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QUASI-OPEN FUNCTIONS IN EUCLIDEAN SPACES

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Abstract. In this paper we consider the semigroups of quasi-open functions. A function f between topological spaces X and Y is quasi-open if for any non-empty open set $U \subset X$, the interior of $f(U)$ in Y is non-empty. We give an abstract characterization of semigroups of quasi-open functions defined on an open set of Euclidean n-space.

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1. Introduction

Investigation of functions by algebraic methods plays a very important role in modern mathematics where we consider various operations on sets of functions. Particularly, many researchers have focused their efforts on the characterization of topological spaces by semigroups of continuous, open, and closed mappings[2],[4]. The purpose of this paper is to investigate semigroups of quasi-open functions. A function f between topological spaces X and Y is quasi-open if for any non-empty open set $U \subset X$, the interior of $f(U)$ in Y is non-empty. If f and g are both quasi-open, then the function composition is also quasi-open. Let $Q(X)$ denote the semigroup of quasi-open functions from a topological space X into itself with composition of functions as multiplication, i.e., for $f, g \in Q(X)$, fg is defined by $fg(x) = f(g(x))$ for all $x \in X$. It is obvious that if X and Y are homeomorphic

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then the semigroups $Q(X)$ and $Q(Y)$ are isomorphic. If $Q(X)$ and $Q(Y)$ are isomorphic, must X and Y be homeomorphic. In general, the answer is no. Let X denote any set with more than two elements containing the elements η, ξ. Consider the topological spaces $Y = (X, \tau_1)$ and $Z = (X, \tau_2)$ with $\tau_1 = \{\oslash, \{\eta\}, X\}$ and $\tau_2 = \{\oslash, \{\eta\}, X \setminus \{\xi\}, X\}$. Evidently $Q(Y)$ and $Q(Z)$ are isomorphic but Y and Z are not homeomorphic. In this paper, we give an abstract characterization of semigroups of quasi-open functions defined on an open set of Euclidean n-space.

2. A Characterization of Semigroups of Quasi-Open Functions

We denote by R^n Euclidean n-space with respect to the standard Cartesian coordinate system.

Theorem 1. *Let X and Y be open subsets of R^n and R^m respectively, $(n, m > 1)$. The semigroups $Q(X)$ and $Q(Y)$ are isomorphic if and only if the spaces X and Y are homeomorphic.*

Proof. It is obvious that if X and Y are homeomorphic then $Q(X)$ and $Q(Y)$ are isomorphic. Specifically, if h is a homeomorphism from X to Y, then $f \to h \circ f \circ h^{-1}$ is an isomorphism from $Q(X)$ onto $Q(Y)$. The proof of the necessary condition follows from Lemma6. Throughout this paper, the symbol φ denotes isomorphism from $Q(X)$ onto $Q(Y)$. $\square$

Lemma 2. *Let X be an open subset of R^m and let $\xi \in X$. Then, there exist two quasi-open functions $f, g \in Q(X)$, such that $f(\xi) \neq g(\xi)$ but $f(x) = g(x)$ for all $x \in X \setminus \{\xi\}$.*

Proof. Let X be an open subset of R^m and let $\xi \in X$. Consider the function $f : X \to X$ defined by

$$f(x) = \begin{cases} \eta_1 & \text{if } x = \xi \\ x & \text{otherwise} \end{cases}$$

and the function $g : X \to X$ defined by

$$g(x) = \begin{cases} \eta_2 & \text{if } x = \xi \\ x & \text{otherwise} \end{cases}$$

where $\eta_1 \neq \eta_2$ are any fixed points in X different than ξ. The transformations f and g are quasi-open, but not continuous and not open. We also have $f(\xi) \neq g(\xi)$ but $f(x) = g(x)$ for all $x \in X \setminus \{\xi\}$. □

Lemma 3. *Let a, b be arbitrary elements of $Q(X)$. The condition*

$$\forall f, g \in Q(X), fa = ga \to fb = gb \tag{1}$$

is necessary and sufficient for $b(X) \subseteq a(X)$.

Proof. If the condition $b(X) \subseteq a(X)$ is satisfied, then for every $x \in X$ there exists a point $\xi \in X$ such that $b(x) = a(\xi)$. Then

$$fb(x) = f(b(x)) = f(a(\xi)) = fa(\xi) = ga(\xi) = g(a(\xi)) = g(b(x)) = gb(x).$$

So the condition 1 holds. Now let the condition 1 hold for some $a, b \in Q(X)$. Suppose that $b(X) \setminus a(X)$ is not empty. It would then follow from Lemma2 that for any point $\xi = b(x)$ in $b(X) \setminus a(X)$ there exist $f, g \in Q(x)$, such that $f(\xi) \neq g(\xi)$ but $f(x) = g(x)$ for all $x \in X \setminus \{\xi\}$. Then for every $x \in X$ the point $a(x)$ is in $X \setminus \{\xi\}$ and therefore $fa(x) = f(a(x)) = g(a(x)) = ga(x)$. But for $\xi = b(x)$ in $b(X) \setminus a(X)$ we have $fb(x) = f(b(x)) = f(\xi) \neq g(\xi) = g(b(x)) = gb(x)$ which contradicts to 1. □

Lemma 4. *Let X and Y be open subsets of R^n and R^m respectively, $(n, m > 1)$. Suppose that φ is an isomorphism from $Q(X)$ to $Q(Y)$ and $a, b \in Q(X)$ If $a(X) = b(X)$ then $(\varphi a)(Y) = (\varphi b)(Y)$.*

Proof. Suppose that $b(X) \subseteq a(X)$. If $f'(\varphi a) = g'(\varphi a)$ for some elements $f', g' \in Q(Y)$ then there exist $f, g \in Q(X)$ such that $f' = \varphi f$ and $g' = \varphi g$. Then $(\varphi f)(\varphi a) = (\varphi g)(\varphi a)$ and since φ is an isomorphism, $\varphi(fa) = \varphi(ga)$ and $fa = ga$. We have $fb = gb$, by Lemma3. Again, since φ is an isomorphism, then $(\varphi f)(\varphi b) = (\varphi g)(\varphi b)$ and therefore $f'(\varphi b) = g'(\varphi b)$. Because $f'(\varphi b) = g'(\varphi b)$ is true for every $f', g' \in Q(Y)$ satisfying the condition $f'(\varphi a) = g'(\varphi a)$ it follows from Lemma3 that $(\varphi b)(Y) \subseteq (\varphi a)(Y)$. In the same way, we could show that if $a(X) \subseteq b(X)$ then $(\varphi a)(Y) \subseteq (\varphi b)(Y)$. □

Lemma 5. *Let X be an open subset of R^m and let A be any open subset of X. Then there exists a quasi-open function $a \in Q(X)$ such that $a(X) = A$.*

Proof. Let X be an open subset of R^m. Assume without loss of generality that X is bounded. Let $g_1 : A \to A$ be a function defined by $g_1(x) = x$ for all $x \in A$, i.e. the identity function. If E_1 is a closed m-ball containing $X \setminus \overline{A}$ and E_2 is a closed m-ball that is contained in A, then there exists a homeomorphism from E_1onto E_2. Consider the restriction of this homeomorphism to $X \setminus \overline{A}$. Denote by g_2 the extension of this function $X \setminus A$ obtained by assigning all boundary points of A to any fixed point in A. The function $f : X \to A$ defined by

$$f(x) = \begin{cases} g_1(x), & \text{if} \quad x \in A \\ g_2(x), & \text{if} \ \ x \in X \setminus A \end{cases}$$

is a quasi-open and onto function. □

Let X be a topological space. Let $C(X)$ denote the family of all open sets of X. The family $C(X)$ is a complete distributive lattice if set inclusion is taken as the ordering. Two topological spaces X and Y are said to be lattice-equivalent if there is a bijective map from $C(X)$ to $C(Y)$ which together with its inverse is order-preserving[3].

Lemma 6. *Let X and Y be open subsets of R^n and R^m respectively, $(n, m > 1)$. The lattices $C(X)$ and $C(Y)$ are lattice-isomorphic.*

Proof. Let A be any open subset of X. By Lemma5 there exists a quasi-open function $a \in Q(X)$ such that $a(X) = A$. Since the semigroups $Q\,(X)$ and $Q\,(Y)$ are isomorphic there exists a quasi-open function $a' \in Q(Y)$ such that $\varphi a = a'$. Let $a'(Y) = A'$. We define a map θ from $C(X)$ to $C(Y)$ by assinging to each open set $A \subset X$ the set $A' \subset Y$. The map θ does not depend on the choice of $a \in Q(X)$. Indeed , if $a(X) = A$ and $b(X) = A$ then Lemma4 says that $(\varphi a)(Y) = (\varphi b)(Y) = A'$. Let A and B be any two different open subsets of X. By Lemma5 there exist two quasi-open functions $a, b \in Q(X)$ such that $a(X) = A$ and $b(X) = B$. Since the semigroups $Q\,(X)$ and $Q\,(Y)$ are isomorphic it follows from Lemma4 that $(\varphi a)Y \neq (\varphi b)Y$. Hence θ is bijective. Now suppose that A' is an arbitrary open set in Y. Since the semigroups $Q\,(X)$ and $Q\,(Y)$ are isomorphic it follows from Lemma4 that there exists an open set $A \subset X$ such that $\theta(A) = A'$. Again it follows from Lemma4 that if $A \subseteq B$ then $\theta(A) \subseteq \theta(B)$. From Theorem2.1 of [3] it follows that the open sets X and Y are homeomorphic. □

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