c[x[(a.a^*a)ec]c^*] = [(a^*ae)cc^*](cx[(a.a^*a)ec])c^* =$
$[(a^*ae)cc^*][(cx(a.a^*a))ec]cc^* = [(a^*ae)cc^*][((cxa)a^*a)ec]c^* = [(a^*ae)ccc^*][(cxa)(a^*ae)c]c^* =$
$[(a^*ae)ccc^*](cxa)[(a^*ae)cc^*] = [a^*(aec)cc^*](cxa)[(a^*(aec)c^*]$.

In the other hand
$(cxa)[a^*(aec)c^*](cxa) = c[xa(a^*(aec)c^*)](cxa) = c[(xaa^*)(aec)c^*)](cxa) =$
$c[x(a.a^*(aec)c^*)](cxa) = c[x((a.a^*a)ec)c^*](cxa) = c[x(aec)c^*](cxa) =$
$c[xa(ecc^*)](cxa) = c[(xa(ecc^*))cx]a = c[((xae)cc^*)cx]a = c[((xae)(cc^*c)x]a =$
$c[(xae)cx]a = c[x(aec)x]a = cxa$
and then cxa is an inverse of $a^*(aec)c^* = (a^*ae)cc^*$.

From the lemma 2.1,(a^*ae) is an idempotent so it commutes with c and the element $(a^*ae)cc^*$ is an idempotent by the same lemma. Now by proposition 2.1, cxa is also an idempotent and then by taking f instead e we get that cxa is also an inverse of $(a^*af)cc^*$ and then $(a^*af)cc^*$ is an idempotent. So

$$[(a^*af)cc^*](cxa)[(a^*af)cc^*] = (a^*af)cc^* \implies a[[(a^*af)cc^*](cxa)[(a^*af)cc^*]]c =$$

$$a[(a^*af)cc^*]c \iff a[[(a^*af)cc^*](cxa)[(a^*af)cc^*]]c =$$

$$a[(a^*af)cc^*]c = (a.a^*a)f(cc^*c) = afc.$$

But

$$a[[(a^*af)cc^*](cxa)[(a^*af)cc^*]]c = a[(a^*af)cc^*]c = a[(a^*af)cc^*][(cxa)[(a^*af)cc^*]c] =$$

$$a(a^*af)[cc^*[(cxa)[(a^*af)cc^*]c]] = (a.a^*a)f[cc^*[(cxa)[(a^*af)cc^*]c]] = af[cc^*[(cxa)[(a^*af)cc^*]c]] =$$

$$af[(cc^*(cxa))[(a^*af)cc^*]c] = af[((cc^*c)xa))[(a^*af)cc^*]c] = af[(cxa))[(a^*af)cc^*]c] =$$

$$af[(cxa)(a^*af)(cc^*c)] = af[(cxa)(a^*af)c] = af[cx[(a.a^*a)fc]] = af[cx[afc]] = (afc)x(afc).$$

So

$$(afc)x(afc) = afc(a).$$

In the other hand

$$(afc)[x(afc)x](afc) = [(afc)x(afc)]x(afc) = (afc)x(afc) = afc$$

and

$$[x(afc)x](afc)[x(afc)x] = x[(afc)x(afc)][x(afc)x] = x(afc)[x(afc)x] =$$

$$x[(afc)x(afc)]x = x(afc)x,$$

so $x(afc)x$ is an inverse of afc. By using the same methods we can get

$$(aec)[x(afc)x](aec) = aec.$$

Hence,

$$x(aec)[(x(afc)x)(aec)x] = x[(aec)[x(afc)x](aec)]x = x(aec)x = x$$

since x is an inverse of aec. But

$$x(aec)[(x(afc)x)(aec)x] = [x(aec)(x(afc)x)](aec)x = [(x(aec)x)(afc)x](aec)x =$$

$$[x(afc)x](aec)x = x(afc)(x(aec)x) = x(afc)x,$$

so $x(afc)x = x$. Therefore, it follows that x is an inverse of afc.

Definition 2.2. Let S be a ternary regular semigroup. If a mapping $\Phi : S \longrightarrow S$ is such that

$$\forall x \in S; x\Phi(x)x = \Phi(x)x\Phi(x),$$

the mapping Φ is called an inverse operator. It is obvious that S has at least one inverse operator, the mapping in which the image of x is its inverse.

Proposition 2.4. Let S be a regular ternary semigroup. The inverse operator Φ is unique if and only if S is inversive semigroup.

Proof. The proof is trivial.

Definition 2.3. Let S, S' be two ternary semigroups. A mapping $\Phi : S \longrightarrow S'$ is a ternary semigroup morphism if for all $a, b, c \in S, \Phi(abc) = \Phi(a)\Phi(b)\Phi(c)$ and $\Phi(1_S) = 1_{S'}$. We can see that if a is an idempotent then $\Phi(a)$ is also an idempotent.

Proposition 2.5. Let S be a ternary commutative inversive semigroup. Any inverse operator Φ is a ternary semigroup morphism.

Proof. Let Φ be an inverse operator and let $a, b, c \in S$. We will prove that $\Phi(abc) = \Phi(a)\Phi(b)\Phi(c)$. To get this assertion it suffices to prove that $\Phi(a)\Phi(b)\Phi(c)$ is an inverse of abc and by uniqueness of inverse of abc we get $\Phi(abc) = \Phi(a)\Phi(b)\Phi(c)$. The commutativity implies that

$$(\Phi(a)\Phi(b)\Phi(c))(abc)(\Phi(a)\Phi(b)\Phi(c)) = (\Phi(c)\Phi(b)\Phi(a))(abc)(\Phi(a)\Phi(b)\Phi(c)) =$$

$$\Phi(c)\Phi(b)[\Phi(a)(abc)(\Phi(a)\Phi(b)\Phi(c))] = \Phi(c)\Phi(b)[\Phi(a)a(bc(\Phi(a)\Phi(b)\Phi(c)))] =$$

$$\Phi(c)\Phi(b)[\Phi(a)a[(bc\Phi(a))\Phi(b)\Phi(c)]] = \Phi(c)\Phi(b)[\Phi(a)a[(\Phi(a)cb)\Phi(b)\Phi(c)]] =$$

$$\Phi(c)\Phi(b)[(\Phi(a)a\Phi(a))(cb\Phi(b))\Phi(c)] = \Phi(c)\Phi(b)[a(cb\Phi(b))\Phi(c)] =$$

$$\Phi(c)\Phi(b)[a(\Phi(b)bc)\Phi(c)] = \Phi(c)\Phi(b)[(\Phi(b)bc)a\Phi(c)] = \Phi(c)[\Phi(b)(\Phi(b)bc)a]\Phi(c) =$$

$$\Phi(c)[(\Phi(b)\Phi(b)b)ca]\Phi(c) = \Phi(c)[(\Phi(b)b\Phi(b))ca]\Phi(c) = \Phi(c)[bca]\Phi(c) =$$

$$\Phi(c)\Phi(c)[cba] = (\Phi(c)\Phi(c)c)ba = (\Phi(c)c\Phi(c))ba = cba = abc.$$

Using the same arguments we can prove that

$$(abc)(\Phi(a)\Phi(b)\Phi(c))(abc) = abc.$$

So $(\Phi(a)\Phi(b)\Phi(c))$ is an inverse of abc and by uniqueness of the inverse we deduce that $(\Phi(a)\Phi(b)\Phi(c)) = \Phi(abc)$. In the other hand and from $111 = 1$ then 1 and $\Phi(1)$ are two inverses of 1 so they are equal and $\Phi(1) = 1$.

Definition 2.4. A subset T of a ternary semigroup is said to be characteristic if $\Phi(T) = T$ for any automorphism of Φ of S.

Proposition 2.6. The set $E(S)$ of idempotents of a ternary semigroup S is characteristic. Let $\Phi : S \longrightarrow S$ be an automorphism of S.

As we have remarked previously, if Φ is any endomorphism of S then $\Phi((E(S)) \subset E(S)$.

Now let $a \in S$ and as Φ is onto then there exists $b \in S$ such that $\Phi(b) = a$. So $a^3 = (\Phi(b))^3 = \Phi(b^3)$ but $a^3 = a = \Phi(b)$ so $\Phi(b^3) = \Phi(b)$ and by injectivity $b^3 = b$ and $E(S) \subset \Phi(E(S))$ which means that $E(S)$ is characteristic.

Proposition 2.7. If $\Phi : S \longrightarrow S$ is both an inverse operator and a morphism of an inversive ternary semigroup S, then it is bijective.

Proof. For any $a \in S$, a and $\Phi(\Phi(a))$ are both inverses of $\Phi(a)$ (that is, $\Phi(\Phi(a))\Phi(a)\Phi(\Phi(a)) = \Phi(\Phi(a)a\Phi(a)) = \Phi(\Phi(a))$ and $\Phi(a)\Phi(\Phi(a))\Phi(a) = \Phi(a\Phi(a)a) = \Phi(a))$, so they are equal, by the uniqueness of inverse. Then there exists $b = \Phi(a) \in S$ such that $\Phi(b) = a$ and Φ is onto.

Suppose that $\Phi(a) = \Phi(b)$ so a is is the inverse of $\Phi(b)$ and b is also the inverse of $\Phi(b)$ then $a = b$ by the uniqueness.

Now let S be a regular ternary semigroup and Ω be the set of all inverse operators in S. On S we define the relation $\mathscr{R}$as follows:

$$a\mathscr{R}b \iff \{\varphi(cad), \varphi \in \Omega\} = \{\varphi(cbd), \varphi \in \Omega\} \forall\, c, d \in S.$$

Proposition 2.8. Let S be a regular ternary semigroup. The relation $\mathscr{R}$ defined previously is a congruence relation in the following meaning:

$$a\mathscr{R}b \implies cda\mathscr{R}cdb$$

and

$$a\mathscr{R}b \implies acd\mathscr{R}bcd.$$

Proof. Let $a\mathscr{R}b, c,d \in S$ and $x \in \{\varphi(\alpha(cda)\beta), \varphi \in \Omega\}$, where $_alpha, \beta \in S$. As $a\mathscr{R}b$, we have the equality $\{\varphi(\alpha(cda)\beta), \varphi \in \Omega\} = \{\varphi((\alpha cd)a\beta), \varphi \in \Omega\} = \{\varphi((\alpha cd)b\beta), \varphi \in \Omega\}$. But $\{\varphi((\alpha cd)b\beta), \varphi \in \Omega\} = \{\varphi(\alpha(cdb)\beta), \varphi \in \Omega\}$ and then $\{\varphi(\alpha(cda)\beta), \varphi \in \Omega\} \subset \{\varphi(\alpha(cdb)\beta), \varphi \in \Omega\}$ It is easy to prove the converse inclusion.

The same method can be used to prove the second implication.

Finally we can deduce the following,

$$a\mathscr{R}b \implies (cda)ef\mathscr{R}(cdb)ef \forall c,d,e,f \in S.$$

Proposition 2.9. If $\mathscr{R}$ is as in proposition 2.8 on a regular ternary semigroup S. The quotient set $S/\mathscr{R}$ induced the quotient ternary operation is a regular ternary semigroup.

Proof. Let $a \in \overline{x}, b \in \overline{y}$ and $c \in \overline{z}$. We have to prove that $abc \in \overline{xyz} = \overline{xyz}$ and then the ternary product of classes is a well defined.

Let $\alpha, \beta \in S$;

$$\{\varphi(\alpha(abc)\beta)\} = \{\varphi(\alpha a(bc\beta))\} = \{\varphi(\alpha x(bc\beta))\} = \{\varphi((\alpha xb)c\beta)\} = \{\varphi((\alpha xb)z\beta)\}.$$

As $b\mathscr{R}y$, by the previous proposition, one has $(\alpha xb)z\beta\mathscr{R}(\alpha xy)z\beta$ so $\{\varphi((\alpha xb)z\beta))\} = \{\varphi((\alpha xy)z\beta)\} = \{\varphi(\alpha(xyz)\beta)\}$. We can then conclude that $abc \in \overline{xyz}$.

It is easy to see that a^* is an inverse of a where a^* is an inverse of a.

Definition 2.5. A semigroup S is left (resp. right) singular if $a^2b = b$ (resp. $ba^2 = b$), for all $a,b \in S$. These semigroups are bands.

Question 1. Is any band S a direct product $A \times B \times C$ where A,C are left singular semigroups and B right singular semigroups? If yes is this decomposition unique?

Proposition 2.10. If S is a commutative rectangular semigroup then $|S| \neq 3$.

Proof. Suppose that S has exactly 3 distinct elements a,b,c and suppose that $abc = a$, then $abc = ab(cac) = c^2(aba) = c^2a = cac = c$ so $a = c$ which is a contradiction. We get the same contradiction if we suppose that $abc = b$ or $abc = c$.

Proposition 2.11. If S is commutative, then S is rectangular if and only if $abc = a$ for all $a,b,c \in S$.

Proof. Sufficiency. Set $c = a$ in the above identity. Necessity. Assume that S is a rectangular band, then $a(bcc)a = a$. Therefore $abc = ab(cac) = ab(cca) = a(bcc)a = a$.

Proposition 2.12. If S is left (resp. right) singular,rectangular semigroup and $|S| \geq 2$, then S is not commutative.

Proof. Suppose that S is a commutative left singular semigroup and $|S| \geq 2$. Let $a \neq b$ in S, then $a^2b = b$ but $a^2b = aba = a$ since S is also rectangular. Contradiction.

Definition 2.7. A ternary semigroup is total if any element of S can be written as the product of three elements of S, that is $S^3 = S$.

Proposition 2.13.. A total semigroup S is rectangular if and only if it satisfies the identity

$$abc = c, \ \forall \ a, b, c \in S.$$

Proof. Sufficiency. If $abc = c, \ \forall \ a, b, c \in S$, then this relation remains true for $c = a$ and then S is rectangular.

Necessity. Let $x \in S$, as S is total then there exist $a, b, c \in S$ such that $x = abc$. For any $y \in S, xyx = (abc)y(abc) = ab[c(yab)c] = abc = x$ since $c(yab)c = c$ by the hypothesis and then S is rectangular.

Remark 2.1. If S satisfies $abc = c \ \forall \ a, b, c \in S$, then S is a band (hint: take $b = c = a$ in the equality).

Proposition 2.18. Let S be a ternary semigroup which satisfies the identity $abc = a^2b \ \forall \ a, b, c \in S$. The mapping $f : S \longrightarrow \ S$ defined by $f(x) = x^3$ is a endomorphism of S.

Proof. We remark that $abc = aab = a^2a = a^3$ so if $x, y, z \in S$, then $f(xyz) = (xyz)(xyz)(xyz) = (x^3)(x^3)(x^3) = (x^3)^3$. In the other hand

$$f(x)f(y)f(z) = (x^3)(y^3)(z^3) = (x^3)^3 \ \text{ So } f(xyz) = f(x)f(y)f(z).$$

Theorem 2.19. Let S be an inversive ternary semigroup. The set of all idempotents of S; $E(S)$ is a subsemigroup. Moreover if $e, f, g \in E(S)$, then

$$efg = fge = gef.$$

Proof. Let x be the only inverse of efg. Now,

$$efg=(efg)x(efg)=\begin{cases}(efg)(xee)(efg)\\(efg)(ggx)(efg)\end{cases}$$

and

$$(xee)(efg)(xee)=x(efg)(xee)=[x(efg)x]ee=xee,$$

$$(ggx)(efg)(ggx)=gg[x(efg)(ggx)]=gg[(xef)(ggg)x]=gg[(xef)gx]=gg[x(efg)x]=ggx.$$

So $x=xee=ggx$. But $x^3=(xee)(ggx)(xee)=x[ee(ggx)](xee)=[x(ee(ggx))x]ee=xee=x$. Then $(efg)^*$ is in $E(S)$ but as the inverse of any idempotent is itself so $efg\in E(S)$ and $E(S)$ is a subsemigroup of S.

In the other hand as

$$(efg)(fge)(efg)=e[fg(fge)](efg)=e[(fgf)ge](efg)=e[fge](efg)=(efg)e(efg)=efg$$

and

$$(fge)(efg)(fge)=[(fge)(efg)f]ge=[f(ge(efg))f]ge=fge$$

then fge is also an inverse of efg and by the uniqueness of the inverse element

$$fge=efg.$$

The consequence is that efg is closed by the cycle (132) of S_3, so is fge and finally $fge=gef$.

Lemma 2.2. Let S be a regulary ternary semigroup and $\alpha:S\longrightarrow P$ be an onto semigroup homomorphism. If $e\in E(P),x,y\in S$ are such that $\alpha(x)=e$ and y is an inverse of x^3,then $f=(xxy)xx\in E(S),f^3=fxf$ and $x?=xfx$.

Proof. Let $x\in S$ be such $\alpha(x)=e$ and let y be an inverse of x^3. Then we have $x?=x^3yx^3$ and $y=yx^3y$. If $f=(xxy)xx$ then:

$$\alpha(f)=(\alpha(x)\alpha(x)\alpha(y))\alpha(x)\alpha(x)=(\alpha(x)(\alpha(x)^3)\alpha(y))(\alpha(x)^3)\alpha(x)=$$

$$(\alpha(x)(\alpha(x^3))\alpha(y))(\alpha(x^3))\alpha(x)=\alpha(x)[(\alpha(x^3))\alpha(y)(\alpha(x^3))]\alpha(x)=$$

$$\alpha(x)[(\alpha(x^3))\alpha(y)(\alpha(x^3))]\alpha(x)=\alpha(x)[\alpha((x^3)y(x^3))]\alpha(x)=$$

$$\alpha(x)[\alpha(x^3)]\alpha(x)=e(e^3)e=eee=e\text{ and}$$

$$f^3=[(xxy)xx][(xxy)xx][(xxy)xx]=(xxy)x[x[(xxy)xx]][(xxy)xx]=$$

$$(xxy)x[x(xxy)[xx[(xxy)xx]]] = (xxy)x[(x^3)y[xx[(xxy)xx]]] =$$

$$(xxy)x[(x^3)y[(xx(xxy))xx]] = (xxy)x[(x^3)y[[(x^3)xy]xx]] =$$

$$(xxy)x[((x^3)y[(x^3)xy])xx] = (xxy)x[(((x^3)y(x?))xy)xx] =$$

$$(xxy)x[((x^3)xy)xx] = [(xxy)x((x^3)xy)]xx = [((xxy)x(x^3))xy]xx =$$

$$[((xxy)x(x^3))xy]xx = [(((xxy)xx)xx)xy]xx = [(fxx)xy]xx = [fx(xxy)]xx =$$

$$fx[(xxy)xx] = fxf.$$

A simple calculus can show that; $x^3 = xfx$.

Theorem 2.20.. Let S be a regulary ternary semigroup and $\alpha : S \longrightarrow P$ be a semigroup homomorphism. $\alpha(S)$ is a regular subsemigroup and in particular, if α is an epimorphism then P is also regular.

Proof. The proof is trivial.

3. Greens relations in ternary semigroups

Definition 3.1.. Let S be a ternary semigroup. We define on S the following preorder relations:

$$a \leq_{\mathscr{L}} b \iff a = xyb \text{ for some } x, y \in S.$$

$$a \leq_{\mathscr{R}} b \iff a = bxy \text{ for some } x, y \in S.$$

$$a \leq_{\mathscr{I}} b \iff a = xby \text{ for some } x, y \in S.$$

$$a \leq_{\mathscr{H}} b \iff a \leq_{\mathscr{L}} b \text{ and } a \leq_{\mathscr{R}} b.$$

Proposition 3.1. [13, 14]. Let S be a ternary semigroup.

(1) Let $a \in S$ be an idempotent and b be an element of S. Then

$$b \leq_{\mathscr{R}} a \iff b = aab.$$

$$b \leq_{\mathscr{L}} a \iff b = baa.$$

(2) If $a \leq_{\mathscr{R}} axy$, then $a\mathscr{R}axy$.

(3) If $a \leq_{\mathscr{L}} axy$, then $a\mathscr{L}xya$.

Where $a\mathscr{R}b \iff a \leq_{\mathscr{R}} b$ and $b \leq_{\mathscr{R}} a$ and $\mathscr{L}$ is defined in same sense.

Definition 3.2. Let T be a subset of a ternary semigroup. We say that T is left (resp. right, twosided) (S,S)-invariant or ideal if

$$\forall x,y \in S,\ \forall\ a \in T, xya \in T(resp.axy \in T,(xya \in T \text{ and } axy \in T)).$$

Theorem 3.2. Let T be a Von Neumann regular subset of S.

(1) If T is left $(S,S)-$invariant, then $\mathscr{L}^T = \mathscr{L}^S \cap (T\times T)$.
(2) If T is right $(S,S)-$invariant, then $\mathscr{R}^T = \mathscr{R}^S \cap (T\times T)$.
(3) If T is left $(S,S)-$invariant, then $\mathscr{H}^T = \mathscr{H}^S \cap (T\times T)$.

Greens relations on ternary semigroups are invariant under morphisms.

Proposition 3.3. [13, 14] Let $\varphi : S \longrightarrow T$ be a ternary semigroup morphism and $\mathscr{R}$ be one of the relations $\mathscr{L},\mathscr{R},\mathscr{H},\mathscr{D},\mathscr{I}$. If $a\mathscr{R}^S b$ then $\varphi(a)\mathscr{R}^T\varphi(b)$.

Proposition 3.4. [13, 14]. If a,b are two idempotent elements, the following conditions are equivalent.

(1) $a \leq_{\mathscr{H}} b$,
(2) $aab = a = baa$,
(3) $bab = a$.

Proposition 3.5. [13, 14]. If α is an inverse of a, then

(1) $a = (a\alpha a)(\alpha a\alpha)(a\alpha a)$,
(2) $\alpha = (\alpha a\alpha)(a\alpha a)(\alpha a\alpha)$.

From the previous equalities we can deduce that $(a\alpha a)$ is an inverse of $(\alpha a\alpha)$.

Proposition 3.6. [13, 14]. If $E(S)$ denotes the set of all idempotent elements of S the restriction of the preorder $\leq_{\mathscr{H}}$ to $E(S)$ is an order, called the naturel order on $E(S)$ and denoted "$\leq$".

Definition 3.3. Let S be a ternary semigroup and T be a subsemigroup of S. T is called a $\mathscr{G}-$subsemigroup if

$$\mathscr{L}^T = \mathscr{L}^S \cap (T\times T), \mathscr{R}^T = \mathscr{R}^S \cap (T\times T), \mathscr{H}^T = \mathscr{H}^S \cap (T\times T),$$

$$\mathscr{D}^T = \mathscr{D}^S \cap (T\times T), \mathscr{I}^T = \mathscr{I}^S \cap (T\times T).$$

Lemma 3.1. The relation $\mathscr{L}$ is right $(S,S)-$invariant and the relation $\mathscr{R}$ is left $(S,S)-$invariant.

Proof. The proof is trivial.

Proposition 3.7.(Green's lemma). Let a,b be two $\mathscr{R}-$equivalent elements of a ternary semigroup S. If $a=buv$ and $b=acd$ for some $u,v,c,d\in S^1$, then the map $\varphi : x \longmapsto xuv$ is a bijection from $L(b)$ onto $L(a)$ and the map $\psi : x \longmapsto xcd$ is a bijection from $L(a)$ onto $L(b)$. Further, these bijections are inverse each other and are such, for $\alpha,\beta\in S$:

$$\alpha\mathscr{L}\beta \iff \varphi(\alpha)\mathscr{L}\varphi(\beta) \text{ and } \alpha\mathscr{L}\beta \iff \psi(\alpha)\mathscr{L}\psi(\beta).$$

Proof. Let $n\in L(a)$. Since $\mathscr{L}$ is right $(S,S)-$invariant then $ncd\in L(acd)$. But $n=xya$ so $(ncd)uv=[(xya)cd]uv=[xy(acd)]uv=[xyb]uv=xy(buv)=xya=n$. In the other hand if $m\in L(b)$ using the same argument we can prove that $(muv)cd=m$ so the maps $x \longmapsto xuv$ and $x \longmapsto xcd$ are inverse of each other then they are bijections between the given sets.

Let $\alpha\mathscr{L}\beta$ then $\alpha=xy\beta$ and $\beta=x'y'\alpha$. So $\varphi(\alpha)=\alpha uv=(xy\beta)uv=xy(\beta uv)=xy\varphi(\beta)$ and $\psi(\alpha)=\alpha uv=(xy\beta)uv=xy(\beta uv)=xy\psi(\beta)$, so

$$\alpha\mathscr{L}\beta \implies \varphi(\alpha)_m athcalL\varphi(\beta) \text{ and } \psi(\alpha)\mathscr{L}\psi(\beta).$$

Conversely, suppose that $\varphi(\alpha)\mathscr{L}\varphi(\beta)$. Then $\psi(\varphi(\alpha))\mathscr{L}\psi(\varphi(\beta))$ and so $\alpha\mathscr{L}\beta$. By using the same argument, we prove the other implication.

The next dual version of the proposition is proved similarly.

Proposition 3.8.(Green's lemma). Let a,b be two $\mathscr{L}-$equivalent elements of a ternary semigroup S. If $a=uvb$ and $b=cda$ for some $u,v,c,d\in S^1$, then the map $\varphi : x \longmapsto uvx$ is a bijection from $R(b)$ onto $R(a)$ and the map $\psi : x \longmapsto cdx$ is a bijection from $R(a)$ onto $R(b)$. Further, these bijections preserve the $\mathscr{R}-$classes and are inverse each other, that is, for $\alpha,\beta\in S$,

$$\alpha\mathscr{R}\beta \iff \varphi(\alpha)\mathscr{R}\varphi(\beta) \text{ and } \alpha\mathscr{R}\beta \iff \psi(\alpha)\mathscr{R}\psi(\beta).$$

Proof. The proof is exactly the same as in proposition 3.7, whith an adaptation to the right classes.

Corollary 3.9. If a,b are $\mathscr{H}-$equivalent then, $\forall\ \alpha,\beta\in S$, we have

$$\alpha\mathscr{H}\beta \iff \varphi(\alpha)\mathscr{H}\varphi(\beta) \text{ and } \alpha\mathscr{H}\beta \iff \psi(\alpha)\mathscr{H}\psi(\beta).$$

Proposition 3.10. Let $x, y \in S$. If $R(y) \cap L(x)$ contains an idempotent e then $xey \in R(x) \cap L(y)$.

Proof. If $e \in R(y) \cap L(x)$ then $eey = y$ and $xee = x$ (Hint: $e \in R(y) \implies y = eab \implies eey = ee(eab) = (eee)ab = eab = y$). So;

$e\mathscr{R}y \implies xee\mathscr{R}xey$ and then $x\mathscr{R}xey$.

$e\mathscr{L}x \implies eey\mathscr{L}xey$ and then $y\mathscr{L}xey$.

Finally $xey \in L(y) \cap R(x)$.

Proposition 3.11. Let $e, f \in E(S)$ the set of all idempotents of S. For all $x \in R(e) \cap L(f)$ there exists $y \in R(f) \cap L(e)$ such $xfy = e$ and $yex = f$.

Proof. If $x \in R(e) \cap L(f)$ then $x = eex$ and $x = xff$. There also are $u, v, a, b \in S^1$ such $e = xuv$ and $f = abx$. Let $y = fuv$ then;

$f = abx = ab(eex) = ab((xuv)ex) = ((abx)uv)ex = (fuv)ex = yex$

and

$e = xuv = (xff)uv = xf(fuv) = xfy$.

In the other hand;

$y = fuv$ and $y = yex$ imply that $y \in R(f)$. So

$e = xfy$ and $y = fuv = (abx)uv = ab(xuv) = abe$ imply $y \in L(e)$. Consequently $y \in R(f) \cap L(e)$.

Corollary 3.12. Let e be an idempotent. For all $x \in H(e)$ there exists $y \in H(e)$ such $xey = e = yex$. Then xef and yex are in $E(S)$.

Proof. take $e = f$ in the previous proposition.

The following question, if it has a positive answer, will be equivalent to Lallemant s theorem in the ternary semigroup case.

Question 2. Let S be a regulary ternary semigroup and $\alpha : S \longrightarrow P$ be an onto semigroup homomorphism. If $e \in E(P)$, is $\alpha^{-1}(e) \cap E(S) \neq \emptyset$?

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] C. E. Clark and J. H. Carruth, Generalized Greens theories. Semigroup Forum 20(2) 1980, pp.95-127.

[2] A. H. Clifford and G. B. Preston, The algebraic theory of semigroups, American Mathematical Society, 1961 (volume 1), 1967 (volume 2). Greens relations are introduced in Chapter 2 of the first volume.

[3] Deng, Lun-Zhi, Zeng, Ji-Wen, You, Tai-Jie , Greens relations and regularity for semigroups of transformations that preserve order and a double direction equivalence. Semigroup Forum, Feb 2012, Vol. 84 Issue 1, p.59 .

[4] Gomes, G.M.S., Pin, J.E., Silva, J.E. (2002). Semigroups, algorithms, automata, and languages. Proceedings of workshops held at the International Centre of Mathematics, CIM, Coimbra, Portugal, May, June and July 2001. World Scientific. ISBN 978-981-238- 099-9. Zbl 1005.00031.

[5] J. A. Green, On the structure of semigroups, Annals of Mathematics (second series) 54(1), July 1951, pp.163-172 .

[6] M. P. Grillet, Greens relations in a semiring, Portugal. Math. 29, 1970, pp 181-195.

[7] T.E.Hall,On regular semigroups whose idempotents form a subsemigroup, Bull. Austral. Math. Soc, vol.1, 1969.

[8] T.E.Hall,On orthodox semigroups and uniform and antiuniform bands, Journal of Algebra. 16, 1970.

[9] T. Harju, Lecture Notes on semigroups, University of Turku Finland, 1996.

[10] J. M. Howie, An introduction to semigroup theory, Academic Press, 1976. ISBN 0-12 - 356950-8. An updated version is available as Fundamentals of semigroup theory, Oxford University Press, 1995. ISBN 0-19-851194-9.

[11] J. M. Howie, Semigroups, Past, Present and Future, Proceedings of the International Conference on Algebra and its Applications, 2002.

[12] Huisheng, Zou Dingyu, Almeida, Jorge , Greens Equivalences on Semigroups of Transformations Preserving Order and an Equivalence Relation. Semigroup Forum, Sep/Oct 2005, Vol.71, Issue 2, p241.

[13] R.Kellil,Greens relations on ternary semigroups, Journal of Semigroup Theory and Applications, 2013.

[14] R.Kellil, Some aspects of derivation in ternary semirings, International Journal of Multidisciplinary Sciences and Engineering,Vol.4,Issue 5, June 2013.

[15] Lawson, Mark V. . Finite automata. Chapman and Hall/CRC. ISBN1-58488-255- 7.(2004). Zbl 1086.68074.

[16] Lehmer, D.H, A ternary analogue of abelian groups, Amer jour of Math.599 (1932),329- 338.

[17] Petraq Petro, Greens relations and minimal quasi-ideals in rings,. Comm. Algebra 30(10), 2002, pp.4677-4686.

[18] Pin,J.E, Mathematical Foundations of Automata Theory (Course edition October 2012).

[19] S. Satoh, K. Yama, and M. Tokizawa, Semigroups of order 8, Semigroup Forum 49, 1994, pp.7-29.

[20] M.Yamada, On regular semigroup in which the idempotent form a band, Pacific Journal of Mathematics, vol.33,N.1, 1970.