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EPIMORPHISMS, DOMINIONS AND REGULAR SEMIGROUPS

NOOR ALAM* AND NOOR MOHAMMAD KHAN

Department of Mathematics, Aligarh Muslim University, Aligarh-202002, India

Abstract. We show that a regular semigroups satisfying certain conditions in the containing semigroup is closed. As immediate corollaries, we have got that the special semigroup amalgam $\mathcal{U} = [\{S, S'\}; U; \{i, \alpha \mid U\}]$ within the class of left [right] quasi-normal orthodox semigroups, $\mathcal{R}[\mathcal{L}]$-unipotent semigroups and left[right] Clifford semigroups is embeddable in a left [right] quasi-normal orthodox semigroup, $\mathcal{R}[\mathcal{L}]$-unipotent semigroup and left[right] Clifford semigroup respectively. Finally we have shown that the class of all semigroups satisfying the identity $xyz = xz$ and the class of all semigroups satisfying the identity $xy = xyx[yx = xyx]$ are closed within the class of all semigroups satisfying the identities $xyz = xz$ and $xy = xyx[yx = xyx]$ respectively.

Keywords: Epimorphism, dominion, left[right] regular band, left[right] Clifford semigroup, $\mathcal{R}$-unipotent semigroup, $\mathcal{L}$-unipotent semigroup, left[right] quasi-normal band, closed semigroup, zigzag equations.

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1. Introduction

In [11], Scheiblich has proved, by using zigzag manipulations, that the variety of all normal bands is closed. In [1], this result is generalized to the variety of all left[right] regular bands. Higgins [5] has shown that a generalized inverse subsemigroup of a semigroup satisfying a certain condition is closed in the containing semigroup. In this paper, we extend this result to a regular semigroup satisfying some conditions in the containing semigroup. Next, we prove a similar result about a regular subsemigroup satisfying some conditions in the containing regular semigroup. As immediate corollaries of this result, we have got that

*Corresponding author

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the classes of all left[right] quasi-normal orthodox semigroups and that of all $\mathcal{R}$-unipotent[$\mathcal{L}$-unipotent], left[right] Clifford semigroups are closed within the classes of all left[right] quasi-normal orthodox and that of all $\mathcal{R}$-unipotent[$\mathcal{L}$-unipotent], left[right] Clifford semigroups respectively.

Finally we show that the class of all semigroups satisfying the identity $xyz = xz$, which contains the class of all rectangular bands, and the class of semigroups satisfying an identity $xy = xyx[yx = xyx]$, which contains the class of left[right] regular bands are closed within the class of semigroups satisfying the identities $xyz = xz$ and $xy = xyx[yx = xyx]$ respectively.

2. Preliminaries

Let U be a subsemigroup of a semigroup S. Following Isbell [9], we say that U dominates an element d of S if for every semigroup T and for all homomorphisms $\beta, \gamma : S \to T$, $u\beta = u\gamma$ for all $u \in U$ implies $d\beta = d\gamma$. The set of all elements of S dominated by U is called the *dominion* of U in S, and we denote it by $Dom(U, S)$. It may be easily seen that $Dom(U, S)$ is a subsemigroup of S containing U. Let $\mathcal{C}$ be a class of semigroups. A semigroup U is said to be *$\mathcal{C}$-closed* if $U \in \mathcal{C}$ and for all $S \in \mathcal{C}$ such that U is a subsemigroup of S, $Dom(U, S) = U$.

A morphism $\alpha : A \to B$ in the category $\mathcal{C}$ of semigroups is called an *epimorphism* (epi for short) if for all $C \in \mathcal{C}$ and for all morphisms $\beta, \gamma : B \to C$, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$. It may be easily seen that a morphism $\alpha : S \to T$ is epi if and only if the inclusion mapping $i : S\alpha \to T$ is epi, and an inclusion map $i : U \to S$ is epi if and only if $Dom(U, S) = S$.

A most useful characterization of semigroup dominions is provided by Isbell's Zigzag Theorem.

Result 1.1([9, Theorem 2.3] or [7, Theorem VII.2.13]). Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:

$$d = a_0y_1 = x_1a_1y_1 = x_1a_2y_2 = x_2a_3y_2 = \cdots = x_ma_{2m-1}y_m = x_ma_{2m}, \tag{1}$$

where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $x_i, y_i \in S$ $(i = 1, 2, \ldots, m)$, and

$$a_0 = x_1a_1, \qquad a_{2m-1}y_m = a_{2m},$$

$$a_{2i-1}y_i = a_{2i}y_{i+1}, \qquad x_ia_{2i} = x_{i+1}a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorization is called a *zigzag* in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$. We refer to the equations in Result 1.1, in whatever follows, as *the zigzag equations*.

Let S be a semigroup and let $E(S)$ denotes the set of all idempotents of S. Recall that S is *regular* if for each $a \in S$ there exists $x \in S$ such that $a = axa$. A *Clifford semigroup* is a regular semigroup whose idempotents lie in its center. Generalizing Clifford semigroups among regular semigroups, Zhu et al. [15] introduced the concept of a left Clifford semigroup. According to them S is a *left*[*right*] *Clifford semigroup* if S is regular and $aS \subseteq Sa[Sa \subseteq aS]$ for all $a \in S$. Infact, they gave the following characterization of left Clifford semigroups.

Result 1.2 Let S be an semigroup with a band $E(S)$ of idempotents. Then the following statements on S are equivalent:

(i) S is a left Clifford semigoup;

(ii) $(\forall e \in E(S))\ eS \subseteq Se$;

(iii) $(\forall e \in E(S))\ (\forall a \in S) eae = ea$.

Characterization of right Clifford semigroup may be stated dually.

A *band* (semigroup whose every element is an idempotent) is *rectangular* which satisfies an identity $xyz = xz$. A $\mathcal{R}$-*unipotent* semigroup S is a regular semigroup whose set of idempotents form a left regular band (i.e. $E(S)$ is a subsemigroup satisfying the identity $efe = ef$). $\mathcal{L}$-*unipotent* semigroups are defined dually. Structure theorems for $\mathcal{R}$-unipotent semigroups may be found in [13] and [14]. A *left quasi normal orthodox* semigroup S is a regular semigroup whose idempotents form a left qusi-normal band (i.e. $efg = efeg\ \forall e, f, g \in E(S)$, see [10] for more details). This class of regular semigroups contains both the classes of $\mathcal{R}$-unipotent semigroups and that of generalized inverse semigroups. Dually the class of right quasi-normal orthodox semigroups contains the class of $\mathcal{L}$-unipotent semigroups and that of generalized inverse semigroups.

We require the following well known properties of a left quasi-normal orthodox semigroup. In the following, we shall denote by a', u', etc. as arbitrary inverses of a, u, etc.

Result 1.3 Let S be a left quasi-normal orthodox semigroup. Let $a \in S$ and e be an idempotent of S.

(i) If a' is an inverse of a, then aea' and $a'ea$ are idempotents.

(ii) $(\forall e, f \in E)(\forall a \in S)(\forall a' \in V(a))\ aef = aea'af$.

Property of right quasi-normal orthodox semigroup may be stated dually.

The following result is a part of left-right dual of Theorem 1 in [12].

Result 1.4 Let S be a regular semigroup. Then the following statements are equivalent.

(a) S is $\mathcal{R}$-unipotent;

(b) $(\forall e \in E)(\forall a \in S)(\forall a' \in V(a))\ ae = aea'a$.

Result 1.5([7, Theorem VII.2.3]). Let U be a subsemigroup of a semigroup S. Let S' be a semigroup disjoint from S and let $\alpha : S \to S'$ be an isomorphism. Let $P = S *_U S'$, be the free product of the amalgam

$$\mathcal{U} = [\{S, S'\};\, U;\, \{i, \alpha \mid U\}],$$

where i is the inclusion mapping of U into S, and let μ, μ' be the natural monomorphisms from S, S' respectively into P. Then

$$(S\mu \cap S'\mu')\mu^{-1} = Dom(U, S).$$

The above stated amalgam $\mathcal{U}$ is called *special semigroup amalgam*.

3. Main results

The following theorem extends [5, Proposition 3] to regular semigroups.

Theorem 3.1. *Let S be a semigroup and U be a regular subsemigroup of S. If $se = ses$ $\forall s \in S$ and $\forall e \in E(U)$, then U is closed in S .*

Proof. Take any $d \in Dom(U, S) \setminus U$. Then, by Result 1.1, we may let (1) be a zigzag of minimal length m in S over U with value d and spine $a_0, a_1, a_2, \ldots, a_{2m}$. Now

$$\begin{aligned}
d &= a_0 y_1 \\
&= x_1 a_1 y_1 && \text{(by zigzag equations)} \\
&= x_1 a_1 a_1' a_1 y_1 && \text{(as } U \text{ is a regular semigroup)} \\
&= x_1 a_1 a_1' a_2 y_2 && \text{(by zigzag equations)} \\
&= x_1 a_1 a_1' x_1 a_2 y_2 && \text{(as } x_1 \in S \text{ and } a_1 a_1' \in E(U)) \\
&= x_1 a_1 a_1' x_2 a_3 y_2 && \text{(by zigzag equations)} \\
&= x_1 a_1 a_1' x_2 a_3 a_3' a_3 y_2 && \text{(as } U \text{ is a regular semigroup)} \\
&= x_1 a_1 a_1' x_1 a_2 a_3' a_3 y_2 && \text{(by the zigzag equations)}
\end{aligned}$$

$$
\begin{aligned}
&= x_1a_1a_1'a_2a_3'a_3y_2 && (\text{as } x_1 \in S \text{ and } a_1a_1' \in E(U))\\
&= a_0a_1'a_2a_3'(a_3y_2) && (\text{by the zigzag equations})\\
&\vdots\\
&= a_0a_1'a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
&= x_1a_1a_1'a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && (\text{by zigzag equations})\\
&= x_1a_1a_1'x_1a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && (\text{as } x_1 \in S \text{ and } a_1a_1' \in E(U))\\
&= x_1a_1a_1'x_2a_3a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && (\text{by zigzag equations})\\
&= x_1a_1a_1'x_2a_3a_3'x_2a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && (\text{as } x_2 \in S \text{ and } a_3a_3' \in E(U))\\
&= x_1a_1a_1'x_2a_3a_3'x_3a_5a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && (\text{by zigzag equations})\\
&= x_1a_1a_1'x_2a_3a_3'x_3a_5a_5'x_3a_6\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && (\text{as } x_3 \in S \text{ and } a_5a_5' \in E(U))\\
&\vdots\\
&= x_1a_1a_1'x_2a_3a_3'x_3a_5a_5'\cdots x_{m-2}a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
&= x_1a_1a_1'x_2a_3a_3'x_3a_5a_5'\cdots x_{m-1}a_{2m-3}a_{2m-3}'(a_{2m-2}y_m) && (\text{by zigzag equations})\\
&= x_1a_1a_1'x_2a_3a_3'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}a_{2m-2}y_m && (\text{as } x_{m-1} \in S \text{ and } a_{2m-3}a_{2m-3}' \in E(U))\\
&= x_1a_1a_1'x_2a_3a_3'x_3a_5a_5'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_ma_{2m-1}y_m && (\text{by zigzag equations})\\
&= x_1a_1a_1'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_ma_{2m-1}a_{2m-1}'a_{2m-1}y_m && (\text{as } U \text{ is regular})\\
&= x_1a_1a_1'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}a_{2m-2}a_{2m-1}'a_{2m} && (\text{by zigzag equations})\\
&= x_1a_1a_1'\cdots x_{m-1}a_{2m-3}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m} && (\text{as } x_{m-1} \in S \text{ and } a_{2m-3}a_{2m-3}' \in E(U))\\
&\vdots\\
&= x_1a_1a_1'x_1a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m}
\end{aligned}
$$

$$= \quad x_1a_1a_1'a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m} \qquad \text{(as } x_1 \in S \text{ and } a_1a_1' \in E(U))$$

$$= \quad a_0a_1'a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m} \in U \qquad \text{(by zigzag equations)}$$

$\Rightarrow$ $d \in U$. Hence $Dom(U,S) = U$. □

Dually, we may prove the following:

Theorem 3.2. *Let S be a semigroup and U be a regular subsemigroup of S. If $es = ses$ $\forall s \in S$ and $\forall e \in E(U)$, then U is closed in S.* □

The following theorem immediately shows that the class of all left[right] quasi-normal orthodox semigroups and the class of all $\mathcal{R}$-unipotent[$\mathcal{L}$-unipotent](left[right] Clifford) semigroups are closed within the class of all left[right] quasi-normal orthodox semigroups and the class of all $\mathcal{R}$-unipotent[$\mathcal{L}$-unipotent] (left[right] Clifford) semigroups respectively.

Theorem 3.3. *Let S be any regular semigroup and U be any regular subsemigroup of S. If $efg = efeg$ $\forall e \in E(S)$ and $\forall f, g \in E(U)$, then U is closed in S .*

Proof. Suppose $d \in Dom(U,S) \setminus U$. Then, by Result 1.1, there exist a zigzag (1) in S over U with value d of minimal length m and spine $a_0, a_1, a_2, \ldots, a_{2m}$. Now

$$\begin{aligned}
d &= a_0y_1 \\
&= x_1a_1y_1 && \text{(by zigzag equations)} \\
&= x_1a_1a_1'a_1y_1 && \text{(as } U \text{ is a regular semigroup)} \\
&= x_1a_1a_1'a_2y_2 && \text{(by zigzag equations)} \\
&= x_1a_1a_1'a_2a_2'a_2y_2 && \text{(as } U \text{ is a regular semigroup)} \\
&= x_1x_1'x_1a_1a_1'a_2a_2'a_2y_2 && \text{(as } S \text{ is a regular semigroup)} \\
&= x_1a_1a_1'x_1'x_1a_2a_2'a_2y_2 && \text{(as } x_1'x_1 \in E(S) \,\&\, a_1a_1', a_2a_2' \in E(U)) \\
&= x_1a_1a_1'x_1'x_1a_2y_2 && \text{(as } U \text{ is a regular semigroup)} \\
&= x_1a_1a_1'x_1'x_2a_3y_2 && \text{(by zigzag equations)}
\end{aligned}$$

$$
\begin{aligned}
&= x_1a_1a_1'x_1'x_2a_3a_3'a_3y_2 && \text{(as } U \text{ is a regular semigroup)}\\
&= x_1a_1a_1'x_1'x_1a_2a_3'a_3y_2 && \text{(by the zigzag equations)}\\
&= x_1a_1a_1'x_1'x_1a_2a_2'a_2a_3'a_3y_2 && \text{(as } U \text{ is a regular semigroup)}\\
&= x_1a_1a_1'a_2a_2'a_2a_3'a_3y_2 && (\text{as } x_1'x_1 \in E(S)\ \&\ a_1a_1', a_2a_2' \in E(U))\\
&= x_1a_1a_1'a_2a_3'a_3y_2 && \text{(as } U \text{ is a regular semigroup)}\\
&= a_0a_1'a_2a_3'(a_3y_2) && \text{(by the zigzag equations)}\\
&\ \vdots\\
&= a_0a_1'a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
&= x_1a_1a_1'a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && \text{(by zigzag equations)}\\
&= x_1a_1a_1'x_1'x_1a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
& \qquad (\text{as } x_1'x_1 \in E(S)\ \&\ a_1a_1', a_2a_2' \in E(U))\\
&= x_1a_1a_1'x_1'x_2a_3a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && \text{(by zigzag equations)}\\
&= x_1a_1a_1'x_1'x_2a_3a_3'x_2'x_2a_4a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
& \qquad (\text{as } x_2'x_2 \in E(S)\ \&\ a_3a_3', a_4a_4' \in E(U))\\
&= x_1a_1a_1'x_1'x_2a_3a_3'x_2'x_3a_5a_5'\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1}) && \text{(by zigzag equations)}\\
&= x_1a_1a_1'x_1'x_2a_3a_3'x_2'x_3a_5a_5'x_3'x_3a_6\cdots a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
& \qquad (\text{as } x_3'x_3 \in E(S)\ \&\ a_5a_5', a_6a_6' \in E(U))\\
&\ \vdots\\
&= x_1a_1a_1'x_1'x_2a_3a_3'\cdots x_{m-2}a_{2m-4}a_{2m-3}'(a_{2m-3}y_{m-1})\\
&= x_1a_1a_1'x_1'x_2a_3a_3'\cdots x_{m-1}a_{2m-3}a_{2m-3}'(a_{2m-2}y_m) && \text{(by zigzag equations)}\\
&= x_1a_1a_1'x_1'x_2a_3a_3'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}'x_{m-1}a_{2m-2}y_m\\
& \qquad (\text{as } x_{m-1}'x_{m-1} \in E(S)\ \&\ a_{2m-3}a_{2m-3}', a_{2m-2}a_{2m-2}' \in E(U))
\end{aligned}
$$

$$= x_1a_1a_1'x_2a_3a_3'x_3a_5a_5'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}'x_ma_{2m-1}y_m \quad \text{(by zigzag equations)}$$

$$= x_1a_1a_1'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}'x_ma_{2m-1}a_{2m-1}'a_{2m-1}y_m \quad \text{(as } U \text{ is regular)}$$

$$= x_1a_1a_1'\cdots x_{m-1}a_{2m-3}a_{2m-3}'x_{m-1}'x_{m-1}a_{2m-2}a_{2m-1}'a_{2m} \quad \text{(by zigzag equations)}$$

$$= x_1a_1a_1'\cdots x_{m-1}a_{2m-3}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m}$$

$$\text{(as } x_{m-1}'x_{m-1}\in E(S)\ \&\ a_{2m-3}a_{2m-3}', a_{2m-2}a_{2m-2}'\in E(U))$$

$\vdots$

$$= x_1a_1a_1'x_1'x_1a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m}$$

$$= x_1a_1a_1'a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m} \quad \text{(as } x_1'x_1\in E(S)\ \&\ a_1a_1', a_2a_2'\in E(U))$$

$$= a_0a_1'a_2a_3'a_4a_5'\cdots a_{2m-4}a_{2m-3}'a_{2m-2}a_{2m-1}'a_{2m}\in U \quad \text{(by zigzag equations)}$$

$\Rightarrow$ $d\in U$. Hence $Dom(U,S)=U$. □

Following corollaries are easy consequences of Theorem 2.3, Result 1.2, Result 1.3 and Result 1.4.

Corollary 3.4. *If U is a left[right] quasi-normal orthodox subsemigroup of a left[right] quasi-normal orthodox semigroup S, then U is closed in S .*

Corollary 3.5. *If U is a $\mathcal{R}$-unipotent[$\mathcal{L}$-unipotent] subsemigroup of a $\mathcal{R}$-unipotent[$\mathcal{L}$-unipotent] semigroup S, then U is closed in S .*

Corollary 3.6. *If U is a left[right] Clifford subsemigroup of a left[right] Clifford semigroup S, then U is closed in S .*

Above corollaries may also be stated as follows:

Corollary 3.7. *Let U be a left [right] quasi-normal orthodox ($\mathcal{R}[\mathcal{L}]$-unipotent, left[right] Clifford) subsemigroup of a left [right] quasi-normal orthodox ($\mathcal{R}[\mathcal{L}]$-unipotent, left[right] Clifford) semigroup S. Let S' be a left [right] quasi-normal orthodox ($\mathcal{R}[\mathcal{L}]$-unipotent, left[right] Clifford) semigroup disjoint from S and let $\alpha : S \to S'$ be an isomorphism. Let $P = S *_U S'$, the free product of the amalgam $\mathcal{U} = [\{S, S'\}; U; \{i, \alpha \mid U\}]$, where i is the inclusion mapping of U into S, and let μ, μ' be the natural monomorphisms from S, S' respectively into P, then $S\mu \cap S'\mu' = U\mu$. Therefore the amalgam $\mathcal{U}$ is embeddable in a left[right] quasi-normal orthodox ($\mathcal{R}[\mathcal{L}]$-unipotent, left[right] Clifford) semigroup .*

Theorem 3.8. *Let S be a semigroup satisfying an identity $xyz = xz$ and U be a subsemigroup of S satisfying an identity $xyz = xz$, then U is closed in S .*

Proof. Let us take $d \in Dom(U,S) \setminus U$. Then, by Result 1.1, we may let (1) be a zigzag in S over U with value d of minimal length m and spine $a_0, a_1, a_2, \ldots, a_{2m}$. Now

$$\begin{aligned}
d &= a_0 y_1 \\
&= x_1 a_1 y_1 && \text{(by zigzag equations)} \\
&= x_1 a_2 y_2 && \text{(by zigzag equations)} \\
&= x_1 a_1 a_2 y_2 && (\text{since } x_1, a_1, a_2 \in S) \\
&= x_1 a_1 a_2 a_3 y_2 && (\text{since } a_2, a_3, y_2 \in S) \\
&= a_0 a_2 a_3 y_2 && \text{(by zigzag equations)} \\
&= \prod_{i=0}^{1} a_{2i} (a_3 y_2) \\
&\vdots \\
&= \prod_{i=0}^{m-2} a_{2i} (a_{2m-3} y_{m-1}) \\
&= a_0 a_2 \cdots a_{2m-4} (a_{2m-2} y_m) && \text{(by zigzag equations)} \\
&= a_0 a_2 \cdots a_{2m-4} a_{2m-2} y_m \\
&= a_0 a_2 \cdots a_{2m-4} a_{2m-2} a_{2m-1} y_m && (\text{since } a_{2m-2}, a_{2m-1}, y_m \in S) \\
&= a_0 a_2 \cdots a_{2m-4} a_{2m-2} a_{2m} && \text{(by zigzag equations)} \\
&= \prod_{i=0}^{m} a_{2i} \in U.
\end{aligned}$$

$\Rightarrow d \in U$. Hence $Dom(U,S) = U$. □

Theorem 3.9. *Let U and S be a semigroups satisfying the identity $xy = xyx$ and with U a subsemigroup of S. Then U is closed in S .*

In order to prove our result, we need the following useful remark which may be well known.

Remark 3.10 Let S be a semigroup satisfying the identity $xy = xyx$. If $E(S)$ be the set of idempotents of S, then $S^2 \subseteq E(S)$.

Proof. Let us take $d \in Dom(U,S) \setminus U$. Then, by Result 1.1, we may let (1) be a zigzag in S over U with value d of minimal length m and spine $a_0, a_1, a_2, \ldots, a_{2m}$. Now

$$
\begin{aligned}
d &= a_0y_1 \\
&= x_1a_1y_1 && \text{(by zigzag equations)} \\
&= x_1a_1x_1a_1y_1 && \text{(since } x_1a_1 \in E(S)) \\
&= x_1a_1x_1a_2y_2 && \text{(by zigzag equations)} \\
&= x_1a_1x_2a_3y_2 && \text{(by zigzag equations)} \\
&= x_1a_1x_2a_3x_2a_3y_2 && \text{(since } x_2a_3 \in E(S)) \\
&= x_1a_1x_1a_2a_3y_2 && \text{(by zigzag equations)} \\
&= x_1a_1a_2a_3y_2 && \text{(since } x_1, a_1 \in S) \\
&= a_0a_2a_3y_2 && \text{(by zigzag equations)} \\
&= \prod_{i=0}^{1} a_{2i}(a_3y_2) \\
&\ \vdots \\
&= \prod_{i=0}^{m-2} a_{2i}(a_{2m-3}y_{m-1}) \\
&= a_0a_2\cdots a_{2m-4}(a_{2m-2}y_m) && \text{(by zigzag equations)} \\
&= x_1a_1a_2\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)} \\
&= x_1a_1x_1a_2\cdots a_{2m-4}a_{2m-2}y_m && \text{(since } x_1, a_1 \in S)
\end{aligned}
$$

$$
\begin{aligned}
&= x_1a_1x_2a_3a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(by zigzag equations)}\\
&= x_1a_1x_2a_3x_2a_4\cdots a_{2m-4}a_{2m-2}y_m && \text{(since } x_2, a_3 \in S)\\
&\vdots\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-2}a_{2m-4}a_{2m-2}y_m\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}a_{2m-2}y_m && \text{(by zigzag equations)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_{m-1}a_{2m-2}y_m && \text{(since } x_{m-1}, a_{2m-3} \in S)\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_ma_{2m-1}y_m && \text{(by zigzag equations)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_ma_{2m-1}x_ma_{2m-1}y_m && \text{(since } x_ma_{2m-1} \in E(S))\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_ma_{2m-1}a_{2m-1}y_m && \text{(since } x_m, a_{2m-1} \in S)\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}x_{m-1}a_{2m-2}a_{2m} && \text{(by zigzag equations)}\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-1}a_{2m-3}a_{2m-2}a_{2m} && \text{(since } x_{m-1}, a_{2m-3} \in S)\\
&= x_1a_1x_2a_3x_2a_4\cdots x_{m-2}a_{2m-4}a_{2m-2}a_{2m} && \text{(by zigzag equations)}\\
&\vdots\\
&= x_1a_1x_2a_3x_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m}\\
&= x_1a_1x_2a_3a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(since } x_2, a_3 \in S)\\
&= x_1a_1x_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by zigzag equations)}\\
&= x_1a_1a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(since } x_1, a_1 \in S)\\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}a_{2m} && \text{(by zigzag equations)}\\
&= \prod_{i=0}^{m} a_{2i} \in U
\end{aligned}
$$

$\Rightarrow d \in U$. Hence $Dom(U, S) = U$. □

Dually, we may prove the following:

Theorem 3.11. *Let U and S be a semigroups satisfying the identity $yx = xyx$ and with U a subsemigroup of S. Then U is closed in S .* □

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