Available online at http://scik.org
J. Semigroup Theory Appl. 2016, 2016:3
ISSN: 2051-2937

PROPER COVERS FOR $\tilde{\mathscr{L}}^U$-AMPLE TYPE B SEMIGROUPS

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Abstract. A $\tilde{\mathscr{L}}^U$-semiadequate semigroup is a semigroup whose projections commute and in which each $\tilde{\mathscr{L}}^U$-class contains a projection. Let (S,U) be a $\tilde{\mathscr{L}}^U$-semiadequate semigroup, where U is the set of projections of S. It is the fact that each $\tilde{\mathscr{L}}^U$-class of (S,U) contains a unique projection. For an element a of (S,U), the projection in the $\tilde{\mathscr{L}}^U$-class containing a is denoted by a^*. Let 1 be an identity of S^1, $U^1 = U \cup \{1\}$. If (S,U) satisfying: (1) for all e, f in U^1 and all elements a in (S,U), $(efa)^* = (ea)^*(fa)^*$; (2) for all elements a in (S,U) and all projections $e \leq a^*$, there is an element f in U^1 such that $e = (fa)^*$, then we say that (S,U) is a $\tilde{\mathscr{L}}^U$-ample type B semigroup. In this paper, we introduce the concept of a proper cover of a $\tilde{\mathscr{L}}^U$-ample type B semigroup and prove that any proper cover for a $\tilde{\mathscr{L}}^U$-ample type B semigroup is a proper cover over a monoid. A structure theorem of proper covers for $\tilde{\mathscr{L}}^U$-ample type B semigroups is obtained. This theorem generalizes the result of Li-Wang for right type B semigroups.

Keywords: $\tilde{\mathscr{L}}^U$-ample type B semigroup; proper cover; monoid.

2010 AMS Subject Classification: 20M10.

1. Introduction

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Received December 9, 2015

Let S be a semigroup and $E(S)$ the set of all idempotents of S. Consider a nonempty subset $U \subseteq E(S)$, namely, the set of projections of S. A relation $\tilde{\mathscr{L}}^U$ on S is defined by the rule that $a\tilde{\mathscr{L}}^U b$ if and only if a and b have the same set of right identities in U, that is $U_a^r = U_b^r$, where $U_a^r = \{x \in U | ax = a\}$. The relation $\tilde{\mathscr{R}}^U$ is defined dually. It can be easily verified that $\mathscr{L} \subseteq \mathscr{L}^* \subseteq \tilde{\mathscr{L}}^U$ and $\mathscr{R} \subseteq \mathscr{R}^* \subseteq \tilde{\mathscr{R}}^U$. Clearly, the relation $\tilde{\mathscr{L}}^U$ on S is a natural generalization of the well known Green's relation $\mathscr{L}$ and also the Green's star relation $\mathscr{L}^*$, adopted by Fountain [1], in studying abundant semigroups.

Recall that a semigroup is said to be $\tilde{\mathscr{L}}^U$-semiabundant if every $\tilde{\mathscr{L}}^U$-class contains a projection. A $\tilde{\mathscr{L}}^U$-semiadequate semigroup is $\tilde{\mathscr{L}}^U$-semiabundant and projections commute. Let (S,U) be a $\tilde{\mathscr{L}}^U$-semiadequate semigroup. It is easy to see that each $\tilde{\mathscr{L}}^U$-class of (S,U) contains a unique projection. For an element a of (S,U), the projection in the $\tilde{\mathscr{L}}^U$-class containing a is denoted by a^*.

Right type A semigroups and right type B semigroups were defined and studied by Fountain [2]. Let T be a right type B semigroup and μ_L be the largest congruence contained in $\mathscr{L}^*$, then T/μ_L is a right type A semigroup. But some examples in [2] show that they are two different kinds of semigroups.

Unlike right (left) type B semigroups and type B semigroups, right (left) type A semigroups and type A semigroups have been extensively studied by many scholars, such as Armstrong [5], Guo-Xie [6], Guo-Guo [7] and many others. Recently, Li-Wang in [10] give a structure theorem of proper covers for right type B semigroups. Li-Wang shows that any proper cover for a right type B semigroup is a proper cover over a left cancellative monoid.

Similar to the definition of right type A semigroups and right type B semigroups in [2]. Now, we define two certain classes of $\tilde{\mathscr{L}}^U$-semiadequate semigroups. For a $\tilde{\mathscr{L}}^U$-semiadequate semigroup (S,U), let 1 be an identity of S^1, $U^1 = U \cup \{1\}$. A $\tilde{\mathscr{L}}^U$-semiadequate semigroup (S,U) is called $\tilde{\mathscr{L}}^U$-ample type A if $ea = a(ea)^*$ for all elements a in S and all projections e in U. A $\tilde{\mathscr{L}}^U$-semiadequate semigroup (S,U) is called $\tilde{\mathscr{L}}^U$-ample type B if it satisfies: (1) for all e, f in U^1 and all elements a in S, $(efa)^* = (ea)^*(fa)^*$; (2) for all elements a in (S,U) and all projections $e \leq a^*$, there is an element f in U^1 such that $e = (fa)^*$. Since the relation $\tilde{\mathscr{L}}^U$ is a natural generalization of the Green's star relation $\mathscr{L}^*$, the $\tilde{\mathscr{L}}^U$-ample type A (B) semigroups can be think as a generalization of right type A (B) semigroups.

In this paper, we concentrate on $\tilde{\mathscr{L}}^U$-ample type B semigroups. First, we introduce the concept of a proper cover of a $\tilde{\mathscr{L}}^U$-ample type B semigroup and then prove that any proper cover for a $\tilde{\mathscr{L}}^U$-ample type B semigroup is a proper cover over a monoid. A structure theorem of proper covers for $\tilde{\mathscr{L}}^U$-ample type B semigroups is obtained. This theorem generalizes the result of Li-Wang for right type B semigroups. We adopt the terminology and notations of [11] and [12].

2. Preliminaries

The object in this section is to introduce the concept of proper covers for $\tilde{\mathscr{L}}^U$-ample type B semigroups. Throughout this paper, a $\tilde{\mathscr{L}}^U$-semiabundant semigroups is denoted by (S,U) unless it is specified otherwise. Before starting our approach, we recall some terminology, notations and results.

From [12], the following lemma gives a basic property of the relation $\tilde{\mathscr{L}}^U$ on a $\tilde{\mathscr{L}}^U$-semiabundant semigroups (S,U).

Lemma 2.1. [12] *Let (S,U) be a $\tilde{\mathscr{L}}^U$-semiabundant semigroup and e be an element of U, then the following are equivalent for $a \in (S,U)$:*

(1) *$a\tilde{\mathscr{L}}^U e$;*

(2) *$ae = a$ and for all $x \in U$, $ax = a$ implies $ex = e$.*

In a $\tilde{\mathscr{L}}^U$-semiabundant semigroups (S,U), each $\tilde{\mathscr{L}}^U$-class contains some projections of U. A $\tilde{\mathscr{L}}^U$-semiadequate semigroup is $\tilde{\mathscr{L}}^U$-semiabundant whose projuctions commute.

Lemma 2.2. *Let (S,U) be a $\tilde{\mathscr{L}}^U$-semiadequate semigroup and e,f be elements of U. If $e\tilde{\mathscr{L}}^U f$, then $e = f$.*

Proof. If $e\tilde{\mathscr{L}}^U f$, by Lemma 2.1, we have $e = ef = fe = f$. □

From the Lemma 2.2 we see that if (S,U) is a $\tilde{\mathscr{L}}^U$-semiadequate semigroup, then each $\tilde{\mathscr{L}}^U$-class of (S,U) contains a unique projection. For an element a of such a semigroup, the projection in the $\tilde{\mathscr{L}}^U$-class containing a is denoted by a^*.

A $\tilde{\mathscr{L}}^U$-semiadequate semigroup (S,U) is called $\tilde{\mathscr{L}}^U$-ample type A if $ea = a(ea)^*$ for all elements a in (S,U) and all projections e in U. For a $\tilde{\mathscr{L}}^U$-semiadequate semigroup (S,U), let 1 be an identity of S^1, $U^1 = U \cup \{1\}$. A $\tilde{\mathscr{L}}^U$-semiadequate semigroup (S,U) is called $\tilde{\mathscr{L}}^U$-ample type B if it satisfies:

(B1) for all e, f in U^1 and all elements a in (S,U), $(efa)^* = (ea)^*(fa)^*$;

(B2) for all elements a in (S,U) and all idempotents $e \leq a^*$, there is an element f in U^1 such that $e = (fa)^*$.

Remark. 1. Dually, we can define the $\tilde{\mathscr{R}}^U$-semiabundant semigroups, the $\tilde{\mathscr{R}}^U$- semiadequate semigroups and the $\tilde{\mathscr{R}}^U$-ample type A (B) semigroups.

2. Since the Green's relation $\tilde{\mathscr{L}}^U$ is a natural generalization of the Green's star relation $\mathscr{L}^*$, the $\tilde{\mathscr{L}}^U$-ample type A (B) semigroups can be think as a generalization of right type A (B) semigroups.

Next, we introduce a congruence σ on a $\tilde{\mathscr{L}}^U$-ample type B semigroups (S,U) which has the property that $(S,U)/\sigma$ is a monoid and $U \subseteq (1_{(S,U)/\sigma})(\sigma^\natural)^{-1}$.

Lemma 2.3. *Let (S,U) be a $\tilde{\mathscr{L}}^U$-ample type B semigroup, a relation σ on (S,U) is given by the rule*

$$a\sigma b \Leftrightarrow \text{ exist } e \in U \text{ such that } ea = eb.$$

Then σ is a congruence, $(S,U)/\sigma$ is monoid and $U \subseteq (1_{(S,U)/\sigma})(\sigma^\natural)^{-1}$. If τ is a monoid congruence on (S,U) such that $U \subseteq (1_{(S,U)/\tau})(\tau^\natural)^{-1}$, then $\sigma \subseteq \tau$. In particular, σ is the minimum monoid congruence on (S,U) such that $U \subseteq (1_{(S,U)/\sigma})(\sigma^\natural)^{-1}$.

Proof. It is clear that σ is an equivalence and right compatible.

[Since $ea = ea$ for all $a \in (S,U)$ and $e \in U$, we have $a\sigma a$. Hence σ is reflexive.

If $a\sigma b$, then exist $e \in U$ such that $ea = eb$. Also, we have $eb = ea$ and so $b\sigma a$. Hence σ is symmetric.

If $a\sigma b$ and $b\sigma c$, then exist $e, f \in U$ such that $ea = eb$ and $fb = fc$. Also, we have $fea = feb$ and $efb = efc$. Since $ef = fe$, we have $fea = feb = efb = efc = fec$ and so $a\sigma c$. Hence σ is transitive.

Now, we have already proved that σ is an equivalence. Let $a,b,c \in (S,U)$ and $a\sigma b$, then

$$\begin{aligned} a\sigma b &\Rightarrow \textit{ exist } e \in U \textit{ such that } ea = eb \\ &\Rightarrow eac = ebc \\ &\Rightarrow ac\sigma bc. \end{aligned}$$

Hence σ is right compatible as required.]

To prove that σ is a congruence, it remain to show that σ is left compatible. Let $a,b,c \in S$ and $a\sigma b$, then

$$\begin{aligned} a\sigma b &\Rightarrow \textit{ exist } e \in U \textit{ such that } ea = eb \\ &\Rightarrow c^*ea = c^*eb \\ &\Rightarrow (fc)^*a = (fc)^*b \quad ((B2)) \\ &\Rightarrow fc.(fc)^*a = fc.(fc)^*b \\ &\Rightarrow fca = fcb \quad (\textit{since } fc\tilde{\mathscr{L}}^U(fc)^*) \\ &\Rightarrow ca\sigma cb. \end{aligned}$$

Hence σ is left compatible as required. Since projections commute, we have $e\sigma f$ for any $e,f \in U$. Let $e \in U$, we claim that $e\sigma$ is an identity of $(S,U)/\sigma$. This is because for any $a\sigma \in (S,U)/\sigma$, we have

$$a\sigma.e\sigma = a\sigma.a^*\sigma = (aa^*)\sigma = a\sigma,$$

and

$$ea = ea \Rightarrow e.ea = e.a \Rightarrow ea\sigma a \Rightarrow ea\sigma = a\sigma \Rightarrow e\sigma.a\sigma = a\sigma.$$

Hence $U \subseteq (1_{(S,U)/\sigma})(\sigma^\natural)^{-1}$. Up to now, we have already established that σ is a congruence, $(S,U)/\sigma$ is a monoid and $U \subseteq (1_{(S,U)/\sigma})(\sigma^\natural)^{-1}$. Now, it remain to show that σ is the least monoid congruence such that $U \subseteq (1_{(S,U)/\sigma})(\sigma^\natural)^{-1}$. Let τ be a monoid congruence with $U \subseteq (1_{(S,U)/\tau})(\tau^\natural)^{-1}$. It is easy to see that for any $e \in U$, $e\tau = 1_{(S,U)/\tau}$. [Since $U \subseteq (1_{(S,U)/\tau})(\tau^\natural)^{-1}$, we have $e\tau = 1_{(S,U)/\tau}$]

Now

$$
\begin{aligned}
a\sigma b &\Rightarrow \ exist\ f \in U\ such\ that\ fa = fb \\
&\Rightarrow \ f\tau.a\tau = f\tau.b\tau \\
&\Rightarrow \ a\tau = b\tau \quad (since\ f\tau = 1_{(S,U)/\tau}) \\
&\Rightarrow \ a\tau b.
\end{aligned}
$$

Hence $\sigma \subseteq \tau$ and so σ is the minimum monoid congruence on (S,U) such that $U \subseteq (1_{(S,U)/\sigma})(\sigma^{\natural})^{-1}$ as required. □

Let (S,U) and (T,V) are two $\tilde{\mathscr{L}}^U$-semiabundant semigroups. Similar to the definition of $\mathscr{L}^*$-homomorphism in [3], a homomorphism ϕ from (S,U) to (T,V) is called $\tilde{\mathscr{L}}^U$-homomorphism if for all $a,b \in S, a\phi = b\phi$ implies $a\tilde{\mathscr{L}}^U b$ and $\phi|_U : U \longrightarrow V$.

Definition 2.1. *Let (T,V) be a $\tilde{\mathscr{L}}^U$-ample type B semigroup, some definitions on (T,V) are as follow:*

$(D1)$ if $\tilde{\mathscr{L}}^U \cap \sigma = 1_{(T,V)}$, then we call (T,V) is proper;

$(D2)$ if (T,V) is proper, ϕ is a $\tilde{\mathscr{L}}^U$-homomorphism from (T,V) onto a $\tilde{\mathscr{L}}^U$-ample type B semigroup (S,U) and for any $e \in U$, there exist $f \in V$ such that $f\phi = e$, then we call (T,V) is a proper cover for (S,U);

$(D3)$ if (T,V) is a proper cover for (S,U), M is a monoid, $(T,V)/\sigma \cong M$ and $V \subseteq 1_M\alpha^{-1}(\sigma^{\natural})^{-1}$, where α is an isomorphism from $(T,V)/\sigma$ onto M, then we call (T,V) is a proper cover for (S,U) (over M).

Example 2.1. *Let N denote the set of natural numbers and put $I = N \times N$. On $S = N \cup I$ define an operation $\circ$ as follows:*

$$
\begin{aligned}
&for\ m,n,h,j \in \ N, \\
m \circ n &= \ m+n \\
m \circ (h,k) &= \ (m+h,k) \\
(h,k) \circ m &= \ (h,k+m) \\
(h,k) \circ (m,n) &= \ (h,k+m+n).
\end{aligned}
$$

It is readily verified that $\circ$ is associative, that the idempotents of S are $0,(0,0)$. Let $U = \{0,(0,0)\}$, it is not difficult to check that the $\tilde{\mathscr{L}}^U$-classes of (S,U) are N and I so that (S,U)

is $\tilde{\mathscr{L}}^U$-semiadequate. It is not hard to show that (S,U) is $\tilde{\mathscr{L}}^U$-ample type B semigroup but not proper.

Example 2.2. *Let N denote the set of natural numbers and put $I = N \times N$. On $S = N \cup I$ define an operation $\circ$ as follows:*

$$\begin{aligned} for\, m,n,h,j \in&\ N, \\ m \circ n =&\ m+n \\ m \circ (h,k) =&\ (m+h,k) \\ (h,k) \circ m =&\ (h+m,k) \\ (h,k) \circ (m,n) =&\ (h+m,k+n). \end{aligned}$$

It is readily verified that $\circ$ is associative, that the idempotents of S are $0,(0,0)$. Let $U = \{0,(0,0)\}$, it is not difficult to check that the $\tilde{\mathscr{L}}^U$-classes of (S,U) are N and I so that (S,U) is $\tilde{\mathscr{L}}^U$-semiadequate. It is not hard to show that (S,U) is $\tilde{\mathscr{L}}^U$-ample type B semigroup and proper.

From [11], a subset A of a semigroup S is called left unitary if for all $a \in A$ and $s \in S$, $as \in A$ implies $s \in A$. Dually, we can define right unitary. S is called unitary if it is both left unitary and right unitary.

Lemma 2.4. *Let (S,U) be a $\tilde{\mathscr{L}}^U$-ample type B semigroup, if (S,U) is proper, then it is U-unitary.*

Proof. Let $e \in U$ and $a \in (S,U)$. On the one hand, if $ea \in U$, then

$$eae.a = ea = eaa^* = e.eaa^* = eae.a^*.$$

It is clear that $eae \in U$ and so $a\sigma a^*$. Then we have $a(\tilde{\mathscr{L}}^U \cap \sigma)a^*$. Since (S,U) is proper, we have $\tilde{\mathscr{L}}^U \cap \sigma = 1_{(S,U)}$ and so $a = a^* \in U$.

On the other hand, if $ae \in U$, then $a = aa^* \sigma ae \sigma a^*$. Hence $a(\tilde{\mathscr{L}}^U \cap \sigma)a^*$. Since (S,U) is proper, we have $\tilde{\mathscr{L}}^U \cap \sigma = 1_{(S,U)}$ and so $a = a^* \in U$. $\square$

3. The main result

In this section, we will show that any proper cover for a $\tilde{\mathscr{L}}^U$-ample type B semigroup is a proper cover over a monoid. A structure theorem of proper covers for $\tilde{\mathscr{L}}^U$-ample type B semigroup is obtained.

Definition 3.1. *Let (S,U) be a $\tilde{\mathscr{L}}^U$-ample type B semigroup, M be a monoid. A surjective relational morphism θ from M to (S,U) is a mapping $\theta : M \longrightarrow 2^{(S,U)}$ such that:*

$(A1)$ $m\theta \neq \emptyset$ *for all* $m \in M$;

$(A2)$ $m_1\theta.m_2\theta \subseteq (m_1m_2)\theta$ *for all* $m_1, m_2 \in M$;

$(A3)$ $\cup_{m\in M} m\theta = (S,U)$;

$(A4)$ $1\theta = U$;

$(A5)$ $|\tilde{\mathscr{L}}^U_a \cap m\theta| \leq 1$ *for all* $m \in M, a \in (S,U)$;

$(A6)$ $m\theta \subseteq a\sigma$ *for all* $m \in M, a \in m\theta$.

Theorem 3.1. *Let (S,U) be a $\tilde{\mathscr{L}}^U$-ample type B semigroup, M be a monoid, θ be a surjective relational morphism from M to (S,U). Let*

$$T = \{(s,m) \in (S,U) \times M | s \in m\theta\},$$

and define a multiplication on T by

$$(s_1,m_1)(s_2,m_2) = (s_1s_2,m_1m_2).$$

Then T is a semigroup and

(1) $V = \{(e,1)|e \in U\}$ *is a subset of* $E(T)$ *and* $V \cong U$;

(2) *for all* $a,b \in S, g,h \in M, (a,g)\tilde{\mathscr{L}}^V(b,h) \Leftrightarrow a\tilde{\mathscr{L}}^U b$;

(3) (T,V) *is a* $\tilde{\mathscr{L}}^V$*-ample type B semigroup;*

(4) *for all* $(a,g),(b,h) \in (T,V), (a,g)\sigma_{(T,V)}(b,h) \Leftrightarrow a\sigma_{(S,U)}b, g = h$.

Proof. Let T be as in the statement of the theorem. It is clearly that T is a semigroup. Now we proof the rest.

(1) Since θ is a surjective relational morphism, by $(A4)$, we have V is a subsemigroup of T and $T \subseteq E(T)$. Then it follows that $V \cong U$.

(2) It is benefit for us to prove the following useful Lemma.

Lemma 3.1. *Let $(a,g) \in (T,V)$, then $(a,g)\tilde{\mathscr{L}}^V(a^*,1)$.*

Proof. Let $(a,g) \in (T,V)$. By (1) we have $(a^*,1) \in V$. It is clear that $(a,g)(a^*,1) = (a,g)$. Now, for any $(f,1) \in V$ if $(a,g)(f,1) = (a,g)$, then

$$\begin{aligned}(a,g)(f,1) = (a,g) &\Rightarrow\ a.f = a \text{ and } g.1 = g\\ &\Rightarrow\ a^*.f = a^* \quad (\text{since } a\tilde{\mathscr{L}}^U a^*)\\ &\Rightarrow\ (a^*,1)(f,1) = (a^*,1).\end{aligned}$$

By the Lemma 2.1 we have $(a,g)\tilde{\mathscr{L}}^V(a^*,1)$ as required. $\square$

Returning now to the main proof. Let $a,b \in (S,U), g,h \in M$. On the one hand, if $(a,g)\tilde{\mathscr{L}}^V(b,h)$, by the Lemma 3.1 we have

$$(a^*,1)\tilde{\mathscr{L}}^V(a,g)\tilde{\mathscr{L}}^V(b,h)\tilde{\mathscr{L}}^V(b^*,1),$$

and so $(a^*,1)\tilde{\mathscr{L}}^V(b^*,1)$. By the Lemma 2.2, it follows that $(a^*,1) = (b^*,1)$ and so $a^* = b^*$. Hence $a\tilde{\mathscr{L}}^U b$ as required.

On the other hand, if $a\tilde{\mathscr{L}}^U b$, then $a^*\tilde{\mathscr{L}}^U b^*$ and so $a^* = b^*$. By the Lemma 3.1, we have

$$(a,g)\tilde{\mathscr{L}}^V(a^*,1) = (b^*,1)\tilde{\mathscr{L}}^V(b,h).$$

Hence $(a,g)\tilde{\mathscr{L}}^V(b,h)$ as required.

(3) From (1) and (2) we have (T,V) is $\tilde{\mathscr{L}}^U$-semiabundant and projections commute and so is $\tilde{\mathscr{L}}^U$-semiadequate. Let $(e,1),(f,1) \in V^1$, and $(a,h) \in (T,V)$, where $e,f \in U^1$. Since (S,U) is a $\tilde{\mathscr{L}}^U$-ample type B semigroup, we have

$$\begin{aligned}[(e,1)(f,1)(a,h)]^* &= (efa,h)^* = ((efa)^*,1)\\ &= ((ea)^*(fa)^*,1) = ((ea)^*,1)((fa)^*,1)\\ &= (ea,h)^*(fa,h)^*\\ &= [(e,1)(a,h)]^*[(f,1)(a,h)]^*.\end{aligned}$$

Hence (T,V) satisfies the condition $(B1)$.

For $(e,1) \in V^1$ and $(a,h) \in (T,V)$, where $e \in U^1$. If $(e,1) \leq (a,h)^* = (a^*,1)$, then

$$(e,1) = (e,1)(a^*,1) = (ea^*,1) = (a^*e,1)$$

and so $e = ea^* = a^*e$. That is $e \leq a^*$. Since (S,U) is a $\tilde{\mathscr{L}}^U$-ample type B semigroup, there exist $f \in U^1$ such that $e = (fa)^*$. And then

$$(e,1) = ((fa)^*,1) = (fa,h)^* = [(f,1)(a,h)]^*.$$

As $(f,1) \in V^1$ and so (T,V) satisfies the condition $(B2)$. Hence (T,V) is a $\tilde{\mathscr{L}}^V$-ample type B semigroup.

(4) Let $(a,g),(b,h) \in (T,V)$, if $(a,g)\sigma_{(T,V)}(b,h)$, there exist $(e,1) \in V$ such that $(e,1)(a,g) = (e,1)(b,h)$ and so $(ea,g) = (eb,h)$. That is $ea = eb$ and $g = h$. Hence $a\sigma_{(S,U)}b$ and $g = h$. Conversely, if $a\sigma_{(S,U)}b$ and $g = h$, then exist $e \in U$ such that $ea = eb$ and so $(e,1)(a,g) = (ea,g) = (eb,h) = (e,1)(b,h)$. Since $(e,1) \in V$, we have $(a,g)\sigma_{(T,V)}(b,h)$. □

Theorem 3.2. *Let (S,U) be a $\tilde{\mathscr{L}}^U$-ample type B semigroup, M be a monoid, θ be a surjective relational morphism from M to (S,U). Let*

$$T = \{(s,m) \in (S,U) \times M | s \in m\theta\},$$

and define a multiplication on T by

$$(s_1,m_1)(s_2,m_2) = (s_1s_2,m_1m_2).$$

Let $V = \{(e,1)|e \in U\}$, then (T,V) is a proper cover of (S,U) over M. Conversely, any proper cover of (S,U) can be constructed in this way.

Proof. From the Theorem 3.1 we have (T,V) is a $\tilde{\mathscr{L}}^V$-ample type B semigroup. Let $(a,g),(b,h) \in (T,V)$ and $(a,g)(\tilde{\mathscr{L}}^V \cap \sigma_{(T,V)})(b,h)$. By the Theorem 3.1 (2), (4) we have $a\tilde{\mathscr{L}}^U b$ and $g = h$ and so $a,b \in h\theta = g\theta$. Since θ is a surjective relational morphism, by $(A5)$ we have $a = b$. Hence $(a,g) = (b,h)$. That is $\tilde{\mathscr{L}}^V \cap \sigma_{(T,V)} = 1_{(T,V)}$. Thus (T,V) is proper.

A mapping β from (T,V) to (S,U) is defined by the rule as follow:

$$\beta : (T,V) \longrightarrow (S,U), (a,g) \longmapsto a.$$

It is clear that β is a surjective homomorphism and $\beta|_V : V \longrightarrow U$. On the one hand, if $[(a,g)]\beta = [(b,h)]\beta$ for $(a,g),(b,h) \in (T,V)$, then $a = b$ and so $a\tilde{\mathscr{L}}^U b$. By the Theorem 3.1 (2) we have $(a,g)\tilde{\mathscr{L}}^V(b,h)$. Hence β is a $\tilde{\mathscr{L}}^V$-homomorphism from (T,V) onto (S,U). On the other hand, for any $e \in U$, by the Theorem 3.1 (1) we have $(e,1) \in V$ and so $[(e,1)]\beta = e$. Thus (T,V) is a proper cover for (S,U).

Since $\sigma_{(T,V)}$ is the least monoid congruence with $V \subseteq (1_{(T,V)/\sigma_{(T,V)}})(\sigma^{\natural}_{(T,V)})^{-1}$, A mapping α from $(T,V)/\sigma_{(T,V)}$ to M is defined by the rule as follow:

$$\alpha : (T,V)/\sigma_{(T,V)} \longrightarrow M, (a,g)\sigma_{(T,V)} \longmapsto g.$$

We claim that α is a one-one mapping. Let $(a,g)\sigma_{(T,V)}, (b,h)\sigma_{(T,V)} \in (T,V)/\sigma_{(T,V)}$ and $(a,g)\sigma_{(T,V)} = (b,h)\sigma_{(T,V)}$. Since $(a,g)\sigma_{(T,V)}(b,h)$, by the Theorem 3.1 (4) we have $a\sigma_{(S,U)}b$ and $g = h$. That is $[(a,g)\sigma_{(T,V)}]\alpha = g = h = [(b,h)\sigma_{(T,V)}]\alpha$, hence α is a mapping. It is clear that α is surjective. If $[(a,g)\sigma_{(T,V)}]\alpha = [(b,h)\sigma_{(T,V)}]\alpha$ for $(a,g)\sigma_{(T,V)}, (b,h)\sigma_{(T,V)} \in (T,V)/\sigma_{(T,V)}$. Then $g = h$ and so $a, b \in g\theta$. By $(A6)$, we have $a\sigma_{(S,U)}b$. So by the theorem 3.1 (4) we have $(a,g)\sigma_{(T,V)}(b,h)$. Thus α is one-one as required. On the one hand, let $(a,g)\sigma_{(T,V)}, (b,h)\sigma_{(T,V)} \in (T,V)/\sigma_{(T,V)}$, since

$$\begin{aligned}[(a,g)\sigma_{(T,V)}.(b,h)\sigma_{(T,V)}]\alpha &= (ab,gh)\sigma_{(T,V)}]\alpha = gh \\ &= [(a,g)\sigma_{(T,V)}]\alpha.[(b,h)\sigma_{(T,V)}]\alpha.\end{aligned}$$

Hence α is an isomorphic. On the other hand, since $V \subseteq (1_{(T,V)/\sigma_{(T,V)}})(\sigma^{\natural}_{(T,V)})^{-1}$ and $(T,V)/\sigma_{(T,V)} \cong M$, we have $V \subseteq (1_M\alpha^{-1})(\sigma^{\natural}_{(T,V)})^{-1}$. Thus (T,V) is a proper cover for (S,U) over M. Up to now we have already established the first statement in this Theorem.

Conversely, let (T,V) be a proper cover for (S,U). Then there is a $\widetilde{\mathscr{L}}^U$-homomorphism ϕ from (T,V) onto (S,U) satisfying for any $e \in U$, there exist $f \in V$ such that $f\phi = e$. Let $M = (T,V)/\sigma_{(T,V)}$, by the Lemma 2.3 M is a monoid with $V \subseteq 1_M(\sigma^{\natural}_{(T,V)})^{-1}$.

A relation morphism θ from M to (S,U) is defined by the rule as follow:

$$\theta : M \longrightarrow 2^{(S,U)}, g \longmapsto g\theta,$$

for any $g \in M, g\theta = \{s \in (S,U) | exist\ t \in (T,V), s = t\phi, t\sigma_{(T,V)} = g\}$. It remain to prove that θ is a surjective relational morphism and $(T,V) \cong (T',V')$, where

$$T' = \{(s,g) \in (S,U) \times M | s \in g\theta\},$$

$$V' = \{e,1) \in (S,U) \times M | e \in 1\theta\}.$$

Let $g \in M$, since the natural morphism $\sigma^{\natural}_{(T,V)} : (T,V) \longrightarrow (T,V)/\sigma_{(T,V)} = M$ is surjective, we have $g\theta \neq \emptyset$ and so θ satisfies condition $(A1)$.

Let $g,h \in M, s_1 \in g\theta, s_2 \in h\theta$, then exist $u,v \in (T,V)$ such that

$$s_1 = u\phi, u\sigma_{(T,V)} = g, s_2 = v\phi, v\sigma_{(T,V)} = h.$$

Then we have $s_1s_2 = (uv)\phi, (uv)\sigma_{(T,V)} = gh$ and so $s_1s_2 \in (gh)\theta$. Hence θ satisfies condition $(A2)$.

It is clear that $\cup_{g\in M} g\theta = (S,U)$. Hence θ satisfies condition $(A3)$.

Let $s \in 1\theta$, then exist $t \in (T,V)$ such that $s = t\phi, t\sigma_{(T,V)} = 1$. Note that $t^*\sigma_{(T,V)} = 1$, we have $t(\tilde{\mathscr{L}}^U \cap \sigma_{(T,V)})t^*$. Since (T,V) is proper, we have $t = t^* \in V$. Hence $1\theta \subseteq U$. Conversely, if $e \in U$, then there exist $f \in V$ such that $e = f\phi$. Since $f\sigma_{(T,V)} = 1$, we have $e \in 1\theta$ and so $U \subseteq 1\theta$. Hence θ satisfies condition $(A4)$.

Now, we prove that $(T,V) \cong (T',V')$ first. A mapping ψ from (T,V) to (T',V') is defined by the rule as follow:

$$\psi : (T,V) \longrightarrow (T',V'), t \longmapsto (t\phi, t\sigma_{(T,V)}),$$

it is clear that ψ is a surjective morphism. Let $t,u \in (T,V)$ and $(t\phi, t\sigma_{(T,V)}) = t\psi = u\psi = (u\phi, u\sigma_{(T,V)})$. Then we have $t\phi = u\phi, t\sigma_{(T,V)} = u\sigma_{(T,V)}$. Since ϕ is a $\tilde{\mathscr{L}}^U$-homomorphism, we have $t\tilde{\mathscr{L}}^V u$ and so $t(\tilde{\mathscr{L}}^V \cap \sigma_{(T,V)})u$. Since (T,V) is proper, we have $t = u$. Hence ψ is an isomorphism. Finally, since $\psi|_V = V'$, we have $V \cong V'$ and so $(T,V) \cong (T',V')$. That is (T',V') is proper.

To show that θ satisfies condition $(A5)$. Let $s,s' \in g\theta$ and $s\tilde{\mathscr{L}}^U s'$, then $(s,g),(s',g) \in (T',V')$. By the Theorem 3.1 (2), we have $(s,g)\tilde{\mathscr{L}}^{V'}(s',g)$. Since ψ is a isomorphism, there exist $t,t' \in (T,V)$ such that

$$t\psi = (t\phi, t\sigma_{(T,V)}) = (s,g), t'\psi = (t'\phi, t'\sigma_{(T,V)}) = (s',g).$$

Then $t\sigma_{(T,V)}t'$ and so there exist $e \in V$ such that $et = et'$. Then

$$e\psi.(s,g) = e\psi.t\psi = (et)\psi = (et')\psi = e\psi.t'\psi = e\psi.(s',g),$$

and so $(s',g)\sigma_{(T',V')}(s,g)$. Then we have $(s',g)(\tilde{\mathscr{L}}^{V'} \cap \sigma_{(T',V')})(s,g)$. Since T' is proper, we have $(s',g) = (s,g)$ and so $s = s'$. Hence θ satisfies condition $(A5)$ as required.

Finally, we show that θ satisfies condition $(A6)$. Let $s,t \in (S,U)$ and $m \in M$ such that $s,t \in m\theta$, then we have $(s,m),(t,m) \in (T',V')$. Similar to the prove of θ satisfies condition $(A5)$, we have $(s,m)\sigma_{(T',V')}(t,m)$. Similar to the prove of the Theorem 3.1 (4), we have $s\sigma_{(S,U)}t$. Hence θ satisfies condition $(A6)$ as required.

Remark. 1. By the Theorem 3.1 and the direct part proof of the Theorem 3.2, we have the following diagram

$$\begin{array}{ccccc} V & \longrightarrow & (T,V) & & \\ \beta\downarrow & & \beta\downarrow & \searrow^{\sigma^{\natural}_{(T,V)}\circ\alpha} & \\ U & \longrightarrow & (S,U) & & M \end{array}$$

where β is a $\tilde{\mathscr{L}}^U$-homomorphism from (T,V) onto (S,U), $\sigma^{\natural}_{(T,V)}$ is a natural morphism and $\beta|_V$ is an isomorphism.

2. From the converse part proof of the Theorem 3.2, we have the following diagram

$$\begin{array}{ccccc} V & \longrightarrow & (T,V) & & \\ \phi\downarrow & & \phi\downarrow & \searrow^{\sigma^{\natural}_{(T,V)}} & \\ U & \longrightarrow & (S,U) & & M \end{array}$$

where $M = (T,V)/\sigma_{(T,V)}$, ϕ is a $\tilde{\mathscr{L}}^U$-homomorphism from (T,V) onto (S,U), $\sigma^{\natural}_{(T,V)}$ is a natural morphism and $\phi|_V$ is an isomorphism.

3. Since the relation $\tilde{\mathscr{L}}^U$ is a natural generalization of the Green's star relation $\mathscr{L}^*$, the $\tilde{\mathscr{L}}^U$-ample type B semigroups can be think as a generalization of right type B semigroups. From this point, this theorem generalizes the result of Li-Wang [10] for right type B semigroups.

The following result is immediate from the Lemma 2.4 and the Theorem 3.2.

Corollary 3.1. *A $\tilde{\mathscr{L}}^U$-ample type B semigroup has a U-unitary proper cover over a monoid.*

Dually, we have the following result.

Corollary 3.2. *A $\tilde{\mathscr{R}}^U$-ample type B semigroup has a U-unitary proper cover over a monoid.*

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] J. B. Fountain, Abundant semigroups, Proc. London Math. Soc. 44 (1982), 103-129.

[2] J. B. Fountain, Adequate semigroups, Proc. Edinburgh Math. Soc. 22 (1979), 113-125.

[3] J. B. Fountain, A class of right pp monoids, Quart. J. Math. Oxford. 28 (1977), 285-330.

[4] J. B. Fountain, Proper left type A monoids revisited, Glasgow. Math. 35 (1995), 293-306.

[5] S. Armstrong, The structure of type A semigroups, Semigroup Forum. 29 (1984), 319-336.

[6] X. J. Guo, X. Y. Xie, A note on left type A semigroups, Semigroup Forum. 58 (1999), 313-316.

[7] X. J. Guo, Y. Q. Guo, The translational hull of a strongly right type A semigroup, Science in China (series A). 43 (2000), 6-12.

[8] X. J. Guo, K. P. Shum, Y. Q. Guo, Perfect rpp semigroups, Comm. Algebra. 29 (2001), 2447-2459.

[9] X. J. Guo, Y. B. Jun, M. Zhao, Pseudo C-rpp semigroups, Acta Mathematica Sinica, English Series. 26(4) (2010), 629-646.

[10] C. H. Li, L. M. Wang, A proper cover on a right type B semigroups, J. South China Normal Univ. (Natural Science Eition). 44 (2012), 54-57.

[11] J. M. Howie, Fundamentals of Semigroup Theory, Oxford University Press, New York, 1995.

[12] M. V. Lawson, Semigroups and ordered categories. I. the reduced case, J. Algebra. 141 (1991), 422-462.

[13] L. F. Huang, Weakly $\mathscr{J}^*$-covered superabundant semigroups and the covers for $\mathscr{R}^{\circ}$-ample semigroups, Master Degree thesis, Guang Zhou, 2009.

[14] Y. Q. Guo, K. P. Shum, P. Y. Zhu, The structure of left C-rpp semigroups, Semigroup Forum. 50 (1995), 9-23.