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ON LARGE ARBITRARY LEFT PATH WITHIN A SEMIGROUP

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Abstract. This paper is about the construction of semigroups S from some given graph G. Let S be a finite non-commutative semigroup, its commuting graph, denoted by $G(S)$, is a simple graph (which has no loops and multiple edges) whose sets of vertices are elements of S and whose sets of edges are those elements of S which commute with other elements i.e. for any $a,b \in S$ such that $ab = ba$ for $a \neq b$. For some non empty finite set X,denote $T(X)$ by semigroup of full transformations and I_r by ideal of $T(X)$ whose rank is less than or equals to r. Let $a_1 - a_2 - a_3 - \ldots - a_m$ be a path in $G(S)$, this path is said to be left path or $l-$ path if

$$a_1 a_i = a_m a_i \text{ for } i \in \{1,2,3,\ldots m\}$$

In this paper, we construct semigroup S of a complete bipartite graph $K_{n,m}$ and find maximum length of $l-$ path in its commuting graph $G(S)$. Moreover,we see that such type of semigroups have knit degree 2.

Keywords: commuting graph; $l-$ path; ideal; semigroup of full transformations; knit degree.

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1. Introduction

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J. Arajo, Kinyon M. and Konieczny give a construction of band (semigroups of idempotents) in which one can find semigroup of any knit degree n, for some positive integer n, except $n = 3$ [1]. The construction of such type of semigroups also helps for finding semigroups for every $n \geq 2$ such that the diameter of commuting graph $G(S)$ is n. On the other hand, finding out the semigroups from a given graph is very important and difficult task in the theory of semigroups. In our paper, we find the semigroup of complete bipartite graph which will be the partly answer, to some extent, of problem given in [1].

Let S be a finite non-commutative semigroup whose centre is defined as $Z(S) = \{a \in S : ab = ba \quad \forall \quad b \in S\}$. The commuting graph of a finite non-commutative semigroup is a simple graph whose sets of vertices are from $S - Z(S)$ and whose sets of edges are the elements of S which commute with other elements i.e. for any $a, b \in$ S such that $ab = ba$ for $a \neq b$. This paper is actually the construction of band (of course non commutative) from a graph and finding out $l-$ path of length n for any even positive number n within the commuting graph of a semigroup. For the construction of such type of semigroups, our main focus will be on semigroup of full transformations $T(X)$ for a finite set X and $l-$ path in a commuting graph of $G(T(X))$.

Let S be a finite non commutative semigroup and $a_1 - a_2 - a_3 - \ldots - a_m$ is a path in $G(S)$, this path is said to be left path or $l-$ path if $a_1 a_i = a_m a_i$ for $i \in \{1, 2, 3, \ldots m\}$. The length of minimal $l-$ path is called the knit degree of the semi group S, denoted by $Kd(S)$. We also see that in such type of semigroups, $Kd(S) = 2$ and the maximum length of $l-$ path in its commuting graph is $2m - 2$, for any positive number m.

Let $T(X)$ be a semigroup of full transformations for a finite set X under the composition of function. Actually the semigroups $T(X)$ is the set of all functions from a finite set X to X. In this paper we consider transformations $a, b \in T(X)$ and define composition of functions as $(ab)(x) = a(b(x))$ from right instead of left i.e. $(x)(ab) = ((x)a)b$ for $x \in X$.

Suppose that $G(S)$ is commuting graph of some non-commutative semigroup of S, then $G(S) = (V, E)$ where V is a finite vertex set and and E is a set of edges such that $E \subseteq \{\{u, v\} : u, v \in V for\ u \neq v\}$. If $v_1, v_2, \ldots v_k$ are the vertices in $G(S)$ then we write a path λ from v_1 to v_k as $\lambda = v_1 - v_2 - \ldots v_k$ of length $k - 1$. We say that λ is minimal path if there does not exist any path shorter than λ.

A bipartite graph is a simple graph in which vertex set $V(G)$ can be divided into two sets named as V_1 and V_2 such that if $v \in V_1$ it may only be adjacent to the vertices of V_2 and similarly if $v \in V_2$ it may only be adjacent to the vertices of V_1. Moreover $V_1 \bigcap V_2 = \emptyset$ and $V_1 \bigcup V_2 = V(G)$. A complete bipartite graph $K_{m,n}$ is bipartite graph that has each vertex set from one set adjacent to each vertex to another set. In this paper, we only consider complete bipartite graphs in which the orders of the both sets are the same.

For $a \in T(X)$ we write image of a by $im(a)$ and kernel of a is defined as

$$ker(a) = \{(x,y) \in X \times X : a(x) = a(y)\}$$

and rank of a as $rank(a) = |im(a)|$

Also $T(X)$ has n ideals $I_1, I_2 \dots I_n$ where $1 \leq r \leq n$

$$I_r = \{a \in T(X) : rank(a) \leq r\}$$

Clearly the ideal I_1 is of rank 1 i.e. a constant transformation and hence its commuting graph will be isolated vertices.

Definition 1.1. Let S be a semigroup and $e \in S$ is an idempotent if $e^2 = e$. Also we define the sets of idempotents in S to be $E(S) = \{e \in S : e^2 = e\}$ Now, $E(S)$ may be empty or it may be $E(S) = S$. If $E(S) = S$ then S is a band. We construct band in our construction in the monoid $T(X)$.

Definition 1.2. Let $e \in T(X)$ be an idempotent and $\{A_1, A_2, \dots A_k\}$ be a partition of X and unique elements $x_1 \in A_1, x_2 \in A_2, ..x_k \in A_k$ such that for every i we have $A_i e = \{x_i\}$. Then the set $\{x_1, x_2, \dots x_k\}$ is the image set of e.We use the following notation for e,

$$e = (A_1, x_1\rangle (A_2, x_2\rangle \dots (A_k, x_k\rangle$$

If e is a constant transformation with image set $\{x\}$ then we write $(X, x\rangle$ [1].

Definition 1.3. Let $e = (A_1, x_1\rangle (A_2, x_2\rangle \dots (A_k, x_k\rangle$ an idempotent in $T(X)$ and let $b \in T(X)$ then b commutes with e if and only if for every $i \in \{1, 2 \dots k\}$ there is a $j \in \{1, 2 \dots k\}$ such that $bx_i = x_j$ and $bA_i \subseteq A_j$ [1].

Definition 1.4. Let $e, f \in I_r$ be idempotents and suppose there is $x \in X$ such that $x \in im(e) \bigcap im(f)$ then $e - (X, x\rangle - f$ [1]

2. Semigroup of complete bipartite graph

Lemma 2.1. Let $c_x, c_y, e \in T(X)$ such that e is an idempotent, then

(1) $c_x e = e c_x$ if and only if $x \in im(e)$

(2) $c_x e = c_y e$ if and only if $(x,y) \in ker(e)$

Proof: (1) Consider $c_x e = e c_x$. As c_x and e commute with each other, therefore, there should be at least one element common in the images of c_x and e but c_x has only one element in the image set i.e x in $im(c_x)$. So $x \in im(c_x) \bigcap im(e)$ or $x \in \{x\} \bigcap im(e)$. This implies that $x \in im(e)$. Conversely, suppose that $x \in im(e)$. We can write it as $x \in \{x\} \bigcap im(e)$. This implies that $x \in im(c_x) \bigcap im(e)$. Thus we have $c_x e = e c_x$.

(2) Consider $c_x e = c_y e$. As $ker(e)$ is defined as $ker(e) = \{(x,y) \in X \times X : xe = ye\}$. Consider $c_x e = c_z$ and $c_y e = c_t$ for some t and z in$X$. Thus $c_z = c_t \Rightarrow z = t$ and hence $ze = te$. Therefore $(x,y) \in ker(e)$. Conversely, let $(x,y) \in ker(e)$ then by def. of $ker(e)$ we have $xe = ye$, implies $c_x e = c_y e$.

Definition 2.1. Let $K \geq 2$ be an integer and $X = \{y_1, y_2, y_3, \ldots y_{2k}, s\}$. We define the idempotent $a_1, a_2, \ldots a_k, b_1, b_2, \ldots, b_k$ as follows:

For $i \in \{1, 2, \ldots 2k\}$. Consider, $im(a_i) = \{y_1, y_2, y_3, \ldots. y_{2k}\}$ and $im(b_i) = \{y_{2i}\}$

For $i \in \{1, 2, \ldots k\}$. There will be $2k$ kernel classes of each $ker(a_i)$.

We define $ker(a_i)-$ classes by,

$$ker(a_1) = \begin{cases} \{y_1, s\}, \\ \{y_2\}, \{y_3\} \ldots \{y_{2k}\} \end{cases}$$

For $i \geq 2$

$$ker(a_i) = \begin{cases} \{y_{i+j}, s\}, & \text{if } j = 1, 2, 3, \ldots \\ \{y_i\}, \ldots \{y_{2k}\}, & \text{if } j \neq 1, 2, 3, \ldots \end{cases}$$

Also

$ker(b_i)$=X

Theorem 2.1. (Cayley's Theorem)

Every semigroup S can be embedded into the semigroup of full transformations $T(X)$ i.e there will be subsemigroups $S^{'}$ of $T(X)$ such that $S \simeq S^{'}$. For proof see [7].

Example 2.1. Let $S = \{a,b,c,d,e,f\}$ be a semi group and its cayley's table is as follows:

$*$	a	b	c	d	e	f
a	a	b	c	d	e	f
b	b	b	b	b	b	b
c	a	b	c	d	e	f
d	d	d	d	d	d	d
e	a	b	c	d	e	f
f	f	f	f	f	f	f

To make the transformations of the above semigroup S we add identity i in S and get the following set of transformations $S^{'}$.

$S^{'} = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6\}$ Where $X = \{a,b,c,d,e,f,i\}$ we partitioning X as:

$A_1 = \{a,i\}$ $A_2 = \{b\}$ $A_3 = \{c\}$ $A_4 = \{d\}$ $A_5 = \{e\}$ $A_6 = \{f\}$

$\phi_1 = \langle A_1, \{a\}\rangle\langle A_2, \{b\}\rangle\langle A_3, \{c\}\rangle\langle A_4, \{d\}\rangle\langle A_5, \{e\}\rangle\langle A_6, \{f\}\rangle$

$A_1 = \{a\}$ $A_2 = \{b,i\}$ $A_3 = \{c\}$ $A_4 = \{d\}$ $A_5 = \{e\}$ $A_6 = \{f\}$

$\phi_2 = \langle A_1, \{a\}\rangle\langle A_2, \{b\}\rangle\langle A_3, \{c\}\rangle\langle A_4, \{d\}\rangle\langle A_5, \{e\}\rangle\langle A_6, \{f\}\rangle$

$\vdots$

$A_1 = \{a\}$ $A_2 = \{b\}$ $A_3 = \{c\}$ $A_4 = \{d\}$ $A_5 = \{e\}$ $A_6 = \{f,i\}$

$\phi_6 = \langle A_1, \{a\}\rangle\langle A_2, \{b\}\rangle\langle A_3, \{c\}\rangle\langle A_4, \{d\}\rangle\langle A_5, \{e\}\rangle\langle A_6, \{f\}\rangle$

The transformations defined in this way and by Theorem 2.3, S is isomorphic to $S^{'}$ because S is semigroup.i.e $S \simeq S^{'}$

Lemma 2.2. The products of elements of the semigroups are defined as:

Let $1 \leq i < j \leq k$ then

(1) $a_i a_i = a_i$ and $b_i b_i = b_i$

(2) $a_i a_j = a_j$ and $a_j a_i = a_i$

(3) $b_i b_j = cy_i$ and $b_j b_i = cy_j$

(4) $a_i b_i = cy_i$ and $b_i a_i = cy_i$

(5) $a_i b_j = cy_j$ and $a_j b_i = cy_i$

(6) $b_i a_j = cy_i$ and $b_j a_i = cy_j$

Proof: (1) Statement is true because the generators are idempotents. (2) Since each kernel class of $ker(a_j)$ except $\{y_i, s\}$ contains both $im(a_{j-1}) = \{y_1, y_2, \ldots, y_{2k}\}$ and $im(a_i)$. Since $y_{j-1} \in im(a_j)$, therefore, a_j will map elements of each kernel class except $\{y_i, s\}$ to y_j. Thus $a_i a_j = a_j$. Similarly each kernel class of $ker(a_i)$ except $\{y_i, s\}$ contains both $im(a_{i+1}) = \{y_1, y_2, \ldots, y_{2k}\}$ and $im(a_j)$. Since $y_i \in im(a_i)$, therefore, a_i maps element of each kernel class except $\{y_i, s\}$ to y_i. Thus $a_j a_i = a_i$. Therefore both transformations a_i and a_j are making right zero semi groups so they will never commute with each other. (3) Both b_i and b_j are constant transformations and therefore they will map on a single element of X and never commute with each other. The proof of other products are similar as proved in (2). Thus $S =< a_i, b_j >$ will be our required semigroup verified by GRAPE which is a package of GAP[10].

3. Left paths (l-path) in $G(K_{m,m})$

Lemma 3.1. Let $a_i,\ a_j,\ b_i,\ b_j \in G(K_{m,n})$ for $m = n$ *and* $i < j$ and $c_b \in G(K_{m,n})$ such that $a_i - c_b - a_j$ is a path in $G(K_{m,n})$, then $a_i a_j \neq a_j a_i$ and $b_i b_j \neq b_j b_i$.

Proof: Since $a_i - c_b - a_j$ is a path in $G(K_{m,n})$. By Lemma 2.1, we have, $a_i c_b = c_b a_i$ and $a_j c_b = c_b a_j$

$\Rightarrow b \in im(a_i) \bigcap im(c_b)$ and $b \in im(a_j) \bigcap im(c_b) \Rightarrow b \in im(a_i) \bigcap im(b_j)$. But by Lemma 2.2 (2) we have, $a_i a_j = a_j$ and $a_j a_i = a_i$. Therefore $a_i a_j \neq a_j a_i$.That is, the transformations a_i and a_j form right zero semigroups, therefore, they will never commute with each other. By Lemma 2.2

(3), implies $b_ib_j \neq b_jb_i$.That is, any two different constant transformations never commute with each other.

Lemma 3.2. Let $K_{m,n}$ for $m = n$ be a complete bipartite graph, then the paths
$\Pi_1 = a_i - b_k - a_j$ with $i < k < j$ and
$\Pi_2 = a_1 - b_1 - a_2 - \cdots - b_{n-1} - a_n$
are the only $l - paths$ in $G(K_{m,n})$ with endpoints not constants.

Proof: Consider the path $\Pi_1 = a_i - b_k - a_j$. By definition of $l-$ path we have, $a_1a_i = a_ma_i\ \forall\{1,2,3,\ldots,m\}$ and products defined in Lemma 2.2, we can show that $a_ia_i = a_i,\ a_ib_k = b_k,\ a_ia_j = a_j$ and on the other hand, $a_ja_i = a_i,\ a_jb_k = b_k,\ a_ja_j = a_j$. So the path $\Pi_1 = a_i - b_k - a_j$ with $i < k < j$ is a $l - paths$ in $G(K_{m,n})$ and it is minimum left path in $G(K_{m,n})$ of length 2 thus, $Kd(S) = 2$.
Now consider the path $\Pi_2 = a_1 - b_1 - a_2 - \cdots - b_{n-1} - a_n$. Again by definition of l-path and by Lemma 2.2 we have $a_1a_1 = a_1,\ a_1a_j = a_j$ and $a_1a_n = a_n$, $a_1b_1 = b_1,\ a_1b_j = b_j$ and $a_1b_{n-1} = b_{n-1}$
On the other hand $a_na_1 = a_1$, $a_na_i = a_i,\ a_nb_i = b_i$ and $a_na_n = a_n$
Therefore the path $\Pi_2 = a_1 - b_1 - a_2 - \cdots - b_{n-1} - a_n$ is also a $l - paths$ in $G(K_{m,n})$.

Lemma 3.3. There is no $l-$ path of odd length in $G(K_{m,n})$ for $m = n$.

Proof: A path of odd length will be in following two cases.
1)path starts from a_i and ends at b_j for some $i < j$ or
2)path starts from b_i and ends at a_j for some $i < j$
In both cases, we see that by Lemma 3.2 above two paths are not $l-$ paths because for $l-$ path starting and end points should be non-constants.

Proposition 3.1. For any even positive number m, there is a $l-$ path of length m in $G(K_{m,n})$ for $m = n$ and maximum length of $l-$ path will not be more than $2m - 2$.

Proof: For $m \geq 2$, the maximum length of a path in $G(K_{m,n})$ is $2m - 1$. The path of length $2m - 1$ is not $l-$ path by Lemma 3.3. The maximum even length of a path in $G(K_{m,n})$ is $2m - 2$ for $m \geq 2$. By Lemma 3.2, this path is $l-$ path in $G(K_{m,m})$ because its end points are not constants. Thus maximum length of $l-$ path in $G(K_{m,n})$ will not be more than $2m - 2$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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