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SEMIGROUP THEORY APPLIED TO BASKET OPTIONS

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Abstract. In this paper, a generalised solution of the multi-dimensional Black-Scholes partial differential equation corresponding to basket options is proposed. By certain transformations this equation can be treated as a generalised heat equation. Semigroup theory techniques are used to discuss its solution.

Keywords: European option; Basket option; Black Scholes equation; Co-semigroups.

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1. Introduction

In present financial markets, prices fluctuate continuously. An increasingly common strategy to keep up with the competition is to use computational models to gain some insight into what the future holds. As the technology advances, these methods are expected to perform even better and are increasingly well trusted by market analysts.

An option is a security giving the 'right' to buy or sell an asset, subject to certain conditions, within a specified period of time. An 'European option' is one that can be exercised only on a

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specified future date. The price that is paid for the asset when the option is exercised is called the ’exercise price’ or ’strike price’. The last day on which the option may be exercised is called the ’expiry date’ or ’maturity date’.

At the expiry time T, the value $C(x,t)$ of this option is given by
$C(x,T) = max(x-E,0)$
where E is the strike price and x is the price of the underlying asset. Since the holder of this option is not obliged to exercise it, the value is never less than zero.

The value of an option depends on at least five factors. They are the expiry date, the strike price, the interest rate, the price of the underlying asset and the voltality of this asset.

The voltality is the standard deviation of the change in value of an asset within a given time period. This is, in some sense, a measure of the risk associated with the option, since a high voltality indicates a higher uncertainity of the value of the asset at the expiry date.

A call option is the right to buy a particular asset for an agreed amount at a specified time in future. A put option is the right to sell a particular asset for an agreed amount at a specified time in future. Since the theory for the pricing of put option is identical to that of the call option, apart from the final value, only the latter will be addressed in this paper. For more details on option theory refer to [5]

2. Black-scholes equation in one dimension

In [1], Black and Scholes proved that under certain natural assumptions about the market place, the value of a European option C, as a function of the current value of the underlying asset x and time t, such that $C = C(x,t)$ verifies the following cauchy problem:

$$\frac{\partial c}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 c}{\partial x^2} + rx\frac{\partial c}{\partial x} - rc = 0, x \geq 0, t \in [0,T] \tag{2.1}$$

with
$C(0,t) = 0$

$C(x,t) \sim x$ as $x \to \infty$

$$C(x,T) = max(x-E,0), x \in (0,\infty) \tag{2.2}$$

where the value of the call option also depends on the voltality of the underlying asset σ, the exercise price E, the expiry T and the interest rate r. In this work r and σ are constants. An explicit solution for (2.1) and (2.2) can be found in [5]

3. Basket option model

A basket option has two or more underlying assets and the holder of the option has the right to purchase a specified amount of these at the expiry date. When determinig the value of such an option using the Black-Scholes model, the dimensionality of the problem grows linearly with the number of underlying assets.

With multi - asset basket options, on a d-dimensional basket of d assets, equation (2.1) and (2.2) generalises to

$\frac{\partial c}{\partial t} + \frac{1}{2}\Sigma_{i,j=1}^{d} \sigma_i \sigma_j p_{ij} x_i x_j \frac{\partial^2 c}{\partial x_i \partial x_j} + r\Sigma_{i=1}^{d} x_i \frac{\partial c}{\partial x_i} - rc = 0$

with

$C(x_1, x_2, ...x_d, T) = max(\Sigma_{i=1}^{d} \mu_i x_i - k, 0)$

where x_i is the price of the underlying asset i, μ_i is the weight factor of the asset i and p_{ij} is the correlation between the assets i and j.

This final value problem is usually transformed into a well-posed initial boundary value problem by reversion of the time axis $T \to T-t$ and an introduction of the boundary conditions on x_i

Then we arrive at the multi-dimensional Black-Scholes equation

$$\frac{\partial c}{\partial \tau} - \frac{1}{2}\Sigma_{i,j=1}^{d} \sigma_i \sigma_j p_{ij} x_i x_j \frac{\partial^2 c}{\partial x_i \partial x_j} - r\Sigma_{i=1}^{d} x_i \frac{\partial c}{\partial x_i} + rc = 0 \tag{3.1}$$

with

$$C(x_1, x_2, ...x_d, 0) = max(\Sigma_{i=1}^{d} \mu_i x_i - k, 0), \quad for \quad (x_1, x_2, ...x_d) \in R_+^d \tag{3.2}$$

4. Reduction of multi-dimension Black-Scholes equation

We consider a transformation which is closely related to principal component analysis of the covariance matrix [4]

Transform (3.1) into the eigen system of the covariance matrix

$\Sigma_{ij} = p_{ij}\sigma_i\sigma_j$

by an orthogonal transformation with Q bringing Σ to diagonal form

$Q\Sigma Q^T = diag(\lambda i)$

To receive this we transform using

$z = Qy$

and obtain

$\frac{\partial c}{\partial t} - \Sigma_{i,j=1}^{d} q_{ij}(r - \frac{1}{2}\sigma_i^2)\frac{\partial c}{\partial z_i} - \frac{1}{2}\Sigma_{i=1}^{d}\lambda_i \frac{\partial^2 c}{\partial z_i^2} + rc = 0$

for every $(z,t) \in R^d \times (0,T)$

with initial condition

$C(z,0) = max(\Sigma_{i=1}^{d}\mu_i e^{\Sigma_{j=1}^{d} q_{ji} x_j} - k, 0)$

Perform an additional translation

$x = z + tb$

with

$b_i = \Sigma_{j=1}^{d} q_{ij}(r - \frac{1}{2}\sigma_j^2)$

Take $v = e^{rt}$ to arrive at the heat equation

$$\frac{\partial c}{\partial t} - \frac{1}{2}\Sigma_{i=1}^{d}\lambda_i \frac{\partial^2 c}{\partial x_i^2} = 0, \quad for \quad every \quad (x,t) \in R^d \times (0,T) \tag{4.1}$$

with initial condition

$$C(x,0) = max(\Sigma_{i=1}^{d}\mu_i e^{\Sigma_{j=1}^{d} q_{ji} x_j}, 0) \tag{4.2}$$

Throughout this paper, we will deal with (4.1) and (4.2) which are equivalent to (3.1) and (3.2) respectively.

5. Main results

In this section, we develop the mathematical tools to charecterize the ’generalized’ solutions of (4.1) and (4.2) using the theory of Co-semigroups.

Define the operators

$A_k : D(A_k) \to L^2(R^d)$ and

$B_k : D(B_k) \to L^2(R^d)$ by

$$D(A_k) = \left\{ f \in L^2(R^d) : \text{f is absolutely continuous with } \frac{\partial f}{\partial x_k} \in L^2(R^d) \right\}$$

$$D(B_k) = \left\{ f \in L^2(R^d) : f, \frac{\partial f}{\partial x_k} \text{are absolutely continuous with } \frac{\partial f}{\partial x_k}, \frac{\partial^2 f}{\partial x_k^2} \in L^2(R^d) \right\}$$

$$A_k f = \frac{\partial f}{\partial x_k}, B_k f = \frac{\partial^2 f}{\partial x_k^2}$$

Observe that $B_k = A_k^2$

We recall some basic facts in the Co-semigroup theory.

A family $T = T(t)_{t \geq 0}$ of bounded linear operators from a Banach space X into itself is called a Co-semigroup on X if

(i) $T(0) = I$, the identity operator on X

(ii)$T(t+s) = T(t)T(s)$ for all $t, s \geq 0$

(iii)$lim_{t \to 0^+} T(t)x = x$ for all $x \in X$

The family $S(t)_{t \in R}$ of bounded linear operators from X into itself is called a Co-group, if

(i) $S(0) = I$, the identity operator on X

(ii)$S(t+s) = S(t)S(s)$ for all $t, s \in R$

(iii)$lim_{t \to 0} S(t)x = x$ for all $x \in X$

The infinitesimal generator of Co-semigroup $T = (T(t))_{t \geq 0}$ is the operator given by

$D(G) = \left\{ x \in X : lim_{t \to 0^+} \frac{1}{t}(T(t)x - x) \in X \right\}$

$Gx = lim_{t \to 0^+} \frac{1}{t}(T(t)x - x), x \in D(G)$

Growth bound of the Co-semigroup $T = (T(t))_{t \geq 0}$ is given by

$w_0(T) = inf\{w \in R : \exists \ M > 0 : \|T(t)\| \leq Me^{\omega t}\}$

Further we have

$w_0(T) = lim_{t \to \infty} \frac{ln\|T(t)\|}{t} = inf_{t > 0} \frac{ln\|T(t)\|}{t}$

We also need the following properties of a Co-semigroup $T = (T(t))_{t\geq 0}$

(i)$\{\lambda \in C : Re\lambda > \omega_0(T)\} \subset \rho(G)$ and

(ii)$R(\lambda, G)x = \int_0^\infty e^{-\lambda t} T(t)x \ dt$ for all $x \in X$

For more details on semi-group theory, refer [3]

Following [2], it can be observed that A_k is the infinitesimal generator of the Co-group

$(S_k(t)f)(x) = f(x_1, x_2, \ldots\ldots x_{k-1}, x_k + t, \ldots\ldots x_d)$

where $x = (x_1, x_2, \ldots\ldots x_d) \in R^d$, $t \in R$

Further B_k is the infinitesimal generator of a Co-semigroup T_k given by

$(T_k(t)f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-s^2}{4t}} f(x_1, x_2, \ldots\ldots x_{k-1}, x_k + s, \ldots\ldots x_d) ds$

for all $t > 0$, $x \in R^d$, $f \in L^2(R^d)$.

Obviously, the semigroup $(S_k(t))_{t\geq 0}$ commute as do the resolvents of A_k and hence of $B_k = A_k^2$. Then the semigroups $(T_k(t))_{t\geq 0}$ generated by B_k, $k = 1$ to d are analytic and commute. Then $T(t) = T_1(\lambda_1, t)T_2(\lambda_2, t), \ldots.. T_d(\lambda_d, t)$ is a bounded semigroup.

Observe that λ_i, $i = 1$ to d are the eigen values of $\sum_{ij}$ and are positive.

We now obtain the generalised solution of (4.1)

Theorem 5.1. *If $f \in L^2(R^d)$, then the function given by $C_f : R_+^d \times R \to C$ given by $C_f(x,t) = \frac{1}{2}(T(t))f)(x)$ is a solution of (4.1)*

Proof.

Clearly the domain $D(A)$ of the generator A of $(T(t))_{t\geq 0}$ contains

$D(\lambda_1 A_1^2) \cap D(\lambda_2 A_2^2) \cap \ldots\ldots \cap D(\lambda_d A_d^2)$

In particular it contains

$D_0 = \{f \in L^2(R^d) / D^\alpha f \in L^2(R^d)$ for every multi index $\alpha \ \ with |\alpha| \leq 2\}$

and for every $f \in D_0$ the generator is given by

$Af = (\lambda_1 A_1^2 + \ldots\ldots + \lambda_d A_d^2)f = \sum_{k=1}^d \lambda_k \frac{\partial^2 f}{\partial x_k^2}$

Since, $\frac{\partial c_f}{\partial t} = \frac{\partial}{\partial t}(\frac{1}{2}T(t)f) = \frac{1}{2}Af$, it follows

$\frac{\partial c_f}{\partial t} - \frac{1}{2}\sum_{k=1}^{d}\lambda_k \frac{\partial^2 c_f}{\partial x_k^2} = 0$ as desired.

We reproduce Lemma 2.1 of [2] in the generalised form.

Lemma 5.2.

If $g \in C(R^d \times R_+, C)$ such that $\bigcup_{t\geq 0} supp(g(.,t))$ is relatively compact, then the map $h : R_+ \to L^2(R^d)$ defined by $h(t) = g(.,t)$ is continuous, where $supp(g)$ denotes the support of the function g.

Now (ii) of theorem 2.1 of [2] gives the following theorem.

Theorem 5.3.

If c is a solution of (4.1) which has in addition the property that $c(.,t)$, $\frac{\partial c}{\partial x}(.,t)$, $\frac{\partial^2 c}{\partial x^2}(.,t) \in L^2(R^d)$ and $sup_{t\in[0,a]}\|\frac{\partial^2 c}{\partial x^2}(.,t)\| < \infty$ for all $a > 0$, then
$c(x,t) = (T(t)c_0)(x)$, $(x,t) \in R_+^d - \{0\} \times R$ where $c_0 = c(.,0)$

Conflict of Interests

The author declare that there is no conflict of interests.

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