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EXISTENCE AND REGULARITY FOR SOME PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. In this work, we study the existence and regularity of solutions for some partial functional integrodifferential equations with infinite delay in Banach spaces. Firstly, we show the existence of the mild solutions. Secondly, we give sufficient conditions ensuring the existence of the strict solution. The method used treats the equations in the domain of A with the graph norm employing results from linear semigroup theory. To illustrate our abstract result, we conclude this work with an application.

Keywords: mild and strict solutions; partial functional integrodifferential equations; C_0-semigroup, infinitesimal generator; infinite delay; phase space.

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1. Introduction

In this work, we study the existence and regularity of solutions for the following partial functional integrodifferential equation with infinite delay

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$$
\text{(1.1)} \qquad \begin{cases} u'(t) & = Au(t) + \displaystyle\int_0^t \alpha(t-s, u(s))\, ds + f(t, u_t) \quad \text{for} \quad t \geq 0, \\ u_0 & = \varphi \in \mathscr{P}, \end{cases}
$$

where $A : \mathscr{D}(A) \subset \mathscr{X} \to \mathscr{X}$ is the infinitesimal generator of a linear semigroup in a Banach space $\mathscr{X}$, α is in general a nonlinear operator from $\mathbb{R}^+ \times \mathscr{D}(A)$ to $\mathscr{X}$, $f : \mathbb{R}^+ \times \mathscr{P} \to \mathscr{X}$ is a continuous function and the phase space $\mathscr{P}$ is a linear space of functions mapping $]-\infty, 0]$ into $\mathscr{D}(A)$ endowed with the graph norm namely for $x \in \mathscr{D}(A)$, $|x|_{\mathscr{D}(A)} = |x| + |Ax|$ then $\left(\mathscr{D}(A), |.|_{\mathscr{D}(A)}\right)$ is a Banach space, for every $t \geq 0$, the history function $u_t \in \mathscr{P}$ is defined by

$$
u_t(\theta) = u(t+\theta) \quad \text{for} \quad \theta \in]-\infty, 0].
$$

As in [36], we consider a nonlinear Volterra integrodifferential equation of parabolic type

$$
\text{(1.2)} \qquad \begin{cases} \dfrac{\partial}{\partial t} w(t,x) = \dfrac{\partial^2}{\partial x^2} w(t,x) + \displaystyle\int_0^t k\left(t-s, \dfrac{\partial^2}{\partial x^2} w(s,x)\right) ds + h(t,x), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad t > 0 \quad \text{and} \quad 0 < x < 1, \\ w(t,0) = w(t,1) = 0, \quad \text{for} \quad t > 0, \\ w(0,x) = w_0(x), \quad \text{for} \quad 0 < x < 1. \end{cases}
$$

The abstract version of the initial boundary value problem (1.2) is given by

$$
\text{(1.3)} \qquad \begin{cases} u'(t) = Au(t) + \displaystyle\int_0^t \alpha(t-s, u(s))\, ds + F(t), \quad \text{for} \quad t \geq 0, \\ u(0) = x \in \mathscr{X}. \end{cases}
$$

Some results are proved concerning local existence, global existence, continuous dependence upon initial values and asymptotic stability for $Eq.(1.3)$ under some suitable assumptions. A vast literature has investigated this equation in various aspects. $Eq.(1.3)$ has many physical applications and arises in such problems as heat flow in materials with memory [7], [8]. As a model see $Eq.(1.2)$. For his study, we also refer the reader to [[3], [9], [23], [28]].

Partial functional differential equations arise in a variety of areas of biological, physical and engineering applications, see, for example, the books and the papers in the following references [[18], [19], [26], [31], [37]] , [17, 29] and the references therein. Recently, the following differential equations with delay have been studies by many authors ([35], and references therein):

$$(1.4)\qquad \begin{cases} u'(t) = Au(t) + F(t, u_t), \quad \text{for} \quad t \in [0, T], \\ u_0 = \varphi \in \mathscr{P}. \end{cases}$$

There has been a great deal of work contributed to the study of partial differential equations with delay by using different methods under different conditions. The most classical work is due to Travis and Webb [35].

The investigation of functional differentials with infinite delay in an abstract admissible phase space was initiated by Hale and Kato [20], Kappel and Schappacher [24], and Schumacher [34]. The method of using admissible phase space enables one to treat a large class of functional differential equations with infinite delay at the same time and obtain general results. For a detailed discussion on this topic, we refer to the book by Hino and al. [22].

$Eq.(1.1)$ is the mixed type of $Eq.(1.3)$ and $Eq.(1.4)$. It well enable us to study the nonlinear Volterra integrodifferential equation with delay. On the basis of the results in $Eq.(1.4)$ we generalize the method used in [36] to derive global existence and regularity of $Eq.(1.1)$. The result obtained is a generalization and a continuation of [36]. The method used treats the equations in the domain of A with the graph norm employing results from linear semigroup theory concerning abstract inhomogeneous linear differential equations. In our work the nonlinear term is treated as a perturbation of the linear equation.

The organization of this work is as follows, in Section 2, we recall some preliminary results about $Eq.(1.3)$ and $Eq.(1.4)$. Some basic notations and assumptions are also given in this section. In Section 3, we prove global existence and regularity of solution to $Eq.(1.1)$ which are the main results of this paper. Moreover, some properties of solutions are also studied. In Section 4, we give an example of application to show that our results valuable.

2. Preliminary results

In this section, we recall some fundamental results needed to establish our results. Throughout the paper, $\mathscr{X}$ is a Banach space, A is closed linear operator on $\mathscr{X}$. $\mathscr{Y}$ represent the Banach space $\mathscr{D}(A)$ equipped with the graph norm defined by $|y|_{\mathscr{Y}} = |y|_{\mathscr{X}} + |Ay|_{\mathscr{X}}$ for $y \in \mathscr{Y}$. $\mathscr{P}$ is the space of continuous function from $]-\infty, 0]$ to $\mathscr{Y}$. It is well know by the Hille-Yosida theorem that A is the infinitesimal generator of a c_0-semigroup of bounded linear operators in $\mathscr{X}$ if and only if

(i) $\overline{\mathscr{D}(A)} = \mathscr{X}$,

(ii) there exist $M \geq 1$, $w \in \mathbb{R}$ such that for $\lambda > w$, $(\lambda I - A)^{-1} \in \mathscr{B}(\mathscr{X})$ and

$$\left|(\lambda I - A)^{-n}\right| \leq \frac{M}{(\lambda - w)^n} \quad \text{for} \quad \lambda > w \quad \text{and} \quad n \in \mathbb{N},$$

where $\mathscr{B}(\mathscr{X})$ is the space of bounded linear operators on $\mathscr{X}$.

Definition 2.1. A continuous function $u : [0, +\infty[\to \mathscr{D}(A)$ is said to be strict solution of $Eq.(1.3)$ if

(i) $u \in \mathscr{C}^1([0, +\infty[; \mathscr{X}) \cap \mathscr{C}([0, +\infty[; \mathscr{Y})$

(ii) u satisfies $Eq.(1.3)$ for all $t \geq 0$.

Remark 2.2. From this definition, we deduce that $u(t) \in \mathscr{D}(A)$, the function $t \mapsto \alpha(t-s, u(s))$ is integrable for all $t \geq 0$ and $s \in [0, s]$.

Theorem 2.3. [33] If u is a strict solution of $Eq.(1.3)$ then u satisfies

$$u(t) = T(t)x + \int_0^t T(t-s) \int_0^s \alpha(s-r, u(r))drds + \int_0^t T(t-s)F(s)ds. \tag{2.1}$$

Remark 2.4. If u satisfies the formula (2.1) u is not in general a strict solution. That is why we give the definition of the mild solution.

Definition 2.5. A continuous function $u : [0, +\infty[\to \mathscr{D}(A)$ is called a mild solution of $Eq.(1.3)$ if u satisfies the formula (2.1).

3. **Existence and regularity of the solutions for Eq.(1.1)**

In this section, we prove global existence and regularity of solution to $Eq.(1.1)$, which are the main results of this paper. Firstly, we show the existence of the mild solutions. Secondly, we give sufficient conditions ensuring the existence of the strict solutions.

3.1 Global existence of the mild solutions

Definition 3.1. We say that a continuous function $u :]-\infty,+\infty[\to \mathscr{D}(A)$ is a strict solution of $Eq.(1.1)$ if the following conditions hold

(i) $u \in \mathscr{C}^1([0,+\infty[;\mathscr{X}) \cap \mathscr{C}([0,+\infty[;\mathscr{Y})$,

(ii) u satisfies $Eq.(1.1)$ on $[0,+\infty[$,

(iii) $u(\theta) = \varphi(\theta)$ for $-\infty < \theta \leq 0$.

Proposition 3.2. If u is a strict solution of $Eq.(1.1)$, then u is given by

$$\begin{aligned} u(t) &= T(t)\varphi(0) + \int_0^t T(t-s)\int_0^s \alpha(s-r,u(r))drds \\ &\quad + \int_0^t T(t-s)f(s,u_s)ds. \end{aligned} \tag{3.1}$$

Proof. It is just a consequence of Theorem.(2.3). In fact, let us suppose $F(t) = f(t,u_t)$ for $t \geq 0$. Then we get the desired result.

Definition 3.3. We say that a continuous function $u :]-\infty,+\infty[\to \mathscr{D}(A)$ is a mild solution of $Eq.(1.1)$ if u satisfies the formula (3.1) and $u_0 = \varphi$.

In this work, we assume that the phase space $(\mathscr{P}, |.|_{\mathscr{P}})$ is a normed linear space of functions mapping $]-\infty,0]$ into $\mathscr{D}(A)$ and satisfying the following fundamental axioms (cf. Hale and Kato in [20]).

$(\mathbf{A_1})$ There exist positive constant H and functions $K(.), M(.) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $u :]-\infty,\sigma+a] \to \mathscr{D}(A)$, $u_\sigma \in \mathscr{P}$, and $u(.)$ is continuous on $[\sigma,\sigma+a]$, then for every $t \in [\sigma,\sigma+a]$ the following conditions holds

(i) $u_t \in \mathscr{P}$,

(ii) $|u(t)|_{\mathscr{Y}} \leq H\,|u_t|_{\mathscr{P}}$, which is equivalent to $|\varphi(0)|_{\mathscr{Y}} \leq H\,|\varphi|_{\mathscr{P}}$ for every $\varphi \in \mathscr{P}$, (**iii**) $|u_t|_{\mathscr{P}} \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} |u(s)|_{\mathscr{Y}} + M(t-\sigma)\,|u_t|_{\mathscr{P}}$,

($\mathbf{A_2}$) For the function $u(.)$ in ($\mathbf{A_1}$), $t \mapsto u_t$ is a $\mathscr{P}-$valued continuous function for $t \in [\sigma, \sigma + a]$.

(**B**) The space $\mathscr{P}$ is a Banach space.

($\mathbf{H_0}$) A is the infinitesimal generator of a c_0-semigroup $(T(t))_{t\geq 0}$ on $\mathscr{X}$.

($\mathbf{H_1}$) $f : \mathbb{R}^+ \times \mathscr{P} \to \mathscr{D}(A)$ is continuous and lipschitzian with respect to the second argument. Let $L_f > 0$ be such that

$$|f(t,\varphi) - f(t,\hat{\varphi})| \leq L_f\,|\varphi - \hat{\varphi}|_{\mathscr{P}} \quad \text{for all} \quad t \geq 0 \quad \text{and} \quad \varphi, \hat{\varphi} \in \mathscr{P}.$$

($\mathbf{H_2}$) The derivative $\frac{\partial \alpha}{\partial t}(t,u)$ exists and is continuous from $\mathbb{R}^+ \times \mathscr{D}(A)$ into $\mathscr{X}$, moreover there exist two nondecreasing continuous functions $b : \mathbb{R}^+ \to \mathbb{R}^+$ and $c : \mathbb{R}^+ \to \mathbb{R}^+$ such that:

$$|\alpha(s,u_1) - \alpha(s,u_2)| \leq b(s)\,|u_1 - u_2|_{\mathscr{Y}}$$

and

$$\left|\frac{\partial \alpha}{\partial s}(s,u_1) - \frac{\partial \alpha}{\partial s}(s,u_2)\right| \leq c(s)\,|u_1 - u_2|_{\mathscr{Y}}$$

for all $s \in \mathbb{R}+$ and $u_1, u_2 \in \mathscr{Y}$.

Theorem 3.4. Assume that ($\mathbf{H_0}$), ($\mathbf{H_1}$) and ($\mathbf{H_2}$) hold. If $\varphi \in \mathscr{P}$, then there exist a unique continuous function $u :]-\infty, +\infty[\to \mathscr{Y}$ which solves (3.1).

Proof. Let $t_1 > 0$. Define the set $\Gamma_{t_1}(\varphi) := \{u \in \mathscr{C}([0,t_1];\mathscr{Y}) : u(0) = \varphi(0)\}$. $\Gamma_{t_1}(\varphi)$ is a closed subset of $\mathscr{C}([0,t_1];\mathscr{Y})$, where $\mathscr{C}([0,t_1];\mathscr{Y})$ is the space of continuous functions from $[0,t_1]$ to $\mathscr{Y}$ equipped with the uniform norm topology. Next, for each $u \in \Gamma_{t_1}(\varphi)$ we define $\tilde{u}$ its extension over $]-\infty, t_1]$ by

$$\tilde{u}(t) = \begin{cases} \varphi(t) & \text{for} \quad t \in]-\infty, 0], \\ u(t) & \text{for} \quad t \in [0, t_1]. \end{cases}$$

Define the operator $\mathscr{K} : \Gamma_{t_1}(\varphi) \to \mathscr{C}(]-\infty,0],\mathscr{X})$ by

$$(\mathscr{K}u)(t) = T(t)\varphi(0) + \int_0^t T(t-s)\left[\int_0^s \alpha(s-r,\tilde{u}(r))\,dr + f(s,\tilde{u}_s)\right] ds \tag{3.2}$$

The first steep is to show that $\mathscr{K}(\Gamma_{t_1}(\varphi)) \subset \Gamma_{t_1}(\varphi)$. In fact, we have

$$(\mathscr{K}u)(t) = T(t)\varphi(0) + \int_0^t T(t-s)\int_0^s \alpha(s-r,\tilde{u}(r))drds + \int_0^t T(t-s)f(s,\tilde{u}_s)ds \quad 0 \le t \le t_1,$$

and

$$(A\mathscr{K}u)(t) = AT(t)\varphi(0) + A\int_0^t T(t-s)\int_0^s \alpha(s-r,\tilde{u}(r))drds + A\int_0^t T(t-s)f(s,\tilde{u}_s)ds \quad 0 \le t \le t_1.$$

Since A is closed, then

$$\begin{aligned}(A\mathscr{K}u)(t) &= AT(t)\varphi(0) + A\int_0^t T(t-s)\int_0^s \alpha(s-r,\tilde{u}(r))drds \\ &+ \int_0^t T(t-s)Af(s,\tilde{u}_s)ds \quad 0 \le t \le t_1.\end{aligned}$$

For the next, we need the following lemmas.

Lemma 3.5. Let $u : [0,t_1] \to \mathscr{X}$ be continuously differentiable. Assume that $(\mathbf{H_2})$ hold. Then,

$$k(t) = \int_0^t \alpha(t-s,u(s))ds$$

is continuously differentiable from $[0,t_1]$ to $\mathscr{X}$.

Proof. Let $k(t) = \int_0^t \alpha(t-s,u(s))ds$ for all $t \in [0,t_1]$. Let $h > 0$.

$$\begin{aligned}\frac{k(t+h)-k(t)}{h} &= \frac{1}{h}\left[\int_0^{t+h} \alpha(t+h-s,u(s))ds) - \int_0^t \alpha(t-s,u(s))ds\right] \\ &= \frac{1}{h}\int_0^t (\alpha(t+h-s,u(s)) - \alpha(t-s,u(s)))ds + \frac{1}{h}\int_t^{t+h} \alpha(t+h-s,u(s))ds\end{aligned}$$

passing to the limit we obtain
$k'(t) = \dfrac{k(t+h)-k(t)}{h} \longrightarrow \int_0^t \dfrac{\partial}{\partial t}\alpha(t-s,u(s))ds + \alpha(0,u(t))$ when $h \longrightarrow 0^+$.

By virtue of the hypothesis we have placed on α, we see that $k(t)$ is continuously differentiable from $[0,t_1]$ to $\mathscr{X}$.

We require the following Lemma, which is proved in $[25, p.488]$.

Lemma 3.6. [25] Let $k:[0,t_1]\to\mathscr{X}$ be continuously differentiable and q be defined by

$$q(t)=\int_0^t T(t-s)k(s)ds \quad \text{for} \quad t\in[0,t_1].$$

Then $q(t)\in\mathscr{D}(A)$ for $t\in[0,t_1]$, q is continuously differentialble, and

$$Aq(t)=q'(t)-k(t)=\int_0^t T(t-s)k'(s)ds+T(t)k(0)-k(t).$$

By virtue of the hypothesis $(\mathbf{H_2})$ then, by Lemmas 3.5 and 3.6, for $u\in\mathscr{Y}$, we deduce that,

$$\begin{aligned}(A\mathscr{K}u)(t) &= AT(t)\varphi(0)+\int_0^t T(t-s)\alpha\left(0,\tilde{u}(s)\right)ds\\ &\quad+\int_0^t T(t-s)\int_0^s \frac{\partial\alpha}{\partial s}\left(s-r,\tilde{u}(r)\right)drds-\int_0^t \alpha\left(t-s,\tilde{u}(s)\right)ds\\ &\quad+\int_0^t T(t-s)Af(s,\tilde{u}_s)ds \quad 0\le t\le t_1.\end{aligned} \tag{3.3}$$

From the axioms $(\mathbf{A_1}-\mathbf{i}),\mathbf{A_2}$ and assumption $(\mathbf{H_1})$, it follows that the maps $t\mapsto f(t,\tilde{u}_t)$ is continuous. Moreover, from $(\mathbf{H_2})$ and $(\mathbf{A_1})$ we infer that for every $u\in\Gamma_{t_1}(\varphi)$ the function $s\mapsto\alpha(s,\tilde{u})$ is continuous on $[0,t_1]$ and so by assumption $(\mathbf{H_0})$ that $t\mapsto\int_0^t T(s)f(s,\tilde{u}_s)ds$ is continuous on $[0,t_1]$. Thus, for $u\in\Gamma_{t_1}(\varphi)$, $\mathscr{K}u$ and $A\mathscr{K}u$ are both continuous from $[0,t_1]$ to $\mathscr{X}$, $\mathscr{K}$ maps $\Gamma_{t_1}(\varphi)$ into $\Gamma_{t_1}(\varphi)$. Then $\mathscr{K}u\in\mathscr{C}([0,t_1];\mathscr{Y})$ and consequently $\mathscr{K}(\Gamma_{t_1}(\varphi))\subset\Gamma_{t_1}(\varphi)$.

We claim that $\mathscr{K}$ is a strict contraction in $\Gamma_{t_1}(\varphi)$. In fact, let $u,v\in\Gamma_{t_1}(\varphi)$. In fact,

$$\begin{aligned}|(\mathscr{K}u)(t)-(\mathscr{K}v)(t)|_{\mathscr{X}} &\le \left|\int_0^t T(t-s)\int_0^s\left(\alpha(s-r,\tilde{u}(r))-\alpha(s-r,\tilde{v}(r)\right)drds\right|_{\mathscr{X}}\\ &\quad+\left|\int_0^t T(t-s)\left(f(s,\tilde{u}_s)-f(s,\tilde{v}_s)\right)ds\right|_{\mathscr{X}}\\ &\le M\int_0^t e^{w(t-s)}\int_0^s|\alpha(s-r,\tilde{u}(r))-\alpha(s-r,\tilde{v}(r))|_{\mathscr{X}}drds\\ &\quad+M\int_0^t e^{w(t-s)}|f(s,\tilde{u}_s)-f(s,\tilde{v}_s)|_{\mathscr{X}}ds\\ &\le M\int_0^t e^{w(t-s)}\int_0^s|\alpha(s-r,\tilde{u}(r))-\alpha(s-r,\tilde{v}(r))|_{\mathscr{X}}drds\\ &\quad+M\int_0^t e^{w(t-s)}|f(s,\tilde{u}_s)-f(s,\tilde{v}_s)|_{\mathscr{Y}}ds.\end{aligned}$$

Without loss of generality, we assume that $w>0$. By $(\mathbf{H_1})$ and $(\mathbf{H_2})$, we obtain that

$$|(\mathscr{K}u)(t)-(\mathscr{K}v)(t)|_{\mathscr{X}} \le Me^{wt_1}\int_0^t\int_0^s b(s-r)\,|\tilde{u}(r)-\tilde{v}(r)|_{\mathscr{Y}}drds+ML_fe^{wt_1}\int_0^t|\tilde{u}_s-\tilde{v}_s|_{\mathscr{Y}}ds,$$

$$
\begin{aligned}
|(A\mathscr{K}u)(t)-(A\mathscr{K}v)(t)|_{\mathscr{X}} &\leq M\int_0^t e^{w(t-s)}\,|\alpha(0,\tilde{u}(s))-\alpha(0,\tilde{v}(s))|_{\mathscr{X}}\,ds \\
&+M\int_0^t e^{w(t-s)}\int_0^s \left|\frac{\partial\alpha}{\partial s}(s-r,\tilde{u}(r))-\frac{\partial\alpha}{\partial s}(s-r,\tilde{v}(r))\right|_{\mathscr{X}} drds \\
&+\int_0^t |\alpha(t-s,\tilde{u}(s))-\alpha(t-s,\tilde{v}(s))|_{\mathscr{X}}\,ds+M\int_0^t e^{w(t-s)}\,|Af(s,\tilde{u}_s)-Af(s,\tilde{v}_s)|_{\mathscr{Y}}\,ds \\
&\leq Mb(0)e^{wt_1}\int_0^t |\tilde{u}(s)-\tilde{v}(s)|_{\mathscr{Y}}\,ds+Me^{wt_1}\int_0^t\int_0^s c(s-r)\,|\tilde{u}(r)-\tilde{v}(r)|_{\mathscr{Y}}\,drds \\
&+\int_0^t b(t-s)\,|\tilde{u}(s)-\tilde{v}(s)|_{\mathscr{Y}}\,ds+ML_f e^{wt_1}\int_0^t |\tilde{u}_s-\tilde{v}_s|_{\mathscr{Y}}\,ds.
\end{aligned}
$$

Which implies that

$$
\begin{aligned}
|(\mathscr{K}u)(t)-(\mathscr{K}v)(t)|_{\mathscr{Y}} &\leq Mb(0)e^{wt_1}\int_0^t |\tilde{u}(s)-\tilde{v}(s)|_{\mathscr{Y}}\,ds \\
&+Me^{wt_1}\int_0^t\int_0^s [b(s-r)+c(s-r)]\,|\tilde{u}(r)-\tilde{v}(r)|_{\mathscr{Y}}\,drds \\
&+\int_0^t b(t-s)\,|\tilde{u}(s)-\tilde{v}(s)|_{\mathscr{Y}}\,ds+2ML_f e^{wt_1}\int_0^t |\tilde{u}_s-\tilde{v}_s|_{\mathscr{Y}}\,ds.
\end{aligned}
$$

Define

$$
\beta_1(t)=\int_0^t e^{-ws}(b(s)+c(s))ds \quad \text{and} \quad \beta_2(t)=\max_{0\leq s\leq t} e^{-ws}b(s) \quad \text{for} \quad \text{all} \quad t\geq 0.
$$

$$
\begin{aligned}
|(\mathscr{K}u)(t)-(\mathscr{K}v)(t)|_{\mathscr{Y}} &\leq Mb(0)e^{wt_1}\int_0^{t_1} |\tilde{u}(s)-\tilde{v}(s)|_{\mathscr{Y}}\,ds+M\beta_1(t)e^{wt_1}\int_0^{t_1} |\tilde{u}(s)-\tilde{v}(s)|_{\mathscr{Y}}\,ds \\
&+M\beta_2(t)e^{wt_1}\int_0^{t_1} |\tilde{u}(s)-\tilde{v}(s)|_{\mathscr{Y}}\,ds+2ML_f e^{wt_1}\int_0^{t_1} |\tilde{u}_s-\tilde{v}_s|_{\mathscr{Y}}\,ds
\end{aligned}
$$

$$
|(\mathscr{K}u)(t)-(\mathscr{K}v)(t)|_{\mathscr{Y}} \leq Mt_1 e^{wt_1}\left[b(0)+\beta_1(t)+\beta_2(t)+2L_f\right]|\tilde{u}-\tilde{v}|_{\mathscr{Y}}.
$$

If we choose t_1 such that $Mt_1\left[b(0)+\beta_1(t)+\beta_2(t)+2L_f\right]e^{wt_1}<1$, then $\mathscr{K}$ is a strict contraction in $\Gamma_{t_1}(\varphi)$, then by applying the Banach fixed point theorem, we deduce that there exists a unique fixed point $u=u(.,\varphi)$ for $\mathscr{K}$ in $\Gamma_{t_1}(\varphi)$, which implies that $Eq.(1.1)$ has a unique mild solution on $]-\infty,t_1]$. A similar argument can be used for $[t_1,2t_1],...,[nt_1,(n+1)t_1]$, for all $n\geq 0$, which implies that the mild solution exists uniquely in $]-\infty,+\infty[$. This completes the proof. $\diamondsuit$

Proposition 3.7. (Dependence continuous with respect to the initial data)
Assume that $(\mathbf{H_0})$, $(\mathbf{H_1})$ and $(\mathbf{H_2})$ hold. Let $\varphi\in\mathscr{P}$. Then there exist continuous functions $\beta_1:\mathbb{R}^+\to\mathbb{R}^+$ and $\beta_2:\mathbb{R}^+\to\mathbb{R}^+$ such that if u and v satisfy $Eq.(1.1)$ for $0\leq t\leq t_1$ with $u_0=\varphi_1, v_0=\varphi_2$. then

$$\begin{cases} |u_t - v_t|_{\mathscr{Y}} \leq M\,|\varphi_1 - \varphi_2|\, e^{[w+M(b(0)+\beta_1(t)+\beta_2(t)+k)]t} & \text{if} \quad w \geq 0 \\ \\ |u_t - v_t|_{\mathscr{Y}} \leq Me^{-wr}\,|\varphi_1 - \varphi_2|\, e^{\left[w+M(b(0)+\beta_1(t)+\beta_2(t)+k)e^{-wr}\right]t} & \text{if} \quad w < 0, \end{cases}$$

where k is the Lipschitz constant of f.

Proof. Define

$$\beta_1(t) = \int_0^t e^{-ws}(b(s)+c(s))ds \quad \text{and} \quad \beta_2(t) = \max_{0\leq s\leq t} b(s)e^{-ws} \quad \text{for} \quad t \geq 0.$$

Using (3.2) and (3.3) we have

$$\begin{aligned} |u(t)-v(t)|_{\mathscr{X}} &\leq Me^{wt}\,|\varphi_1-\varphi_2|_{\mathscr{P}} + M\int_0^t e^{w(t-s)}\int_0^s |\alpha(s-r,u(r))-\alpha(s-r,v(r))|_{\mathscr{X}}\,drds \\ &+M\int_0^t e^{w(t-s)}\,|f(s,u_s)-f(s,v_s)|_{\mathscr{X}}\,ds \end{aligned}$$

$$\begin{aligned} |u(t)-v(t)|_{\mathscr{X}} &\leq Me^{wt}\,|\varphi_1-\varphi_2|_{\mathscr{P}} + M\int_0^t e^{w(t-s)}\int_0^s |\alpha(s-r,u(r))-\alpha(s-r,v(r))|_{\mathscr{X}}\,drds \\ &+M\int_0^t e^{w(t-s)}\,|f(s,u_s)-f(s,v_s)|_{\mathscr{Y}}\,ds \end{aligned}$$

$$\begin{aligned} |u(t)-v(t)|_{\mathscr{X}} &\leq Me^{wt}\,|\varphi_1-\varphi_2|_{\mathscr{P}} + Me^{wt}\int_0^t e^{-ws}\int_0^s b(s-r)\,|u(r)-v(r)|_{\mathscr{Y}}\,drds \\ &+ML_f e^{wt}\int_0^t e^{-ws}\,|u_s-v_s|_{\mathscr{Y}}\,ds \end{aligned}$$

On the other hand, we have

$$\begin{aligned} |(Au)(t)-(Av)(t)|_{\mathscr{X}} &\leq Me^{wt}\,|A(\varphi_1-\varphi_2)|_{\mathscr{P}} \\ &+M\int_0^t e^{w(t-s)}\left[|\alpha(0,u(s))-\alpha(0,v(s))|_{\mathscr{X}} + \int_0^s \left|\frac{\partial\alpha}{\partial s}(s-r,u(r)) - \frac{\partial\alpha}{\partial s}(s-r,v(r))\right|_{\mathscr{X}} dr\right] \\ &+\int_0^t |\alpha(t-s,u(s))-\alpha(t-s,v(s))|_{\mathscr{X}}\,ds + M\int_0^t e^{w(t-s)}\,|Af(s,u_s)-Af(s,v_s)|_{\mathscr{X}}\,ds \end{aligned}$$

$$\begin{aligned}|(Au)(t)-(Av)(t)|_{\mathscr{X}} &\leq Me^{wt}\,|A(\varphi_1-\varphi_2)|_{\mathscr{P}}\\ &+M\int_0^t e^{w(t-s)}\left[b(0)\,|u(s)-v(s)|_{\mathscr{Y}}+\int_0^s c(s-r)\,|u(r)-v(r)|_{\mathscr{Y}}\,dr\right]ds\\ &+\int_0^t b(t-s)\,|u(s)-v(s)|_{\mathscr{Y}}\,ds+ML_f\int_0^t e^{w(t-s)}\,|u_s-v_s|_{\mathscr{Y}}\,ds.\end{aligned}$$

$$\begin{aligned}|u(t)-v(t)|_{\mathscr{Y}} &\leq Me^{wt}\,|\varphi_1-\varphi_2|+Me^{wt}\int_0^t e^{-ws}\int_0^s(b(s-r)+c(s-r))\,|u(r)-v(r)|_{\mathscr{Y}}\,drds\\ &+Mb(0)e^{wt}\int_0^t e^{-ws}\,|u(s)-v(s)|_{\mathscr{Y}}\,ds+Me^{-wt}\int_0^t b(t-s)\,|u(s)-v(s)|_{\mathscr{Y}}\,ds\\ &+2ML_fe^{wt}\int_0^t e^{-ws}\,|u_s-v_s|_{\mathscr{Y}}\,ds\end{aligned}$$

$$\begin{aligned}|u(t)-v(t)|_{\mathscr{Y}} &\leq Me^{wt}\,|\varphi_1-\varphi_2|+Me^{wt}\beta_1(t)\int_0^t|u(s)-v(s)|_{\mathscr{Y}}\,ds+Mb(0)e^{wt}\int_0^t e^{-ws}\,|u(s)-v(s)|_{\mathscr{Y}}\,ds\\ &+M\beta_2(t)\int_0^t|u(s)-v(s)|_{\mathscr{Y}}\,ds+2ML_fe^{wt}\int_0^t e^{-ws}\,|u_s-v_s|_{\mathscr{Y}}\,ds\end{aligned}$$

$$|u(t+\theta)-v(t+\theta)|_{\mathscr{Y}}\leq\begin{cases}|\varphi_1-\varphi_2| \quad \text{if} \quad t+\theta\leq 0,\\ Me^{w(t+\theta)}\,|\varphi_1-\varphi_2|+Me^{w(t+\theta)}\beta_1(t+\theta)\int_0^{t+\theta}|u(s)-v(s)|_{\mathscr{Y}}\,ds\\ +Mb(0)e^{w(t+\theta)}\int_0^{t+\theta}e^{-ws}\,|u(s)-v(s)|_{\mathscr{Y}}\,ds\\ +M\beta_2(t+\theta)\int_0^{t+\theta}|u(s)-v(s)|_{\mathscr{Y}}\,ds\\ +2ML_fe^{w(t+\theta)}\int_0^{t+\theta}e^{-ws}\,|u_s-v_s|_{\mathscr{Y}}\,ds, \quad \text{if} \quad t+\theta\geq 0.\end{cases}$$

If $w \geq 0$, then

$$e^{-wt}|u_t - v_t|_{\mathscr{Y}} \leq M|\varphi_1 - \varphi_2| + M\beta_1(t)\int_0^t e^{-ws}|u_s - v_s|_{\mathscr{Y}}\,ds + Mb(0)\int_0^t e^{-ws}|u_s - v_s|_{\mathscr{Y}}\,ds$$

$$+M\beta_2(t)\int_0^t e^{-ws}|u_s - v_s|_{\mathscr{Y}}\,ds + 2ML_f\int_0^t e^{-ws}|u_s - v_s|_{\mathscr{Y}}\,ds$$

$$e^{-wt}|u_t - v_t|_{\mathscr{Y}} \leq M|\varphi_1 - \varphi_2| + M\left[b(0) + \beta_1(t) + \beta_2(t) + 2L_f\right]\int_0^t e^{-ws}|u_s - v_s|_{\mathscr{Y}}\,ds.$$

If $w < 0$, then

$$e^{-wt}|u_t - v_t|_{\mathscr{Y}} \leq Me^{-wr}|\varphi_1 - \varphi_2| + Me^{-wr}\left[b(0) + \beta_1(t) + \beta_2(t) + 2L_f\right]\int_0^t e^{-ws}|u_s - v_s|_{\mathscr{Y}}\,ds.$$

By Gronwall's Lemma, the result follows.

Proposition 3.8. Suppose the hypothesis of Theorem 3.4 and $\varphi \in \mathscr{P}$. Suppose there exist constants β_1^0 and β_2^0 such that $\int_0^t e^{-ws}(b(s)+c(s))ds \leq \beta_1^0$, $b(t)e^{-wt} \leq \beta_2^0$ for $t \geq 0$, and $M(\beta_1^0 + \beta_2^0 + b(0) + k) + w =^{def} \lambda < 0$ for some $w < 0$. Then the solutions of $Eq.(1.1)$ are exponentially asymptotically stable in the following sens: if u, v are the solutions of $Eq.(1.1)$ for $u_0 = \varphi_1, v_0 = \varphi_2$, respectively, then

$$|u_t - v_t|_{\mathscr{Y}} \leq Me^{-wr}|\varphi_1 - \varphi_2|e^{\lambda t}, \quad \text{for} \quad t \geq 0.$$

Proof. The proof following Proposition 3.7 by obseving that $\beta_1(t)$ and $\beta_2(t)$ satisfy $\beta_1(t) \leq \beta_1^0$ and $\beta_2(t) \leq \beta_2^0$. ◊

3.2 Existence of strict solutions

In this section we recall some fundamental results needed to establish our results. The following results were established in [30]. We consider the inhomogeneous initial value problem

$$\text{(3.4)} \qquad \begin{cases} u'(t) = Au(t) + F(t) & \text{for} \quad t \geq 0, \\ u(0) = x \in \mathscr{X} \end{cases}$$

where $F : [0, a] \longrightarrow \mathscr{X}$, be continuous.

Theorem 3.9. [30] Let A be the infinitesimal generator of a c_0-semigroup $(T(t))_{t\geq 0}$. let $F \in L^1(0,a;\mathscr{X})$ be continuous on $[0,a]$ and let

$$v(t)=\int_0^t T(t-s)F(s)ds, \quad t\in[0,a].$$

The $Eq.(3.4)$ has a solution u on $[0,a]$ for every $x\in\mathscr{D}(A)$ if one of the following conditions is satisfied;

(1) $v(t)$ is continuously differentiable on $[0,a]$.

(2) $v(t)\in\mathscr{D}(A)$ for $t\in[0,a]$ and $Av(t)$ is continuous on $[0,a]$.

If $Eq.(3.4)$ has a strict solution u on $[0,a]$ for some $x\in\mathscr{D}(A)$ then v satisfies both (1) and (2).

Theorem 3.10. Let $u\in\mathscr{C}([0,t_1];\mathscr{D}(A))$ the mild solution be defined by the formula (3.1). If $u_0\in\mathscr{D}(A)$ and $f\in L^1(\mathbb{R}^+\times\mathscr{C};\mathscr{D}(A))$ be continuous from $\mathbb{R}^+\times\mathscr{C}$ to $\mathscr{D}(A)$, then u is a strict solution of $Eq.(1.1)$.

Proof. It is just a consequence of Theorem 3.9. In fact, let us suppose

$$v(t)=\int_0^t T(t-s)\int_0^s \alpha(s-r,u(r))drds+\int_0^t T(t-s)f(s,u_s)ds \quad \text{for} \quad t\geq 0.$$

We show that v satisfies the following two conditions

(i) v is continuously differentiable on $[0,t_1]$ and v' is continuous,

(ii) $v(t)\in\mathscr{D}(A)$ on $[0,t_1]$ and $Av\in L^1([0,t_1];\mathscr{X})$.

Based on the formula (3.1) we have: $v(t)=u(t)-T(t)\varphi(0)$ is differentiable for $t>0$ as the difference of two such differentiable functions and $v'(t)=u'(t)-T(t)A\varphi(0)$ is obviously continuous on $]0,t_1[$. Therefore (i) is satisfied. Also if $\varphi\in\mathscr{D}(A)$ $T(t)\varphi\in\mathscr{D}(A)$ for $t\geq 0$ and therefore $v(t)=u(t)-T(t)\varphi(0)\in\mathscr{D}(A)$ for $t>0$ and $Av(t)=Au(t)-AT(t)\varphi=u'(t)-\int_0^t\alpha(t-s,u(s))ds-f(t,u_t)-T(t)A\varphi$ is continuous on $]0,t_1[$. Thus also (ii) is satisfaied.

On the other hand, it is easy to verify for $h>0$ the identify

$$\begin{aligned}\left(\frac{T(h)-I}{h}\right)v(t) &= \frac{v(t+h)-v(t)}{h}\\ &-\frac{1}{h}\int_t^{t+h} T(t+h-s)\,[k(s)+f(s,u_s]\,ds.\end{aligned}\tag{3.5}$$

From the continuity of k and f it is clear that the second therm on the right-hand side of (3.5) has the limit $k(t)+f(t,u_t)$ as $h\to 0$. If $v(t)$ is continuously differentiable on $]0,t_1[$ then it follows from (3.5) that $v(t)\in\mathscr{D}(A)$ for $0<t<t_1$ and $Av(t)=v'(t)-[k(t)+f(t,u_t)]$. Since $v(0)=0$ it follows that $u(t)=T(t)\varphi(0)+v(t)$ is the solution of $Eq.(1.1)$ for $\varphi(0)\in\mathscr{D}(A)$. If $v(t)\in\mathscr{D}(A)$ it follows from (3.5) that $v(t)$ is differentiable from the right at t and the right derivative $\mathscr{D}^+v(t)$ of v satisfies $\mathscr{D}^+v(t)=Av(t)+k(t)+f(t,u_t)$. Since $\mathscr{D}^+v(t)$ is continuous, $v(t)$ is continuously differentiable and $v'(t)=Av(t)+k(t)+f(t,u_t)$. Since $v(0)=0$, $u(t)=T(t)\varphi(0)+v(t)$ is the solution of $Eq.(1.1)$ for $\varphi\in\mathscr{D}(A)$ and the proof is complete.

4. **Application**

For illustration, we propose to study the existence of solutions for the following model

$$\begin{cases}\dfrac{\partial}{\partial t}w(t,x) &= \dfrac{\partial^2}{\partial x^2}w(t,x)+\displaystyle\int_0^t \gamma(t-s,\frac{\partial^2}{\partial x^2}w(s,x))ds\\ &+\displaystyle\int_{-\infty}^0 h(\theta,w(t+\theta,x))d\theta \quad \text{for} \quad t\geq 0 \quad \text{and} \quad 0\leq x\leq 1,\\ w(t,0) &= w(t,1)=0 \quad \text{for} \quad t\geq 0,\\ w(\theta,x) &= \varphi_0(\theta,x) \quad \text{for} \quad \theta\in]-\infty,0] \quad \text{and} \quad 0\leq x\leq 1,\end{cases}\tag{4.1}$$

where $h:\mathbb{R}^-\times\mathbb{R}\to\mathbb{R}$ is continuous and Lipschitzian with respect to the second argument, the initial data function $\varphi_0:]-\infty,0]\times[0,1]\to\mathbb{R}$ is a given function, $\gamma:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ is bounded uniformly continuous, continuously differentiable in its first place and the derivative $\frac{\partial\gamma}{\partial t}$ exists and is lipschitzian continuous.

To rewrite $Eq.(4.1)$ in the abstract from, we introduce the space $\mathscr{X}=L^2((0,1);\mathbb{R})$.

Let $A : \mathscr{D}(A) \to \mathscr{X}$ be defined by

$$\begin{cases} \mathscr{D}(A) = H^2(0,1) \cap H_0^1(0,1), \\ \\ Az = z''. \end{cases}$$

It is well known that A is the generator of c_0-semigroup, which implies that $(\mathbf{H_0})$ is satisfied.

Let $\alpha : \mathbb{R}^+ \times \mathscr{D}(A) \to \mathscr{X}$ by $\alpha(t,z) = \gamma(t,Az)$ for $t \geq 0$.

The phase space $\mathscr{P} = BUC(\mathbb{R}^-; \mathscr{D}(A))$ is the space of bounded uniformly continuous functions from $\mathbb{R}^-$ into $\mathscr{D}(A)$ provided with the following norm

$$|\varphi|_{\mathscr{P}} = \sup_{\theta \leq 0} |\varphi(\theta)|_{\mathscr{D}(A)} = \sup_{\theta \leq 0} |\varphi(\theta)|_{L^2(0,1)} + \sup_{\theta \leq 0} \left| \frac{\partial^2}{\partial x^2} \varphi(\theta) \right|_{L^2(0,1)}.$$

Then $\mathscr{P} = BUC(\mathbb{R}^-; \mathscr{D}(A))$ satisfies axioms $(\mathbf{A_1})$, $(\mathbf{A_2})$.

Let $f : \mathbb{R}^+ \times \mathscr{P} \to \mathscr{X}$ be defined by

$$f(t,\varphi)(x) = \int_{-\infty}^{0} h(\theta, \varphi(\theta)(x))d\theta \quad \text{for} \quad 0 \leq x \leq 1 \quad \text{and} \quad t \geq 0.$$

Let us suppose $v(t) = w(t,.)$ and the initial data φ be defined by

$$\varphi(\theta)(x) = \varphi_0(\theta, x), \quad \text{for} \quad \theta \leq 0 \quad \text{and} \quad x \in [0,1].$$

Then $Eq.(4.1)$ takes the following abstract from

$$\begin{cases} \frac{d}{dt} v(t) = Av(t) + \int_0^t \alpha(t-s, v(s))ds + f(t, v_t) \quad \text{for} \quad t \geq 0, \\ \\ v_0 = \varphi. \end{cases} \tag{4.2}$$

We suppose that there exists a function $b_1(.) \in L^1(\mathbb{R}^-, \mathbb{R}^+)$ such that

$(\mathbf{H_3})$

$$|h(\theta, x_1) - h(\theta, x_2)| \leq b_1(\theta)|x_1 - x_2| \quad \text{for} \quad \theta \leq 0 \quad \text{and} \quad x_1, x_2 \in \mathbb{R}.$$

$(\mathbf{H_4})$ $h(\theta, 0) = 0, \quad \text{for} \quad \theta \leq 0.$

Assumptions $(\mathbf{H_3})$ and $(\mathbf{H_4})$ imply that $f(t,\varphi) \in \mathscr{D}(A)$ for $\varphi \in \mathscr{P}$. In fact, let $\varphi \in \mathscr{P}$. Then

$$f(t,\varphi)(x) = \int_{-\infty}^{0} h(\theta, \varphi(\theta)(x))d\theta \quad \text{for} \quad x \in [0,1],$$

and

$$\begin{aligned}
|f(t,\varphi)(x)| &\leq \int_{-\infty}^{0} b_1(\theta)\,|\varphi(\theta)(x)|\,d\theta \\
&\leq \sup_{\theta\leq 0}|\varphi(\theta)(x)|\int_{-\infty}^{0} b_1(\theta)d\theta \\
|f(t,\varphi)(x)|_{L^2(0,1)} &\leq \sup_{\theta\leq 0}|\varphi(\theta)|_{L^2(0,1)}\,|b_1|_{L^1(-\infty,0)}\,.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|Af(t,\varphi)(x)| &\leq \int_{-\infty}^{0} b_1(\theta)\,|A\varphi(\theta)(x)|\,d\theta \\
&\leq \sup_{\theta\leq 0}|A\varphi(\theta)(x)|\int_{-\infty}^{0} b_1(\theta)d\theta \\
|Af(t,\varphi)(x)|_{L^2(0,1)} &\leq \sup_{\theta\leq 0}|A\varphi(\theta)|_{L^2(0,1)}\,|b_1|_{L^1(-\infty,0)}\,.
\end{aligned}$$

Consequently

$$|f(t,\varphi)(x)|_{\mathscr{D}(A)} \leq \mathsf{C}\,|\varphi|_{\mathscr{P}}\,.$$

Moreover assumption ($\mathbf{H_4}$) implies that $f(t,\varphi)(0) = f(t,\varphi)(1) = 0$.

Using the dominated convergence theorem, one can show that $f(t,\varphi)$ is a continuous function on $[0,1]$. Moreover, for every $\varphi_1,\varphi_2 \in \mathscr{P}$, we have

$$\begin{aligned}
|(f(t,\varphi_1)-f(t,\varphi_2))(x)| &\leq \int_{-\infty}^{0} |h(\theta,\varphi_1(\theta)(x)) - h(\theta,\varphi_2(\theta)(x))|\,d\theta \\
&\leq \int_{-\infty}^{0} b_1(\theta)\,|\varphi_1(\theta))(x) - \varphi_2(\theta)(x)|\,d\theta \\
|(f(t,\varphi_1)-f(t,\varphi_2))(x)|_{L^2(0,1)} &\leq \sup_{-\infty\leq\theta\leq 0}|\varphi_1(\theta)(x)-\varphi_2(\theta)(x)|\,|b_1|_{L^1(\mathbb{R}^-)}
\end{aligned}$$

On the other hand, we have

$$|A(f(t,\varphi_1)-f(t,\varphi_2))(x)| \leq \int_{-\infty}^{0} b_1(\theta)\,|A(\varphi_1(\theta))(x)-\varphi_2(\theta))(x)|\,d\theta$$

$$|A(f(t,\varphi_1)-f(t,\varphi_2))(x)|_{L^2(0,1)} \leq \sup_{-\infty\leq\theta\leq 0} |A(\varphi_1(\theta)(x)-\varphi_2(\theta))(x)|\,|b_1|_{L^1(\mathbb{R}^-)}$$

Consequently

$$|f(t,\varphi_1)-f(t,\varphi_2)|_{\mathscr{D}(A)} \leq \mathsf{C}\,|\varphi_1-\varphi_2|_{\mathscr{P}}\,.$$

We conclude that f is Lipschitz continuous.

In addition, we suppose that

(i) γ is bounded uniformly continuous, continuously differentiable in its first place and the derivative $\frac{\partial\gamma}{\partial t}$ exists and is lipschitzian continuous.

(ii) The initial data $\varphi \in \mathscr{P} = BUC\left(]-\infty,0]\times[0,1];\mathscr{D}(A)\right)$,

$\varphi_0(0,0)=\varphi_0(0,1)=0$ is continuous from $]-\infty,0]\times[0,1]$ to $\mathscr{D}(A)$.

From the assumption (i), α satisfies the hypothesis $(\mathbf{H_2})$. Finally, from assumption (ii) and Theorem 3.4, we deduce that $\varphi\in\mathscr{P}, Eq.(4.2)$ has a unique mild solution which is defined for all $t\geq 0$.

To prove that the mild solution of $Eq.(4.2)$ is a strict one, we need the following assumption.

(iii) $h\in L^1(\mathbb{R}^-\times\mathbb{R};\mathbb{R})$ be continuous on $\mathbb{R}^-\times\mathbb{R}$.

(iv) $\varphi_0\in\mathscr{P}$ such that $\varphi_0(0,.)\in\mathscr{D}(A)$. Consequently, by Theorem 3.10, we obtain the following existence result.

Proposition 4.1. Under the above assumptions, $Eq.(4.2)$ has a unique strict solution v and the solution u defined by $u(t,x)=v(t)(x)$ for $t\geq 0$ and $x\in[0,1]$ is a solution $Eq.(4.2)$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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