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ON ARF NUMERICAL SEMIGROUPS

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Abstract. In this study, we obtain an Arf semigroup by means of a sequence. We also establish some results on the Arf semigroup.

Keywords: Numerical semigroup; Arf numerical semigroups; Frobenius number; Genus.

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1. Introduction

A numerical semigroup S is a subset of the non-negative integers $\mathbb{N}$, closed under addition,that contains 0 and has finite complement in $\mathbb{N}$. The condition $\sharp(\mathbb{N}\backslash S) < \infty$ is equivalent to impose that $gcd(A) = 1$ for every system of generators A of S,where $\sharp(A)$ denotes the cardinality of A any set. Every numerical semigroup S is finitely generated and has a unique minimal system of generators $a_1, ..., a_k$; that is

$$S = \langle a_1, ..., a_k \rangle = \{r_1a_1 + ... + r_ka_k; r_i \in \mathbb{N}\}$$

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with $a_1 < ... < a_k$ and $a_i \notin \langle a_1,...,\hat{a}_i,...,a_k \rangle$. Then will say that S is minimally generated by $\{a_1,...,a_k\}$ [2].

An important invariant of S is the largest integer not belonging to S, known as the Frobenius number of S and denoted by $F(S)$. We define $n(S) = \sharp(\{0,1,...,F(S)\} \cap S)$ [5]. It is also well known that $S = \langle a_1,...,a_k \rangle = \{0,s_1,s_2,s_3,...,s_n,F(S)+1,\rightarrow ...\}$ where " $\rightarrow$ " means that every integer greater than $F(S)+1$ belongs to S, $n = n(S)$ and $s_i < s_{i+1}$ for $i = 1,2,...n$. The integer $F(S)+1$ is conductor of S.

The elements of $\mathbb{N} \backslash S$ are called gaps of S. Set of gaps of S is denoted by $H(S)$. Its cardinality is the genus of S, $g(S)$,which is sometimes referred to as the degree of singularity of S. A gap x of numerical semigroup S is said to be fundamental if $\{2x,3x\} \subset S$. We denote by $FH(S)$ the set of all fundamental gaps of S [4].

Let S be a numerical semigroup. It is denoted by $N(S) = \{s \in S : s < F(S)\}$. This set fully determines S . Clearly, $n(S) = \sharp(N(S))$ [5].

A numerical semigroup S is Arf numerical semigroup if for every $x,y,z \in S$ such that $x \geq y \geq z$, one has that $x+y-z \in S$ [1, 3]. However, Barucci and his colleagues determine that an Arf numerical semigroup has maximal embedding dimension [2].But the reverse is not true. For example, the numerical semigroup $S = \langle 3,7,11 \rangle = \{0,3,6,7,9,\rightarrow ...\}$ has maximal embedding dimension, while it is not Arf, because $7+7-6 = 8 \notin S$.

The main goal of this paper is to prove *Theorem 2.2* which gives a class of Arf semigroup by means of a sequence. We also find some special invariants of this class such as gaps, fundamental gaps, genus, Frobenius number and $n(S)$.

2. Main results

Proposition 2.1. *Let S be a proper subset of* $\mathbb{N}$. *Then S is an Arf numerical semigroup if and only if there exist positive integers* $x_1,...,x_n$ *such that*

$$S = \{0,x_1,x_1+x_2,...,x_1+...+x_{n-1},x_1+...+x_n,\rightarrow ...\}$$

and $x_i \in \{x_{i+1},x_{i+1}+x_{i+2},...,x_{i+1}+...+x_n,\rightarrow ...\}$ *for all* $i \in \{1,...,n\}$ [3].

Theorem 2.2. *Let $a \in \mathbb{Z}^+$ and $a \geq 2$. If (a_n) is a sequence with general term*

$$a_n = \begin{cases} 1, & \text{if } n = 1 \\ a^{n-2}, & \text{if } n \neq 1 \end{cases}$$

then $(a_n) = \{1, a^0, a^1, ..., a^{n-2}, ...)$. When t positive integer multiples of its first n elements are selected as $x_n = t, x_{n-1} = a^0 t, x_{n-2} = a^1 t, ..., x_1 = a^{n-2} t$, respectively. $S = \{0, x_1, x_1 + x_2, ..., x_1 + ... + x_n \rightarrow ...\} = \{0, a^{n-2}t, (a^{n-2} + a^{n-3})t, ..., (a^{n-2} + ... + a^0 + 1)t, \rightarrow ...\}$ is an Arf numerical semigroup.

Proof. Let $a \in \mathbb{Z}^+$ and $a \geq 2$. Then

$$\begin{aligned} x_n &= t \\ x_{n-1} &= a^0 t \in \{t, \rightarrow ...\} \\ &\vdots \\ x_1 &= a^{n-2} t \in \{a^{n-3}t, (a^{n-3} + a^{n-4})t, ..., (a^{n-3} + ... + a^0 + 1)t, \rightarrow ...\} \end{aligned}$$

because

$$\begin{aligned} a^{n-3} + ... + a^0 &= a^{n-3} + ... + a^0 \\ (a-1)(a^{n-3} + ... + a^0) &\geq a^{n-3} + ... + a^0, \qquad (a \geq 2) \\ a^{n-2} - 1 &\geq a^{n-3} + ... + a^0 \\ a^{n-2} &\geq a^{n-3} + ... + a^0 + 1 \\ a^{n-2} t &\geq (a^{n-3} + ... + a^0 + 1)t, \qquad (t \in \mathbb{Z}^+) \end{aligned}$$

and $x_i \in \{x_{i+1}, x_{i+1} + x_{i+2}, ..., x_{i+1} + ... + x_n \rightarrow ...\}$ for all $i \in \{1, ..., n\}$. Thus $x_1, ..., x_n$ fulfill the conditions of *Proposition 2.1.* Then $S = \{0, x_1, x_1 + x_2, ..., x_1 + ... + x_n \rightarrow ...\} = \{0, a^{n-2}t, (a^{n-2} + a^{n-3})t, ..., (a^{n-2} + ... + a^0 + 1)t, \rightarrow ...\}$ is numerical semigroup with Arf property.

Example 2.3. *Take $a = 3, n = 5$ and $t = 7$. Then $x_5 = 7, x_4 = 7, x_3 = 21, x_2 = 63$ and $x_1 = 189$ so $S = \{0, 189, 252, 273, 280, 287, \rightarrow ...\}$ is obtained by means of this sequence. The numerical semigroup S is Arf property.*

Proposition 2.4. *Let $a \in \mathbb{Z}^+$, $a \geq 2$ and $t, n > 1$. If $S = \{0, a^{n-2}t, (a^{n-2}+a^{n-3})t, ..., (a^{n-2}+ ... + a^0+1)t, \rightarrow ...\}$, then the set of gaps of S is*

$$\begin{aligned} H(S) = \{&1, ..., a^{n-2}t-1, a^{n-2}t+1, a^{n-2}t+2, ..., (a^{n-2}+a^{n-3})t)-1, \\ &(a^{n-2}+a^{n-3})t+1, (a^{n-2}+a^{n-3})t+2, ..., (a^{n-2}+...+a^{n-i})t-1, \\ &(a^{n-2}+...+a^{n-i})t+1, (a^{n-2}+...+a^{n-i})t+2, ..., \\ &(a^{n-2}+...+a^{n-i-1})t-1, (a^{n-2}+...+a^{n-i-1})t+1, \\ &(a^{n-2}+...+a^{n-i-1})t+2, ..., (a^{n-2}+...+a^0+1)t-1\} \end{aligned}$$

for all $i = 4, 5, ..., n-1$ and genus of S is $g(S) = (a^{n-2}+a^{n-3}+...+a^0+1)t-n$.

Proof. Let $a \in \mathbb{Z}^+$, $a \geq 2$ and $t, n > 1$. Let

$$\begin{aligned} A = \{&1, ..., a^{n-2}t-1, a^{n-2}t+1, a^{n-2}t+2, ..., (a^{n-2}+a^{n-3})t)-1, \\ &(a^{n-2}+a^{n-3})t+1, (a^{n-2}+a^{n-3})t+2, ..., (a^{n-2}+...+a^{n-i})t-1, \\ &(a^{n-2}+...+a^{n-i})t+1, (a^{n-2}+...+a^{n-i})t+2, ..., \\ &(a^{n-2}+...+a^{n-i-1})t-1, (a^{n-2}+...+a^{n-i-1})t+1, \\ &(a^{n-2}+...+a^{n-i-1})t+2, ..., (a^{n-2}+...+a^0+1)t-1\} \end{aligned}$$

If $x \in A$, then

$$\begin{aligned} 0 < x < a^{n-2}t \quad &\Rightarrow \quad x \notin S \\ a^{n-2}t < x < (a^{n-2}+a^{n-3})t \quad &\Rightarrow \quad x \notin S \\ \vdots \\ for\ i = 4, 5, ..., n-1 \\ (a^{n-2}+a^{n-3}+...+a^{n-i})t < x < (a^{n-2}+a^{n-3}+...+a^{n-i-1})t \quad &\Rightarrow \quad x \notin S \\ (a^{n-2}+a^{n-3}+...+a^0)t < x < (a^{n-2}+a^{n-3}+...+a^0+1)t \quad &\Rightarrow \quad x \notin S. \end{aligned}$$

and $x \notin S$. So $x \in H(S)$ and $A \subseteq H(S)$.

Let $y \in H(S)$. Thus $y \notin S$.

$$
\begin{aligned}
y \notin S \quad \Rightarrow \quad & (a)\ y \in A_1 = \{1, 2, ..., a^{n-2}t - 1\}\ or \\
& (b)\ y \in A_2 = \{a^{n-2}t + 1, a^{n-2}t + 2, ..., (a^{n-2} + a^{n-3})t - 1\}\ or \\
& for\ i = 4, 5, ..., n-1 \\
& (c)\ y \in A_i = \{(a^{n-2} + a^{n-3} + ... + a^{n-i})t + 1, (a^{n-2} + a^{n-3} + ... + a^{n-i})t + 2, ..., \\
& \qquad (a^{n-2} + a^{n-3} + ... + a^{n-i-1})t - 1\} \quad or \\
& for\ i = 4, 5, ..., n-1 \\
& (d)\ y \in A_n = \{(a^{n-2} + a^{n-3} + ... + a^0)t + 1, (a^{n-2} + a^{n-3} + ... + a^0)t + 2, ..., \\
& \qquad (a^{n-2} + a^{n-3} + ... + a^0 + 1)t - 1\} \\
\Rightarrow \quad & y \in A_1 \cup A_2 \cup ... \cup A_i \cup ... \cup A_n = A
\end{aligned}
$$

So, $H(S) \subseteq A$. Therefore, since $A \subseteq H(S)$ and $H(S) \subseteq A$, it follows that $H(S) = A$. Using the definition of the genus, we get

$$
\begin{aligned}
g(S) = \sharp(H(S)) \quad = \quad & (a^{n-2}t - 1) + [(a^{n-2} + a^{n-3})t - a^{n-2}t - 1] + ... + \\
& [(a^{n-2} + ... + a^{n-i-1})t - (a^{n-2} + ... + a^{n-i})t - 1] + ... + \\
& [(a^{n-2} + ... + a^0 + 1)t - (a^{n-2} + ... + a^0)t - 1] \\
= \quad & (a^{n-2}t - 1) + (a^{n-3}t - 1) + ... + (a^{n-i-1}t - 1) + ... + \\
& (a^0 t - 1) + (t - 1) \\
= \quad & (a^{n-2} + a^{n-3} + ... + a^0 + 1)t + n(-1) \\
= \quad & (a^{n-2} + a^{n-3} + ... + a^0 + 1)t - n.
\end{aligned}
$$

Remark 2.5. *Let $a \in \mathbb{Z}^+$ and $a \geq 2$. Let S be a Arf numerical semigroup in the form $S = \{0, a^{n-2}t, (a^{n-2} + a^{n-3})t, ..., (a^{n-2} + ... + a^0 + 1)t, \rightarrow ...\}$. For $t = 1$,*

$$\begin{aligned} n(S) &= n-2 \\ F(S) &= (a^{n-2}+a^{n-3}+...+a^1)-1 \\ g(S) &= (a^{n-2}+a^{n-3}+...+a^1)-(n-2). \end{aligned}$$

Theorem 2.6. *Let $a,t,n \in \mathbb{Z}^+$, $a,n > 2$ and t be positive even integer, and let $S = \{0, a^{n-2}t, (a^{n-2}+a^{n-3})t, ..., (a^{n-2}+...+a^0+1)t, \rightarrow ...\} = \{0, s_1, s_2, ..., s_{n-1}, s_n \rightarrow ...\}$. The set of Fundamental gaps of numerical semigroup S is $FH(S) = \{\frac{s_1}{2}, \frac{s_2}{2}, ..., \frac{s_n}{2}, \frac{s_n}{2}+1, \frac{s_n}{2}+2, ..., s_1-1, s_1+1, s_1+2, ..., s_2-1, ..., s_i+1, ..., s_{i+1}-1, s_{i+1}+1, ..., s_n-1\}$ for $i = 2,3,...,n-1$.*

Proof. Let $a,t,n \in \mathbb{Z}^+$, $a,n > 2$ and t be positive even integer, and let $K = \{\frac{s_1}{2}, \frac{s_2}{2}, ..., \frac{s_n}{2}, \frac{s_n}{2}+1, \frac{s_n}{2}+2, ..., s_1-1, s_1+1, s_1+2, ..., s_2-1, ..., s_i+1, ..., s_{i+1}-1, s_{i+1}+1, ..., s_n-1\}$. Firstly, we show $K \subset FH(S)$.

If $x \in K$ and $x \geq s_1$, then

$$\begin{aligned} a^{n-3}+...+a^0 &= a^{n-3}+...+a^0 \\ (a-1)(a^{n-3}+...+a^0) &> a^{n-3}+...+a^0; \quad (a>2) \\ a^{n-2}-1 &> a^{n-3}+...+a^0 \\ a^{n-2} &> a^{n-3}+...+a^0+1 \\ a^{n-2}+a^{n-2} &> a^{n-2}+a^{n-3}+...+a^0+1 \\ 2a^{n-2} &> a^{n-2}+...+a^0+1 \\ 2a^{n-2}t &> (a^{n-2}+...+a^0+1)t, \quad (t \in \mathbb{Z}^+) \\ 2s_1 &> s_n \end{aligned}$$

and we can also obtain that $3s_1 > s_n$. Since $2x, 3x > s_n$ and $2x, 3x \in S$, we get that $x \in FH(S)$.

If $x \in K$ and $\frac{s_1}{2} \leq x \leq \frac{s_n}{2}$, then $x = \frac{s_j}{2}$ for $j = 1, ..., r$.

$2x = 2\frac{s_j}{2} = s_j \in S$ and

$$\begin{aligned}
3x &= 3\frac{s_j}{2} = s_j + \frac{s_j}{2} \\
&= (a^{n-2} + ... + a^{n-j-1})t + \frac{(a^{n-2} + ... + a^{n-j-1})t}{2} \\
&> (a^{n-2} + ... + a^{n-j-1})t + \frac{(a^{n-2} + ... + a^{n-j-1})t}{a}, \quad (a > 2) \\
&= (a^{n-2} + ... + a^{n-j-1})t + (a^{n-3} + ... + \underbrace{a^{n-j-1}}_{a^{n-j-1} > a^{n-j-2} + ... + a^0 + 1} + a^{n-j-2})t \\
&> \underbrace{(a^{n-2} + ... + a^{n-j-1} + a^{n-j-2} + ... + a^0 + 1)t}_{s_n} + k = s_n + k > s_n, \quad (k \in \mathbb{Z}^+)
\end{aligned}$$

$3x = 3\frac{s_j}{2} \in S$. This implies $x \in FH(S)$.

If $x \in K$ and $\frac{s_1}{2} < x < s_1 - 1$, then

$$\begin{aligned}
x \geq \frac{s_n}{2} + 1 \quad &\Rightarrow \quad (2x \geq s_n + 2) \quad and \quad (3x \geq 3\frac{s_n}{2} + 3 = s_n + \underbrace{\frac{s_n}{2}}_{\in \mathbb{Z}^+} + 3 > s_n) \\
&\Rightarrow \quad 2x, 3x \in S.
\end{aligned}$$

This implies $x \in FH(S)$. So, $K \subset FH(S)$.

Now, we show $FH(S) \subset K$. Let $x \in FH(S)$ and $x \notin K$. If $x \notin K$, then either $x \in H(S)$ or $x \notin H(S)$. If $x \notin H(S)$, then $x \in S$, in contradiction with $x \in FH(S)$. Then $x \in H(S)$.

If $x \in FH(S)$, then either $2x = s_i$ for $i \leq n$ or $x > \frac{s_n}{2}$. If $i \leq n$ and $2x = s_i$, then $x \in K$. If $x > \frac{s_n}{2}$, then $\frac{s_n}{2} \leq s_i - 1$ and consequently, it must happen that either $x \in \{\frac{s_n}{2} + 1, \frac{s_n}{2} + 2, ..., s_i - 1\}$ or $x \in \{y \in \mathbb{N} : s_1 \leq y \leq s_n\} \backslash \{s_1, ..., s_n\}$. So $FH(S) \subset K$.

Since $K \subset FH(S)$ and $FH(S) \subset K$, we have $FH(S) = K$.

Corollary 2.7. *Let $a, t, n \in \mathbb{Z}^+$. Take $S = \{0, a^{n-2}t, (a^{n-2} + a^{n-3})t, ..., (a^{n-2} + ... + a^0 + 1)t, \rightarrow ...\} = \{0, s_1, s_2, ..., s_{n-1}, s_n \rightarrow ...\}$ for $a, n > 2$ The set of Fundamental gaps of numerical semigroup S is as follow:*

For $i = 2, 3, ..., n-1$,

(1) If a is positive even integer and t is positive odd integer, then

$FH(S) = \{\frac{s_1}{2}, \frac{s_2}{2}, ..., \frac{s_{n-2}}{2}, \frac{s_n}{2}, \frac{s_n}{2} + 1, \frac{s_n}{2} + 2, ..., s_1 - 1, s_1 + 1, s_1 + 2, ..., s_2 - 1, ..., s_i + 1, ..., s_{i+1} - 1, s_{i+1} + 1, ..., s_n - 1\}$.

(2) If a,t is positive odd integers, then

- *For even positive integer n,*
  $FH(S) = \{\frac{s_2}{2}, \frac{s_4}{2}, ..., \frac{s_{n-2}}{2}, \frac{s_n}{2}, \frac{s_n}{2}+1, \frac{s_n}{2}+2, ..., s_1-1, s_1+1, s_1+2, ..., s_2-1, ..., s_i+1, ..., s_{i+1}-1, s_{i+1}+1, ..., s_n-1\}$.
- *For odd positive integer n,*
  $FH(S) = \{\frac{s_2}{2}, \frac{s_4}{2}, ..., \frac{s_{n-1}}{2}, \frac{s_{n+1}}{2}, \frac{s_{n+1}}{2}+1, \frac{s_{n+1}}{2}+2, ..., s_1-1, s_1+1, s_1+2, ..., s_2-1, ..., s_i+1, ..., s_{i+1}-1, s_{i+1}+1, ..., s_n-1\}$.

Example 2.8. *Take $a=3, n=6$ and $t=6$. Then $S=\{0,162,216,234,240,242,244,\rightarrow ...\}$ and The set of Fundamental gaps of numerical semigroup S is*

$$\begin{aligned} FH(S) = \{&81,108,117,120,121,122,123,124,125,126,127,128,129,130,131,132,133,\\ &134,135,136,137,138,139,140,141,142,143,144,145,146,147,148,149,150,\\ &151,152,153,154,155,156,157,158,159,160,161,163,164,165,166,167,168,\\ &169,170,171,172,173,174,175,176,177,178,179,180,181,182,183,184,185,\\ &186,187,188,189,190,191,192,193,194,195,196,197,198,199,200,201,202,\\ &203,204,205,206,207,208,209,210,211,212,213,214,215,217,218,219,220,\\ &221,222,223,224,225,226,227,228,229,230,231,232,233,235,236,237,238,\\ &239,241,243\}. \end{aligned}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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