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ON CONSTRUCTION OF SOME NEW TYPES OF SEMIGROUPS

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Abstract. This paper is about the classification of the commuting graphs of semigroups to some extent. Let S be a finite non-commutative semigroup, its commuting graph, denoted by $G(S)$, is a simple graph (which has no loops and multiple edges) whose sets of vertices are elements of S and whose sets of edges are those elements of S which commute with other elements i.e. for any $a,b \in S$ such that $ab = ba$ for $a \neq b$. In this paper, we construct some new types of semigroup of cyclic tree type graphs, sunflower like graphs, almost complete type graphs and almost ladder type graphs with the help of Monid package of GAP.

Keywords: commuting graph; cyclic tree graph; idempotent transformation; band.

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1. Introduction

J. Araujo, Kinyon M. and Konieczny give a construction of band (semigroups of idempotents) in which one can find semigroup of any knit degree n, for some positive integer n, except $n = 3$ [1]. The construction of such type of semigroups also helps for finding semigroups for

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every $n \geq 2$ such that the diameter of commuting graph $G(S)$ is n. On the other hand, finding out the semigroups from a given graph is very important and difficult task in the theory of semigroups. In our paper, we find the semigroup of complete bipartite graphs, cyclic tree type graphs, sunflower like graphs, almost complete type graphs and almost ladder type graphs which will be the partly answer, to some extent, of problem given in [1].

Let S be a finite non-commutative semigroup whose centre is defined as $Z(S) = \{a \in S : ab = ba \quad \forall \quad b \in S\}$. The commuting graph of a finite non-commutative semigroup is a simple graph whose sets of vertices are from $S - Z(S)$ and whose sets of edges are the elements of S which commute with other elements i.e. for any $a, b \in$ S such that $ab = ba$ for $a \neq b$. This paper is actually the construction of band (of course non commutative) from a some given graph of different types. For the construction of such type of semigroups, our main focus will be on semigroup of full transformations $T(X)$ for a finite set X.

Suppose that $G(S)$ is commuting graph of some non-commutative semigroup of S, then $G(S) = (V, E)$ where V is a finite vertex set and and E is a set of edges such that $E \subseteq \{\{u, v\} : u, v \in V for\ u \neq v\}$. If $v_1, v_2, ...v_k$ are the vertices in $G(S)$ then we write a path λ from v_1 to v_k as $\lambda = v_1 - v_2 - ...v_k$ of length $k-1$. We say that λ is minimal path if there does not exist any path shorter than λ.

A undirected graph is graph in which for every pair of vertices there is no need of directional character unlike as in directed graphs. A tree is an undirected graph in which any two vertices are connected by exactly one path. Tree has no cycles but we construct such type of semigroups whose graph is a tree with cycles and we name it cyclic tree type graph. Similarly, we construct the semigroups of the graphs of sunflower like, ladder type and almost complete type graphs whose geometry resemblance with the graphs of sunflower, ladder and complete graphs.

Let $T(X)$ be a semigroup of full transformations for a finite set X under the composition of function. Actually the semigroups $T(X)$ is the set of all functions from a finite set X to X. In this paper we consider transformations $a, b \in T(X)$ and define composition of functions as $(ab)(x) = a(b(x))$ from right instead of left i.e $(x)(ab) = ((x)a)b$ for $x \in X$.

For $a \in T(X)$ we write image of a by $im(a)$ and kernel of a is defined as

$$ker(a) = \{(x, y) \in X \times X : a(x) = a(y)\}$$

and rank of a as $rank(a) = |im(a)|$ Also $T(X)$ has n ideals $I_1, I_2 \dots I_n$ where $1 \leq r \leq n$

$$I_r = \{a \in T(X) : rank(a) \leq r\}$$

Clearly the ideal I_1 is of rank 1 i.e a constant transformation and hence its commuting graph will be isolated vertices.

Definition 1.1. Let S be a semigroup and $e \in S$ is an idempotent if $e^2 = e$. Also we define the sets of idempotents in S to be $E(S) = \{e \in S : e^2 = e\}$ Now, $E(S)$ may be empty or it may be $E(S) = S$. If $E(S) = S$ then S is a band. We construct band in our construction in the monoid $T(X)$.

Definition 1.2. Let $e \in T(X)$ be an idempotent and $\{A_1, A_2, \dots .A_k\}$ be a partition of X and unique elements $x_1 \in A_1, x_2 \in A_2, ..x_k \in A_k$ such that for every i we have $A_i e = \{x_i\}$. Then the set $\{x_1, x_2, ...x_k\}$ is the image set of e. We use the following notation for e,

$$e = (A_1, x_1\rangle (A_2, x_2\rangle \dots (A_k, x_k\rangle$$

If e is a constant transformation with image set $\{x\}$ then we write $(X, x\rangle$ [1].

Definition 1.3. Let $e = (A_1, x_1\rangle (A_2, x_2\rangle \dots (A_k, x_k\rangle$ an idempotent in $T(X)$ and let $b \in T(X)$ then b commutes with e if and only if for every $i \in \{1, 2...k\}$ there is a $j \in \{1, 2...k\}$ such that $bx_i = x_j$ and $bA_i \subseteq A_j$ [1].

Definition 1.4. Let $e, f \in I_r$ be idempotents and suppose there is $x \in X$ such that $x \in im(e) \bigcap im(f)$ then $e - (X, x\rangle - f$ [1]

Lemma 1.1. Let $c_x, c_y, e \in T(X)$ such that e is an idempotent, then

(1) $c_x e = e c_x$ if and only if $x \in im(e)$

(2) $c_x e = c_y e$ if and only if $(x, y) \in ker(e)$

Proof (1)Consider $c_x e = e c_x$. As c_x and e commute with each other, therefore, there should be at least one element common in the images of c_x and e but c_x has only one element in the image set i.e x in $im(c_x)$. So $x \in im(c_x) \bigcap im(e)$ or $x \in \{x\} \bigcap im(e)$. This implies that $x \in im(e)$.

Conversely, suppose that $x \in im(e)$. We can write it as $x \in \{x\} \bigcap im(e)$. This implies that $x \in im(c_x) \bigcap im(e)$. Thus we have $c_x e = e c_x$.

(2) Consider $c_x e = c_y e$. As $ker(e)$ is defined as $ker(e) = \{(x, y) \in X \times X : xe = ye\}$. Consider $c_x e = c_z$ and $c_y e = c_t$ for some t and z in$X$. Thus $c_z = c_t \Rightarrow z = t$ and hence $ze = te$. Therefore

$(x,y) \in ker(e)$. Conversely, let $(x,y) \in ker(e)$ then by def. of $ker(e)$ we have $xe = ye$, implies $c_x e = c_y e$.

2. Semigroup of a cyclic tree type graphs

Definition 2.1. Let $X = \{y_o, y_1, y_2 \ldots y_k, s\}$ be a finite set. For $i \in \{1,2,3\ldots k\}$, consider the idempotent transformations, $a_o, a_1, a_2, \ldots, a_k$ with $im(a_o) = \{y_o\}$ and $im(a_i) = \{y_{i-1}, y_i\}$. Define the kernel classes as;

$ker(a_o) = X$, for $i \geq 1$, $ker(a_i)$ as

$Class-1 = \{y_o, y_1, y_2 \ldots y_{i-1}\}$ and $Class-2 = \{y_i, y_{i+1}, y_{i+2} \ldots y_k, s\}$

Lemma 2.1. Let $1 \leq i < j \leq k$ then

1) $a_o a_i = cy_{i-1}$, $a_i a_o = cy_o$

2) $a_i a_{i+1} = cy_i$, $a_{i+1} a_i = cy_i$

3) $a_i a_j = cy_{j-1}$ for $j > i+1$

4) $a_j a_i = cy_i$

Proof: 1) Since the 1st kernel class of a_i is in the $im(a_o)$. Therefore a_i will map all the elements of class-1 to cy_{i-1}. Thus $a_o a_i = cy_{i-1}$. Second part is obvious because a_o is a constant transformation and it will map on some single element of X say, y_0. Thus $a_i a_o = cy_o$.

2) Since $im(a_i) = \{y_{i-1}, y_i\}$ and $im(a_{i+1}) = \{y_i, y_{i+1}\}$

$ker(a_i)$ is define in def.3.1 as $Class-1 = \{y_o, y_1, y_2 \ldots y_{i-1}\}$ and $Class-2 = \{y_i, y_{i+1}, y_{i+2} \ldots y_k, s\}$

$ker(a_{i+1})$ is define as $Class-1 = \{y_o, y_1, y_2 \ldots y_i\}$ and $Class-2 = \{y_{i+1}, y_{i+2}, y_{i+2} \ldots y_k, s\}$

As y_i is in class-1 of $ker(i+1)$. So a_{i+1} will map all elements of class-1 to cy_i. Thus $a_i a_{i+1} = cy_i$. Similarly y_i is in class-2 of $ker(a_i)$. So a_i will map all elements of class-2 to cy_i. Thus $a_{i+1} a_i = cy_i$. Therefore $a_{i+1} a_i = a_i a_{i+1} cy_i$.

3)As $im(a_i) = \{y_{i-1}, y_i\}$ and $im(a_j) = \{y_{j-1}, y_j\}$. Since there is no common element in $im(a_i)$ and $im(a_j)$ and $ker(a_j)$ is $Class-1 = \{y_o, y_1, y_2 \ldots y_{j-1}\}$

$Class-2 = \{y_j, y_{j+1}, y_{i+2} \ldots y_k, s\}$. Since $im(a_i) \subseteq Class-1$ of $Ker(a_j)$. So a_j will map all the elements of class-1 to cy_{j-1}.

Thus $a_i a_j = cy_{j-1}$ for $j > i+1$

4) Since $im(a_j) \subseteq Class-2$ of $Ker(a_i)$. So a_i will map all the elements of class-2 to cy_i.

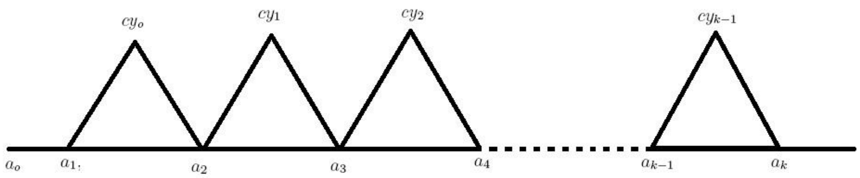

FIGURE 1. Cyclic Tree Type Graph

3. Semigroup of sunflower like graphs

Definition 3.1. Let $X = \{y_o, y_1, y_2 \dots y_k, s\}$ be a finite set. Consider the idempotents $a_o, a_1, a_2 \dots a_k$ transformations with $im(a_o) = \{y_o, y_1\}$ and for $i \in \{1,2,3\dots k\}$ $im(a_i) = \{y_{i-1}, y_i, y_{i+1}\}$.
Now we define the kernel classes of these transformations as;
$ker(a_o)$ as $Class-1 = \{y_o, s\}$ and $Class-2 = \{y_1, y_2, y_3 \dots y_k\}$
For $i \geq 1$ we define $ker(a_i)$ as $Class-1 = \{y_o, y_1, y_2 \dots y_{i-1}\}$
$Class-2 = \{y_i, s\}$, $Class-1 = \{y_{i+1}, y_{i+2}, y_{i+3} \dots y_k\}$

Lemma 3.1. Let for $i < j$ we have
1) $a_i a_{i+2} = cy_{i+1} = a_{i+2} a_i$ for $i \in \{0,1,2\dots k\}$
2) $a_i a_{i+1} = a_{n1}$ $a_{i+1} a_i = a_{n2}$ for $i \in \{1,2\dots k\}$
3) $a_i a_j = cy_{j-1}$ for $j > i+2$
4) $a_j a_i = cy_{i+1}$ for $j > i+2$

Proof: 1) For $i = 0$ since $im(a_o) = \{y_o, y_1\}$ and $im(a_2) = \{y_1, y_2, y_3\}$. As $y_1 \in im(a_0) \bigcap im(a_2)$. Since y_1 lies in class-1 of $ker(a_2)$, so a_2 map all elements of $ker(a_2)$ to y_1. Therefore, $a_0 a_2 = cy_1$, similarly $a_2 a_1 = cy_1$.
For $i \geq 1$, we have $im(a_i) = \{y_{i-1}, y_i, y_{i+1}\}$ and $im(a_{i+2}) = \{y_{i+1}, y_{i+2}, y_{i+3}\}$ $y_{i+1} \in im(a_i) \bigcap im(a_{i+2})$. As $ker(a_{i+2})$ as $Class-1 = \{y_o, y_1, y_2 \dots y_{i+1}\}$ $Class-2 = \{y_{i+2}, s\}$ $Class-1 = \{y_{i+3}, y_{i+4}, y_{i+5} \dots y_k\}$ Since $y_{i+1)}$ is in the class-1 of $ker(a_{i+2}$ so a_{i+2} maps all the elements of class-1 to cy_{i+1}. Thus $a_i a_{i+2} = cy_{i+1}$ for $i \in \{0,1,2\dots k\}$. Now as $a_{i+1})$ is in class-3 of $ker(a_i)$, so $a_i)$ maps all the elements of class-3 to cy_{i+1}. Thus $a_{i+2} a_i = cy_{i+1}$ for $i \in \{0,1,2\dots k\}$. So $a_i a_{i+2} = a_{i+2} a_i = cy_{i+1}$.
2) For $i = 0$ since y_1 is in class-2 of $ker(a_1)$ so $a_1)$ will map all elements of class-2 to $y_{(2)}$.

Thus $a_o a_1 = cy_2$. Similarly, $a_1 a_0 = cy_2$. For $i \geq 1$ we have a new transformation say a_{n1} with $im(a_{n1}) = \{y_i, y_{i+1}\}$. Similarly For $i \geq 1$ we have another a new transformation say a_{n2} with $im(a_{n2}) = \{y_i, y_{i+1}\}$. Hence $a_i a_{i+1} \neq a_{i+1} a_i$.

3) Since class-1 of $ker(a_j)$ contains $im(a_i)$. So a_j will maps all elements of class-1 to y_{j-1}. Thus $a_i a_j = cy_{j-1}$.

4) Since class-3 of $ker(a_i)$ contains $im(a_j)$. So a_i will maps all elements of class-3 to y_{i+1}. Thus $a_j a_i = cy_{i+1}$.

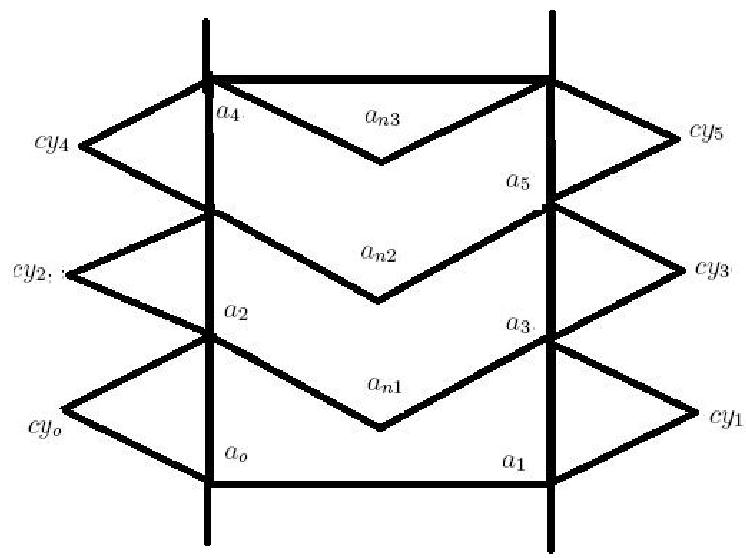

FIGURE 2. Sunflower Like Graph

4. Semigroup of almost complete type graphs

Definition 4.1. Let $X = \{y_o, y_1, y_2 \ldots y_k, s\}$ be a finite set. Consider the idempotents $a_o, a_1, a_2 \ldots a_k$ transformations with $im(a_o) = \{y_o, y_1\}$, $im(a_1) = \{y_1\}$ and $im(a_2) = \{y_o, y_1, y_2\}$, for $i \in \{3, 4, 5 \ldots k\}$, $im(a_i) = \{y_o, y_1, y_2 \ldots y_i\}$.

Now we define the kernel classes of these transformations as; $ker(a_o)$ as $Class-1 = \{y_o, y_2, y_3 \ldots y_k, s\}$ $Class-2 = \{y_1\}$, $ker(a_1)$ as $ker(a_1) = X$, $ker(a_2)$ as $Class-1 = \{y_o\}$, $Class-2 = \{y_1\}$, $Class-2 = \{y_2, y_3 \ldots y_k, s\}$ $\forall \ i \geq 3$, we define $ker(a_i)$ as

$Class-1 = \{y_o, y_1\}$, $class-2 = \{y_2\}$, $class-3 = \{y_3\}$.......... $Class-k = \{y_i, y_{i+1}, y_{i+2} \ldots y_k, s\}$.

Lemma 4.1. The products are defined as,

1) $a_ia_i = a_i$, $a_oa_1 = cy_1$, $a_1a_0 = cy_1$, $a_oa_2 = a_o$, $a_2a_o = a_o$, $a_1a_2 = cy_1$, $a_2a_1 = cy_1$.

2) For $3 \leq i < j < k$, we have, $a_ia_j = a_i$, $a_ja_i = a_i$.

Proof:

1) First statement is true because generators are idempotents. Since $y_1 \in im(a_o) \bigcap im(a_1)$. So $a_oa_1 = cy_1$, similarly $a_1a_0 = cy_1$. Since class-1 and class-2 of $ker(a_2)$ is in $im(a_o)$ so $a_oa_2 = a_o$, similarly $a_2a_o = a_o$. Finally, since class-2 of $ker(a_2)$ is in $im(a_1)$ so $a_1a_2 = cy_1$ and as a_1 is constant therefore $a_2a_1 = cy_1$.

2) Since for all $3 \leq i < j$ the kernel classes of $ker(a_j)$ are in the $im(a_i)$ therefore $a_ia_j = a_i$ similarly, $a_ja_i = a_i$.

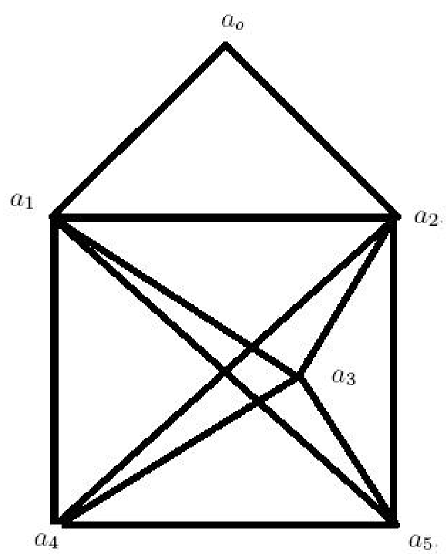

FIGURE 3. Almost Complete Type Graph

5. Semigroup of almost ladder type graphs

Definition 5.1. Let $X = \{y_o, y_1, y_2 \dots y_k, s\}$ be a finite set.Consider the idempotents $a_o, a_1, a_2 \dots a_k$ transformations with $im(a_o) = \{y_o, y_1\}$, $im(a_1) = \{y_1\}$.

For $i \in \{2,3,4,5\dots k\}$ $im(a_i) = \{y_{i-2}, y_{i-1}, y_i\}$.

Now we define the kernel classes of these transformations as;

$ker(a_o)$ as $Class-1=\{y_o,y_2,y_3\dots y_k,s\}$, $Class-2=\{y_1\}$, $ker(a_1)$ as $ker(a_1)=X$

Now $\forall\ i\geq 2$, we define $ker(a_i)$ as

$Class-1=\{y_o,y_1\cdots y_{i-2}\}$,

$class-2=\{y_{i-1}\}$,

$Class-3=\{y_i,y_{i+1},y_{i+2}\dots y_{i+k},s\}$.

Lemma 5.1. The products of the above semigroups are defined as;

1) $a_ia_1=cy_2$ for $i\in\{0,1,2\dots k\}$

2) $a_1a_i=cy_2$ for $i\in\{0,2,3\}$ and $a_1a_i=cy_{i-1}$ for $i\geq 4$

3) For $i\geq 2$, $a_ia_{i-1}=a_{i-1}a_i$ and $a_ia_{i-2}=a_{i-2}a_i$

Proof: 1) First statement is obvious because a_1 is a constant transformation.

2)For $i\in\{0,2,3\}$ the second kernel class of a_i lie in the $im(a_1)$. Therefore all the elements of this class will map on cy_2 and hence all the the elements of X will map on cy_2. Thus $a_1a_i=cy_2$

Similarly, for $i\geq 4$, the first kernel class of a_i lie in the $im(a_1)$. Therefore all the elements of this class will map on cy_{i-2} and hence all the the elements of X will map on cy_{i-2}. Thus $a_1a_i=cy_{i-1}$.

3)For $i\geq 2$, The first and third kernel classes of a_{i-1} contains the first and third kernel classes of a_i. Therefore the products of a_ia_{i-1} and $a_{i-1}a_i$ will be same and thus $a_ia_{i-1}=a_{i-1}a_i$. Similarly, $a_ia_{i-2}=a_{i-2}a_i$. During the study of this paper, we observe the following Conjecture.

Conjecture: There are no semigroups of a circle and straight line.

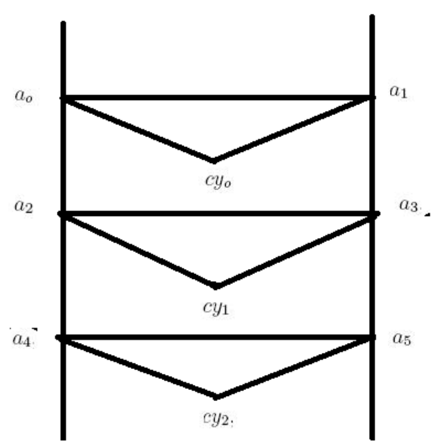

FIGURE 4. Almost Ladder Type Graph

Conflict of Interests

The authors declare that there is no conflict of interests.

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