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CATEGORY OF ASYNCHRONOUS SYSTEMS AND POLYGONAL MORPHISMS

AHMET HUSAINOV

Computer Technology Faculty, Komsomolsk-on-Amur State Technical University, Komsomolsk-on-Amur, Russian Federation

Abstract. A *weak asynchronous system* is a trace monoid with a partial action on a set. A *polygonal* morphism between weak asynchronous systems commutes with the actions and preserves the independence of events. We prove that the category of weak asynchronous systems and polygonal morphisms has all limits and colimits.

Keywords: trace monoid; partial monoid action; limits; colimits; asynchronous transition system.

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1. Introduction

Mathematical models of parallel systems find numerous applications in parallel programming. They are applied for the development and verification of programs, searching for deadlocks and estimation of runtime. These models are widely applied to the description of semantics and the development of languages of parallel programming [23].

There are various models of parallel computing systems [24]. For example, for the solution of the dining philosophers problem, it is convenient to use higher dimensional automata [10], but

E-mail address: husainov51@yandex.ru

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for a readers/writers problem, it is better to consider asynchronous systems [18]. For comparing these models adjoint functors have been constructed [13, 14, 15, 26].

For comparing higher dimensional automata and asynchronous transition systems described in [2] there are open problems concerning the existence of colimits. Here we propose a construction of a cocomplete category of asynchronous systems which avoids this problem and allows comparisons using adjoint functors in a standard way.

Often, a parallel composition of models constructed as the pullback in the category of these models [3, 4, 16]. Therefore, the incompleteness makes difficult to study a construction of parallel composition. We prove that our category of asynchronous systems is complete.

An asynchronous system is a model of a computing system consisting of events (instructions, machine commands) and states. The states are defined by values of variables (or cells of memory). Some events can occur simultaneously. The category of asynchronous systems was first studied by M. Bednarczyk [2] and the work was further developed in [4].

We will consider an asynchronous system as a set with partial trace monoid action. Partial maps provide some difficulties in the study of asynchronous systems. A possible way out of this situation is the modelling the computer systems using functors with values in restriction categories defined in [7].

But we will follow to [26] and shall represent the partial maps as total maps by adding an element $*$ to the sets. A set with partial trace monoid action is considered as a trace monoid acting on a pointed set. Morphisms between trace monoids acting on the pointed sets lead to *polygonal* morphisms of asynchronous systems.

These morphisms have great value for studying homology groups of the asynchronous systems, introduced in [18]. They also help in studying homology groups of the Mazurkiewicz trace languages and Petri nets [19]-[20].

We believe that our complete and cocomplete category of asynchronous systems will be very useful in the study of concurrent processes.

We describe the contents of the paper. The first Section is current. In the second Section, the category $FPCM$ of trace monoids and basic homomorhisms is investigated. It is proved that, in this category, there are limits (Theorem 2.7) and colimits (Theorem 2.9) although even finite products do not coincide with Cartesian products. The subcategory $FPCM^{\|} \subset FPCM$ with independence preserving morphisms is studied. It is proved that this subcategory is complete (Theorem 2.16) and cocomplete (Theorem 2.17). In the third Section, the conditions of existence of limits and colimits in a category of diagrams with values in a fixed category are studied. The fourth Section is devoted to a category of weak asynchronous systems and polygonal morphisms. Main results about completeness and cocompleteness of a category of weak asynchronous systems and polygonal morphisms are proved (Theorems 4.17 and 4.18). The fifth Section contains concludes.

2. Categories of trace monoids

The basis for trace monoid theory was developed in [6]. Applications to computer science were described by A. Mazurkiewicz [22], V. Diekert and Y. Métivier [9]

We shall consider a trace monoid category and basic homomorphisms and its subcategory consisting of independence preserving homomorphisms. We want to prove the existence of limits and colimits of diagrams in these categories.

Trace monoids. A map $f : M \to M'$ between monoids is a *homomorphism*, if $f(1) = 1$ and $f(\mu_1\mu_2) = f(\mu_1\mu_2)$ for all $\mu_1, \mu_2 \in M$. Denote by *Mon* the category of all monoids and homomorphisms.

Let E be an arbitrary set. An *independence relation* on E is a subset $I \subseteq E \times E$ satisfying the following conditions:

- $(\forall a \in E)\ (a,a) \notin I$ (irreflexivity),
- $(\forall a,b \in E)\ (a,b) \in I \Rightarrow (b,a) \in I$ (symmetry).

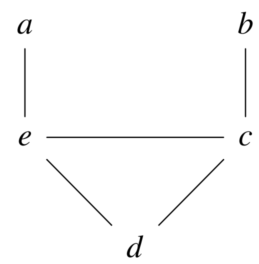
Elements $a,b \in E$ are *independent*, if $(a,b) \in I$.

Let E^* be the free monoid of all words $a_1a_2\cdots a_n$ where $a_1,a_2,\cdots,a_n \in E$ and $n \geq 0$, with operation of concatenation $(a_1\cdots a_n)(b_1\cdots b_m) = a_1\cdots a_nb_1\cdots b_m$. The identity 1 is the empty word.

Let I be an independence relation on E. We define the equivalence relation $\equiv_I$ on E^* putting $v \equiv_I w$ for $v,w \in E^*$, if there exists a finite sequence $u_1,u_2,\ldots,u_k$ of words such that $v = u_1$, $w =$

u_k, and for i, $1 \leq i < k$, there exist words u'_i, u''_i, and letters a_i, b_i satisfying: $u_i = u'_i a_i b_i u''_i$, $u_{i+1} = u'_i b_i a_i u''_i$, and $(a_i, b_i) \in I$.

For example, for the set $E = \{a,b,c,d,e\}$ and for the relation I given by the adjacency graph drawn in the following figure



the sequence of permutations

$$eadcc \overset{(e,a)}{\to} aedcc \overset{(e,d)}{\to} adecc \overset{(e,c)}{\to} adcec \overset{(d,c)}{\to} acdec \overset{(e,c)}{\to} acdce \overset{(d,c)}{\to} accde$$

shows that $adecc \equiv_I accde$.

For every $w \in E^*$, the equivalence class $[w]$ is called the *trace* of w.

Let E be a set and let I be an independence relation. We denote by $M(E,I)$ the monoid of equivalence classes $[w]$ of all $w \in E^*$ with the operation $[w_1][w_2] = [w_1 w_2]$ for $w_1, w_2 \in E^*$.

We emphasize that the set E can be infinite.

In some cases, we omit the square brackets in the notations for elements of $M(E,I)$. If $I = \emptyset$, then $M(E,I)$ is equal to the *free monoid* E^*. If $I = ((E \times E) \setminus \{(a,a) | a \in E\})$, then $M(E,I)$ is the *free commutative monoid*. In this case, we denote it by $M(E)$.

Definition 2.1. A monoid M is called a trace monoid if there exists a set E with an independence relation I such that M is isomorphic to $M(E,I)$.

Let M be a trace monoid. An element $\mu \neq 1$ is *indecomposable* if for all $\mu_1, \mu_2 \in M$ satisfying $\mu_1 \mu_2 = \mu$, we have $\mu_1 = 1 \vee \mu_2 = 1$. Denote by E the set of indecomposable elements. The set E generates the monoid M. We let $I = \{(a,b) \in E \times E |\ ab = ba \ \&\ a \neq b\}$. It is easy to see that $M = M(E,I)$.

Thus, each trace monoid equals to some monoid $M(E,I)$ where E consists of indecomposable elements of the trace monoid and $I = \{(a,b) \in E \times E |\ ab = ba \ \&\ a \neq b\}$.

Example 2.2. Let $M(E_1, I_1)$ and $M(E_2, I_2)$ be trace monoids. We consider their Cartesian product $M(E_1, I_1) \times M(E_2, I_2)$. The monoid $M(E_1, I_1) \times M(E_2, I_2)$ is generated by the set of indecomposable elements $E = E_1 \times \{1\} \cup \{1\} \times E_2$. The independence relation equals

$$I = \{((e_1,1),(1,e_2)) \mid e_1 \in E_1,\ e_2 \in E_2\} \cup \{((1,e_2),(e_1,1)) \mid e_1 \in E_1,\ e_2 \in E_2\} \cup$$
$$\cup \{((e_1,1),(e'_1,1)) \mid (e_1,e'_1) \in I_1\} \cup \{((1,e_2),(1,e'_2)) \mid (e_2,e'_2) \in I_2\}.$$

The Cartesian product $M(E_1,I_1) \times M(E_2,I_2)$ is isomorphic to $M(E,I)$. We see that the Cartesian product of trace monoids is a trace monoid.

The category of trace monoids and basic homomorphisms. Let us introduce basic homomorphisms and we shall show that the category of trace monoids and basic homomorphisms is complete and cocomplete.

Definition 2.3. A homomorphism $f : M(E,I) \to M(E',I')$ is *basic* if $f(E) \subseteq E' \cup \{1\}$.

If $w = e_1 \cdots e_n \in M(E,I)$ for some $e_1 \in E$, ..., $e_n \in E$, then n is called the *length* of the trace w. It is easy to see that a homomorphism is basic if and only if it does not increase the length of elements of $M(E,I)$. Let $FPCM$ be the category of trace monoids and basic homomorphisms.

Consider the problem of the existence of products in $FPCM$. The Cartesian product $M(E_1,I_1) \times M(E_2,I_2)$ does not have universal property in the category $FPCM$, and therefore it is not a product in $FPCM$. For building products and other constructions, we shall consider partial maps as total maps between pointed sets obtained as follows.

For each set E, we take an element $*_E$ such that $*_E \notin E$. By the axiom of regularity, we have $E \notin E$. Hence, we can take $*_E = E$. The element $*_E$ will be denoted by $*$.

Let $E_* = E \cup \{*\}$. We assign to each partial map $f : E_1 \rightharpoonup E_2$, a total map $f_* : E_{1*} \to E_{2*}$ defined as

$$f_*(a) = \begin{cases} f(a), & \text{if } f(a) \text{ defined,} \\ *, & \text{otherwise.} \end{cases}$$

Any basic homomorphism $f : M(E_1,I_1) \to M(E_2,I_2)$ can be given by the pointed total map $f_* : E_{1*} \to E_{2*}$, defined as

$$f_*(x) = \begin{cases} f(x), & \text{if } f(x) \in E_2; \\ *, & \text{if } f(x) = 1 \ \vee \ x = *. \end{cases}$$

For a monoid $M(E,I)$, we identify its identity 1 with $* \in E_*$.

Let E_* be a pointed set. A binary relation of *commutativity* on E is called a subset $T \subseteq E_* \times E_*$ satisfying the following conditions

(1) $(\forall a \in E_*)\ (a,*) \in T\ \&\ (*,a) \in T$,
(2) $(\forall a \in E_*)\ (a,a) \in T$ (reflexivity),
(3) $(\forall a,b \in E_*)(a,b) \in T \Rightarrow (b,a) \in T$ (symmetry).

Let *ComRel* be the category of pairs (E_*,T) where each pair consists of a pointed set with a commutativity relation. Morphisms $(E_{1*},T_1) \xrightarrow{f} (E_{2*},T_2)$ in the category *ComRel* are pointed maps $f : E_{1*} \to E_{2*}$ satisfying $(a_1,b_1) \in T_1 \Rightarrow (f(a_1), f(b_1)) \in T_2$.

For a set E, denote by Δ_{E_*} the relation of commutativity $\{(a,a)|a \in E_*\} \subseteq E_* \times E_*$.

Proposition 2.4. *The category $FPCM$ is isomorphic to ComRel.*

Proof. Define the functor $FPCM \to ComRel$ on objects by $M(E,I) \mapsto (E_*,T)$ where $T = I \cup (E \times \{*\}) \cup (\{*\} \times E) \cup \Delta_{E_*}$. The functor transforms each basic homomorphisms $f : M(E_1,I_1) \to M(E_2,I_2)$ into the map $f_* : E_{1*} \to E_{2*}$ assigning to pairs $(a_1,b_1) \in T_1$ the pairs $(f_*(a_1), f_*(b_1)) \in T_2$.

An inverse functor assigns to each object (E_*,T) of the category *ComRel* the trace monoid $M(E,I)$, where

$$I = T \setminus (\{(a,a)|a \in E_*\} \cup \{(a,*)|a \in E\} \cup \{(*,a)|a \in E\}), \tag{2.1}$$

and to any morhism $(E_{1*},T_1) \xrightarrow{f} (E_{2*},T_2)$ the homomorphism $\widetilde{f} : M(E_1,I_1) \to M(E_2,I_2)$ given on basic elements by $\widetilde{f}(e) = f(e)$ if $f(e) \in E_2$, and $\widetilde{f}(e) = 1$, if $f(e) = *$. This completes the proof.

Consider a family of trace monoids $\{M(E_j,I_j)\}_{j\in J}$. Transform it to family of pointed sets with commutativity relations $\{(E_{j_*},T_j)\}_{j\in J}$. The product of this family in the category *ComRel* equals the Cartesian product $(\prod\limits_{j\in J} E_{j_*}, \prod\limits_{j\in J} T_j)$. The category $FPCM$ is isomorphic to *ComRel*. Therefore, we obtain the following

Proposition 2.5. *The category $FPCM$ has all products.*

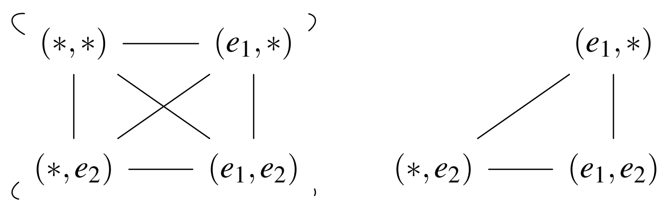
Any object (E_*,T) of *ComRel* corresponds to a trace monoid $M(E,I)$ with the set $E = E_* \setminus \{*\}$ and independence relation defined by formula (2.1).

It follows that the product of $M(E_j,I_j)$, $j \in J$ has the set of generators $E = (\prod_{j\in J} E_{j*}) \setminus \{(*)\}$ where $(*) \in \prod_{j\in J} E_{j*}$ denotes a family of elements each of which equals $* \in E_{j_*}$. Let

$$T_j = I_j \cup (\{(a,a) | a \in E_{j_*}\} \cup \{(a,*) | a \in E_j\} \cup \{(*,a) | a \in E_j\}).$$

The relation I is obtained from $T = \prod_{j\in J} T_j$ by the formula (2.1).

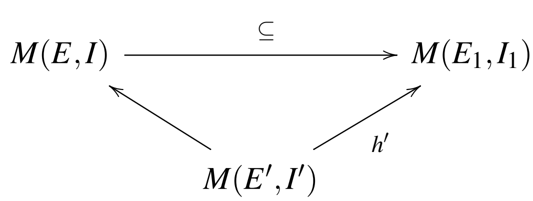
Example 2.6. Let $J = \{1,2\}$, $E_1 = \{e_1\}, E_2 = \{e_2\}$, $I_1 = I_2 = \emptyset$. Then $M(E_1,I_1) \cong M(E_2,I_2) \cong \mathbb{N}$ are isomorphic to the monoid generated by one element. Compute the product $M(E,I) = M(E_1,I_1) \prod M(E_2,I_2)$. The set E_* equals $E_{1*} \times E_{2*}$. In following picture at the left, it is shown the graph of the relation $T \subseteq E_* \times E_*$ and on the right it is shown the graph of the relation I obtained by the formula (2.1)).



We see, that the product is isomorphic to a free commutative monoid $\mathbb{N}^3$ generated by three elements.

Theorem 2.7. *Each diagram D in $FPCM$ has a limit.*

Proof. Since $FPCM$ has all products, it is enough to show the existence of equalizers. Consider a pair $M(E_1,I_1) \underset{g}{\overset{f}{\rightrightarrows}} M(E_2,I_2)$ of basic homomorphisms. Let $E = \{e \in E_1 \mid f(e) = g(e)\}$. The submonoid of $M(E_1,I_1)$ generated by E is a trace monoid $M(E,I)$ with the independence relation $I = I_1 \cap (E \times E)$. Consider an arbitrary basic homomorphism $h' : M(E',I') \to M(E_1,I_1)$ such that $g(h'(e')) = f(h'(e'))$ for all $e' \in E'$. Observe that $h'(e') \in E \cup \{1\}$. It follows that h' maps $M(E',I')$ into $M(E,I)$ and the following triangle is commutative:



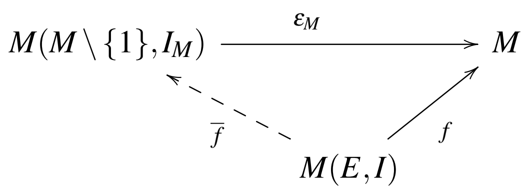
Therefore, the inclusion $M(E,I)$ into $M(E_1,I_1)$ is an equalizer of the pair (f,g). This completes the proof.

Proposition 2.8. *Let* $\mathrm{Ob}\,Mon \to \mathrm{Ob}\,FPCM$ *be the map carrying each monoid* M *to the trace monoid* $M(M\setminus\{1\}, I_M)$ *with*

$$I_M = \{(\mu_1,\mu_2) \in (M\setminus\{1\})\times(M\setminus\{1\}) \mid \mu_1 \neq \mu_2 \;\&\; \mu_1\mu_2 = \mu_2\mu_1\}.$$

This map can be extended to a functor $R : Mon \to FPCM$ *right adjoint to the inclusion* $U : FPCM \to Mon$.

Proof. Define a homomorphism $\varepsilon_M : M(M\setminus\{1\}, I_M) \to M$ setting $\varepsilon(\mu) = \mu$ on the generators of $M(M\setminus\{1\}, I_M) \to M$. It easy to see that for each homomorphism $f : M(E,I) \to M$, there exists unique basic homomorphism $\overline{f}$ making the following diagram commutative



It is defined by $\overline{f}(e) = f(e)$ on elements $e \in E$. This homomorphism is couniversal arrow. By the universal property, the map $M \mapsto (M(M\setminus\{1\}, I_M), \varepsilon_M)$ uniquely extends up to the right adjoint functor. This completes the proof.

Theorem 2.9. *The category* $FPCM$ *is cocomplete and the inclusion functor* $FPCM$ *into the category* Mon *preserves all colimits.*

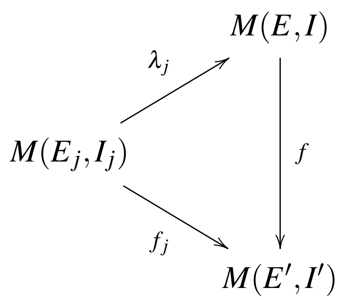
Proof. Let $D : J \to FPCM$ be a diagram with values $D(j) = M(E_j, I_j)$. Consider $\varinjlim^J D$ in the category Mon of all monoids. The colimit is isomorphic to a quotient monoid $\biguplus_{j\in J} M(E_j, I_j)/\equiv$ obtained from the coproduct in Mon by identifications of elements $e_j \equiv D(j\to k)e_j$. It follows that the colimit is generated by the disjoint union $\biguplus_{j\in J} E_j$ and represented by the following equations:

(1) $ee' \equiv e'e$ for every $j \in J$ and for all $e, e' \in E_j$ satisfying $(e,e') \in I_j$,

(2) if $e'_k = D(j\to k)(e_j)$ for some $e_j \in E_j$, $e'_k \in E_k$, then $e_j \equiv e'_k$,

(3) $e_j \equiv 1$ if $M(j\to k)(e_j) = 1$.

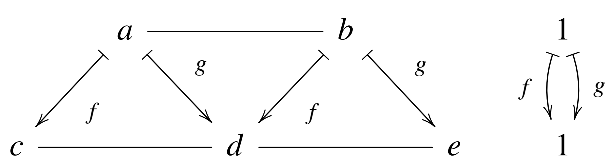
This monoid is generated by a set E obtained as a quotient set of $\biguplus_{j\in J} E_j$ under the equivalence relation containing pairs type (2) by removing the classes containing elements $e_j \equiv 1$. The equations (1) give the relation I. We obtain the trace monoid $\varinjlim^J D = M(E,I)$. The morphisms of colimiting cone are basic homomorphisms sending every e_j to its equivalence class or 1. For any other cone $f_j : M(E_j, I_j) \to M(E', I')$ consisting of basic homomorphisms, the morphism $\varinjlim^J D \to M(E', I')$ assigns to each class $[e_j]$ the element $f_j(e_j)$.



Therefore, $FPCM$ has all colimits.

It follows from Proposition 2.8 that the inclusion $FPCM \subset Mon$ preserves all colimits as having a right adjoint [21]. This completes the proof.

Example 2.10. Consider the free commutative monoid $M(\{a,b\})$ and the trace monoid $M = M(\{c,d,e\}, \{(c,d),(d,c),(d,e),(e,d)\})$. Let $f, g : M\{a,b\} \to M$ be two homomorphisms defined as $f(a) = c$, $g(b) = d$, $g(a) = d$, $g(b) = c$.



(2.2)

The coequalizer of f, g is the trace monoid generated by c, d, e with equations $c = a = d = b$, $cd = dc$, $de = ed$. The top line of the diagram (2.2) shows the relation for $M(\{a,b\})$ and the bottom shows the relation for M. Consequently, the coequalizer is equal to the free commutative monoid generated by one element.

Independence preserving basic homomorphisms. We prove that the category of trace monoids and independence preserving homomorphisms has all limits and colimits.

If $f : M(E,I) \to M(E',I')$ is a basic homomorphism, then for all $(a,b) \in I$, we have $(f(a), f(b)) \in I' \vee f(a) = 1 \vee f(b) = 1 \vee f(a) = f(b)$.

Definition 2.11. A basic homomorphism $f : M(E,I) \to M(E',I')$ is called *independence preserving* if for all $a,b \in E$, the following implication holds

$$(a,b) \in I \Rightarrow (f(a),f(b)) \in I' \vee f(a) = 1 \vee f(b) = 1 . \tag{2.3}$$

It is easy to see, that the class of independence preserving homomorphisms is closed under composition.

Notice that for any basic homomorphism the implication (2.3) is equivalent to the condition $(a,b) \in I \Rightarrow (f(a) \neq f(b)) \vee (f(a) = f(b) = 1)$.

Let $FPCM^{\|} \subset FPCM$ be the subcategory consisting of all trace monoids and independence preserving basic homomorphisms.

Let us prove the existence of the products in the $FPCM^{\|}$. For this purpose, we introduce the following *pointed independence relation*.

Definition 2.12. Let E be a set. A *pointed independence relation on* E is a subset $R \subseteq E_* \times E_*$ satisfying the following conditions:

(1) $(\forall a \in E_*)\ (a,*) \in R\ \&\ (*,a) \in R$;
(2) $(\forall a \in E_*)\ (a,a) \in R \Rightarrow a = *$;
(3) $(\forall a,b \in E_*)\ (a,b) \in R \Leftrightarrow (b,a) \in R$.

Let *IndRel* be the category of pairs (E_*,R) consisting of pointed sets E_* and partial independence relations $R \subseteq E_* \times E_*$. Its morphisms $(E_*,R) \xrightarrow{f} (E'_*,R')$ defined as pointed maps $f : E_* \to E'_*$ satisfying the following conditions:

$$(a,b) \in R \Rightarrow (f(a),f(b)) \in R'.$$

Proposition 2.13. *The category* $FPCM^{\|}$ *is isomorphic to* *IndRel*.

Proof. Let $U : \mathrm{Ob}(FPCM^{\|}) \to \mathrm{Ob}(IndRel)$ assigns to every $M(E,I)$ a pointed set with a pointed independence relation $U(M(E,I)) = (E_*,R)$ where $R = I \cup (E_* \times \{*\}) \cup (\{*\} \times E_*)$. For any morphism $f : M(E,I) \to M(E',I')$ in $FPCM^{\|}$, and for $U(M(E,I)) = (E_*,R)$, $U(M(E',I')) = (E'_*,R')$, we define the values of $U(f) : (E_*,R) \to (E'_*,R')$ on $x \in E_*$ by

$$U(f)(x) = \begin{cases} f(x), & \text{if } f(x) \in E'; \\ *, & \text{if } f(x) = 1 \vee x = *. \end{cases}$$

Then we obtain the functor $U : FPCM^{\|} \to IndRel$. The functor U has an inverse functor $IndRel \to FPCM^{\|}$ which carries any object (E_*, R) to the monoid $M(E,I)$ where $I = R \setminus ((E_* \times \{*\}) \cup (\{*\} \times E_*))$. This completes the proof.

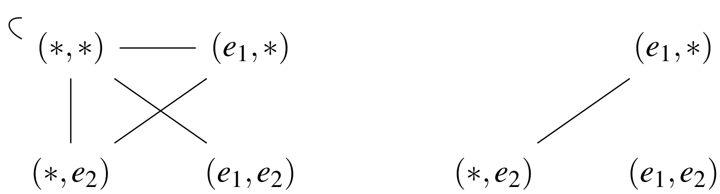
Corollary 2.14. *The category $FPCM^{\|}$ has all products.*

Proof. By Proposition 2.13, it is sufficient to prove that the category *IndRel* has all products. Let $(E_{j_*}, R_j)_{j \in J}$ be a family of objects in *IndRel*. Take the Cartesian product $\prod\limits_{j \in J} E_{j_*}$ with the relation

$$\prod_{j \in J} R_j = \{((a_j)_{j \in J}, (b_j)_{j \in J}) \mid (\forall i \in J)(a_i, b_i) \in R_i\}.$$

Denote by $p_i : (\prod\limits_{j \in J} E_{j_*}, \prod\limits_{j \in J} R_j) \to (E_{i*}, R_i)$ the projections defined as $p_i((a_j)_{j \in J}) = a_i$. For every object (E_*, R) in *IndRel* with a family of morphisms $q_j : (E_*, R) \to (E_{j_*}, R_j)$, we have $(\forall j \in J)(a,b) \in R \Rightarrow (q_j(a), q_j(b)) \in R_j$. It follows that $q : (E_*, R) \to (\prod\limits_{j \in J} E_{j_*}, \prod\limits_{j \in J} R_j)$ defined as $q(a) = (q_j(a))_{j \in J}$ is the unique morphism satisfying $p_j \circ q = q_j$ for all $j \in J$. Thus, the pair $(\prod\limits_{j \in J} E_{j_*}, \prod\limits_{j \in J} R_j)$ with the projections p_j is the product in the category *IndRel*. This completes the proof.

Example 2.15. Let $E_1 = \{e_1\}, E_2 = \{e_2\}$, $I_1 = I_2 = \emptyset$. Denote by $M(E,I)$ the product $M(E_1, I_1) \prod M(E_2, I_2)$ in the category $FPCM^{\|}$. As in Example 2.6, the set E_* equals $E_{1*} \times E_{2*}$. The graph of the relation $R \subseteq E_* \times E_*$ is shown in the following picture at the left.

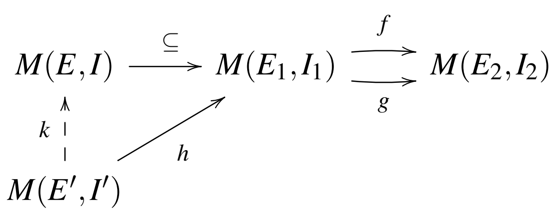


The graph of the relation $I = R \setminus ((E_* \times \{*\}) \cup (\{*\} \times E_*))$ is shown on the right. Hence, the product $M(E_1, I_1) \prod M(E_2, I_2)$ in the category is isomorphic to the free product of monoids $\mathbb{N}^2$ and $\mathbb{N}$. Consequently, the inclusion functor $FPCM^{\|} \subset FPCM$ does not preserve products.

We have seen that the category $FPCM^{\|}$ has products. Moreover, the following is true:

Theorem 2.16. *The category $FPCM^{\|}$ has all limits. The inclusion functor $FPCM^{\|} \subset FPCM$ preserves equalizers.*

Proof. Since $FPCM^{\|}$ has products, it it enough to prove the existence of equalizers. As in the proof of Theorem 2.7, for any pair of basic homorphisms $M(E_1,I_1) \overset{f}{\underset{g}{\rightrightarrows}} M(E_2,I_2)$ in the category $FPCM$, its equalizer is the inclusion $M(E,I) \subseteq M(E_1,I_1)$, where $E = \{e \in E_1 | f(e) = g(e)\}$ and $I = I_1 \cap (E \times E)$. Inclusion preserves independence. Consider a trace monoid $M(E',I')$ with an independence preserving homomorphism $h : M(E',I') \to M(E_1,I_1)$ satisfying $fh = gh$. Since $h(E') \subseteq E$, there is a basic homomorphism k drawn by dashed arrow in the following diagram:



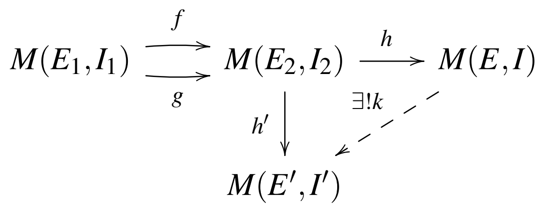
We have $k(e') = h(e')$ for all $e' \in E'$. The homomorphism h preserves independence. Hence, for all $(a',b') \in I'$, the condition $k(a') = 1 \vee k(b') = 1 \vee (k(a'),k(b')) \in I_1$ holds. Thus, k preserves independence. Equalizers are constructed in the category $FPCM$. Therefore the inclusion $FPCM^{\|} \subset FPCM$ preserves equivalizers. This completes the proof.

We turn now to colimits.

Theorem 2.17. *The category $FPCM^{\|}$ is cocomplete.*

Proof. The coproduct of trace monoids $\{M(E_i,I_i)\}_{i\in J}$ is a monoid given by the set of generators $\biguplus_{i\in J} E_i$ and the relations $ab = ba$ for all $(a,b) \in \biguplus_{i\in J} I_i$. It is easy to see that it is the coproduct in the category $FPCM^{\|}$. Hence, it is sufficient to prove the existence of coequalizers. For this purpose, consider an arbitrary pair of morphisms $M(E_1,I_1) \overset{f}{\underset{g}{\rightrightarrows}} M(E_2,I_2)$ in the category $FPCM^{\|}$. Let $h : M(E_2,I_2) \to M(E,I)$ be the coequalizer in the category $FPCM$. Consider a morphism $h' : M(E_2,I_2) \to M(E',I')$ in $FPCM^{\|}$ such that $h' \circ f = h' \circ g$. There exists a unique

k making commutative the triangle of the following diagram



Since h' preserves independence, the following implication is true:

$$(\forall(a,b)\in I_2)(h'(a)=h'(b)\Rightarrow h'(a)=1\ \&\ h'(b)=1). \tag{2.4}$$

Let $\equiv_h$ be the smallest congruence relation on $M(E,I)$ for which $h(a)\equiv_h 1$ and $h(b)\equiv_h 1$ if $(a,b)\in I_2$ satisfies $h(a)=h(b)$. Denote by $cls: M(E,I)\to M(E,I)/\equiv_h$ the homomorphism assigning to any $e\in E_*$ its class $cls(e)$ of the congruence. We have from (2.4) with $kh=h'$ that

$$(\forall(a,b)\in I_2)k(h(a))=k(h(b))\Rightarrow k(h(a))=1\ \&\ k(h(b))=1.$$

We see that k has constant values on each congruence class $cls(e)$ where $e\in E_*$. Hence, we can define a map $k': M(E,I)/\equiv_h\to M(E',I')$ by $k'(cls(e))=k(e)$ for all $e\in E_*$. The homomorphism k' is unique for which $k'\circ cls\circ h=h'$. Therefore, $cls\circ h: M(E_2,I_2)\to M(E,I)/\equiv_h$ is the coequalizer of (f,g). This completes the proof.

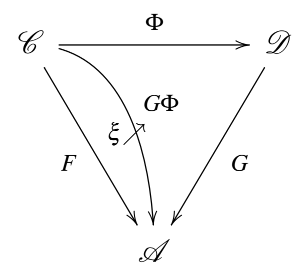
In Example 2.10, we have $h(c)=h(d)=h(e)$. Since $(c,d)\in I_2$ and $(d,e)\in I_2$, we have $cls\circ h(c)=1$, $cls\circ h(d)=1$, $cls\circ h(e)=1$. Therefore, the coequalizer in $FPCM^{\|}$ equals $\{1\}$.

3. Category of diagrams with various domains

This Section does not contain new results but rather recalls some notions which shall be used. A *diagram in a category* $\mathscr{A}$ is a functor $\mathscr{C}\to\mathscr{A}$ defined on some small category $\mathscr{C}$. We shall consider categories of the diagrams accepting values in some fixed category. Let us study the conditions providing completeness or cocompleteness of this category.

Morphisms and objects in a diagram category. Let $\mathscr{A}$ be a category and let $F:\mathscr{C}\to\mathscr{A}$ be a diagram. Denote this diagram by the pair $(\mathscr{C},F)$ in which the domain category is explicitly specified.

Let $(\mathscr{C},F)$ and $(\mathscr{D},G)$ be diagrams in $\mathscr{A}$. A *morphism of the diagrams* $(\Phi,\xi):(\mathscr{C},F)\to(\mathscr{D},G)$ is given by a pair (Φ,ξ) consisting of a functor $\Phi:\mathscr{C}\to\mathscr{D}$ and natural transformation $\xi:F\to G\Phi$:



Define the *identity morphism* by the formula $1_{(\mathscr{C},F)}=(1_{\mathscr{C}},1_F)$ where $1_{\mathscr{C}}:\mathscr{C}\to\mathscr{C}$ is the identity functor and $1_F:F\to F$ is the identity natural transformation. The *composition of morphisms*

$$(\mathscr{C},F)\overset{(\Phi,\xi)}{\to}(\mathscr{D},G)\overset{(\Psi,\eta)}{\to}(\mathscr{E},H)$$

is defined as a pair $(\Psi\Phi,(\eta*\Phi)\cdot\xi)$ where $\eta*\Phi:G\Phi\to H\Psi\Phi$ is a natural transformation given by a family of morphisms specified as a family of morphisms

$$(\eta*\Phi)_c=\eta_{\Phi(c)}:G(\Phi(c))\to H(\Psi(\Phi(c))),\ c\in\mathrm{Ob}\mathscr{C},$$

and $(\eta*\Phi)\cdot\xi$ is the composition of natural transformations $F\overset{\xi}{\to}G\Phi\overset{\eta*\Phi}{\to}H\Psi\Phi$. The composition is associative.

Let Cat be the category of small categories and functors. Denote by $(Cat,\mathscr{A})$ the category of diagrams in $\mathscr{A}$ and morphisms of diagrams.

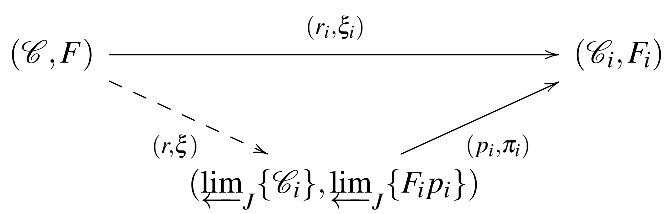
For any subcategory $\mathfrak{C}\subseteq Cat$, we consider diagrams $F:\mathscr{C}\to\mathscr{A}$ defined on categories $\mathscr{C}\in\mathfrak{C}$. Such diagrams with morphisms $(\Phi,\xi):(\mathscr{C},F)\to(\mathscr{D},G)$ where $\Phi\in\mathrm{Mor}\mathfrak{C}$, will be make a subcategory of $(Cat,\mathscr{A})$. Denote this subcategory by $(\mathfrak{C},\mathscr{A})$.

Limits in a category of diagram. Let J be a small category. In some cases, the diagrams are conveniently denoted, specifying their values on objects. For example, we will denote by $\{A_i\}_{i\in J}$ the diagram $J\to\mathscr{A}$ with values A_i on objects $i\in J$ and $A_\alpha:A_i\to A_j$ on morphisms $\alpha:i\to j$ of J. We say that a category $\mathscr{A}$ has J-limits if every diagram $\{A_i\}_{i\in J}$ in $\mathscr{A}$ has a limit. If $\mathscr{A}$ has J-limits for all small categories J, then $\mathscr{A}$ is said to be a *complete category* or a *category with all limits*.

We will consider subcategories $\mathfrak{C} \subseteq Cat$ with J-limits. But the J-limits in $\mathfrak{C}$ need not be isomorphic to the J-limits in Cat.

Proposition 3.1 *Let $\mathscr{A}$ be a complete category and let J be a small category. If a subcategory $\mathfrak{C} \subseteq Cat$ has J-limits, then the category $(\mathfrak{C}, \mathscr{A})$ has J-limits. In particular, if $\mathfrak{C} \subseteq Cat$ is a complete category, then the category $(\mathfrak{C}, \mathscr{A})$ is complete.*

Proof. Let $\{(\mathscr{C}_i, F_i)\}_{i\in J}$ be a diagram in $(\mathfrak{C}, \mathscr{A})$. One given by a diagram $\{\mathscr{C}_i\}_{i\in J}$ with functors $\mathscr{C}_\alpha : \mathscr{C}_i \to \mathscr{C}_j$ and natural transformations $\varphi_\alpha : F_i \to F_j\mathscr{C}_\alpha$. Let $p_i : \varprojlim_J\{\mathscr{C}_i\} \to \mathscr{C}_i$ is the limit cone of the diagram $\{\mathscr{C}_i\}_{i\in J}$ in $\mathfrak{C}$. The compositions $F_i \circ p_i$ belong to the category $\mathscr{A}^{\varprojlim_J\{\mathscr{C}_i\}}$. The natural transformations $F_ip_i \xrightarrow{\varphi_\alpha * p_i} F_j\mathscr{C}_\alpha p_i \xrightarrow{=} F_jp_j$ give the functor $J \to \mathscr{A}^{\varprojlim_J\{\mathscr{C}_i\}}$. Let $\varprojlim_J\{F_ip_i\} \in \mathscr{A}^{\varprojlim_J\{\mathscr{C}_i\}}$ be its limit. Denote by $\pi_i : \varprojlim_J\{F_ip_i\} \to F_ip_i$ the limit cone. It easy to see that morphisms $(p_i, \pi_i) : (\varprojlim_J\{\mathscr{C}_i\}, \varprojlim_J\{F_ip_i\}) \to (\mathscr{C}_i, F_i)$ of diagrams make the cone over the diagram in $(\mathfrak{C}, \mathscr{A})$. Considering an another cone $(r_i, \xi_i) : (\mathscr{C}, F) \to (\mathscr{C}_i, F_i)$ it can be seen that there exists the unique morphism (r, ξ) making the commutative triangle



It follows that the limit is isomorphic to $(\varprojlim_J\{\mathscr{C}_i\}, \varprojlim_J\{F_ip_i\})$. This completes the proof.

Colimits in a category of diagrams. Let $\mathfrak{C} \subseteq Cat$ be a subcategory. Consider an arbitrary category $\mathscr{A}$. We shall prove that if the colimits exist in $\mathfrak{C}$, then those exist in $(\mathfrak{C}, \mathscr{A})$. For any functor $\Phi : \mathscr{C} \to \mathscr{D}$, we denote by $\mathrm{Lan}^\Phi : \mathscr{A}^\mathscr{C} \to \mathscr{A}^\mathscr{D}$ the left Kan extension functor [21]. Its properties are well described in [21]. This functor is characterized as a left adjoint to the functor $\Phi^* : \mathscr{A}^\mathscr{D} \to \mathscr{A}^\mathscr{C}$ assigning to each diagram $F : \mathscr{D} \to \mathscr{A}$ the composition $F \circ \Phi$, and to the natural transformation $\eta : F \to G$ the natural transformation $\eta * \Phi$.

Proposition 3.2. *Let $\mathfrak{C} \subseteq Cat$ be a category with all colimits. Then, for any cocomplete category $\mathscr{A}$, the category $(\mathfrak{C}, \mathscr{A})$ has all colimits.*

Proof. Consider a diagram $\{(\mathscr{C}_i, F_i)\}_{i\in J}$ in $(\mathfrak{C}, \mathscr{A})$. As above, each morphism $\alpha : i \to j$ is mapped to the natural transformation $\varphi_\alpha : F_i \to F_j\mathscr{C}_\alpha$. Let $\varinjlim^J \{\mathscr{C}_i\}$ be the colimit of the diagram in $\mathfrak{C}$. Denote by $q_i : \mathscr{C}_i \to \varinjlim^J \{\mathscr{C}_i\}$ morphisms of the colimit cone. Consider the Kan extensions $\mathrm{Lan}^{q_i} F_i$ and the units of adjunction

$$\begin{array}{ccc} \mathscr{C}_i & \xrightarrow{\quad q_i \quad} & \varinjlim^J \{\mathscr{C}_i\} \\ & \nearrow \eta_i & \\ {\scriptstyle F_i} \searrow & & \swarrow {\scriptstyle \mathrm{Lan}^{q_i} F_i} \\ & \mathscr{A} & \end{array}$$

We get the diagram in the category $\mathscr{A}^{\varinjlim^J \{\mathscr{C}_i\}}$ consisting of objects $\mathrm{Lan}^{q_i} F_i$ and morphisms given at $\alpha : i \to j$ by the compositions $\mathrm{Lan}^{q_i} F_i \xrightarrow{\mathrm{Lan}^{q_i}(\varphi_\alpha)} \mathrm{Lan}^{q_i} F_j\mathscr{C}_\alpha \xrightarrow{=} \mathrm{Lan}^{q_j}\mathrm{Lan}^{\mathscr{C}_\alpha} F_j\mathscr{C}_\alpha \xrightarrow{\mathrm{Lan}^{q_j}(\varepsilon_\alpha)} \mathrm{Lan}^{q_j} F_j$ where $\varepsilon_\alpha : \mathrm{Lan}^{\mathscr{C}_\alpha}(F_j\mathscr{C}_\alpha) \to F_j$ are counits of adjunction. Let $\varinjlim^J \{\mathrm{Lan}^{q_i} F_i\}$ be the colimit of this diagram.

Prove that $(\varinjlim^J \{\mathscr{C}_i\}, \varinjlim^J \{\mathrm{Lan}^{q_i} F_i\})$ is a colimit of the diagram $\{(\mathscr{C}_i, F_i)\}_{i\in J}$ in $(\mathfrak{C}, \mathscr{A})$. For this purpose, consider an arbitrary (direct) cone $(\mathscr{C}_i, F_i) \to (\mathscr{C}, F)$ over $\{(\mathscr{C}_i, F_i)\}_{i\in J}$ in the category $(\mathfrak{C}, \mathscr{A})$. One is given by some functors

$$\begin{array}{ccc} \mathscr{C}_i & \xrightarrow{\quad r_i \quad} & \mathscr{C} \\ & \nearrow \psi_i & \\ {\scriptstyle F_i} \searrow & & \swarrow {\scriptstyle F} \\ & \mathscr{A} & \end{array}$$

and natural transformations $\psi_i : F_i \to Fr_i$ for which the following diagrams are commutative

$$\begin{array}{ccc} (\mathscr{C}_i, F_i) & \xrightarrow{(r_i, \psi_i)} & (\mathscr{C}, F) \\ {\scriptstyle (\mathscr{C}_\alpha, \varphi_\alpha)} \downarrow & \nearrow {\scriptstyle (r_j, \psi_j)} & \\ (\mathscr{C}_j, F_j) & & \end{array} \qquad \begin{array}{ccc} F_i & \xrightarrow{\psi_i} & Fr_i \\ {\scriptstyle \varphi_\alpha} \downarrow & & \| \\ F_j\mathscr{C}_\alpha & \xrightarrow[\psi_j * \mathscr{C}_\alpha]{} & Fr_j\mathscr{C}_\alpha \end{array}$$

Since $\varinjlim^J \{\mathscr{C}_i\}$ is the colimit in $\mathfrak{C}$, the unique functor $r : \varinjlim^J \{\mathscr{C}_i\} \to \mathscr{C}$ is corresponded to the functors of this cone $r_i : \mathscr{C}_i \to \mathscr{C}$, such that $r_i = rq_i$ dor all $i \in J$ where $q_i : \mathscr{C}_i \to \varinjlim^J \{\mathscr{C}_i\}$ is the colimit cone.

For any $i \in J$, the functor Lan^{q_i} is left adjoint to q_i^*. Hence, there exists a bijection between natural transformations

$$F_i \overset{\psi_i}{\to} Fr_i = Frq_i \quad \text{and} \quad \mathrm{Lan}^{q_i} F_i \overset{\overline{\psi_i}}{\to} Fr\,.$$

This bijection can be obtained by universality of the Kan extension. It maps each commutative triangle in $\mathscr{A}^{\mathscr{C}_i}$ to the commutative triangle in $\mathscr{A}^{\varinjlim^J \{\mathscr{C}_i\}_{i\in J}}$:

$$\begin{array}{ccccccc}
F_i & \xrightarrow{\psi_i} & Frq_i & \qquad\mapsto\qquad & \mathrm{Lan}^{q_i}F_i & \xrightarrow{\overline{\psi_i}} & Fr \\
\downarrow{\scriptstyle \varphi_\alpha} & & \| & & \downarrow & \nearrow{\scriptstyle \overline{\psi_j}} & \\
F_j\mathscr{C}_\alpha & \xrightarrow[\psi_j * \mathscr{C}_\alpha]{} & Frq_j\mathscr{C}_\alpha & & \mathrm{Lan}^{q_j}F_j & &
\end{array}$$

For the diagram

$$\begin{array}{ccc}
F_i & \xrightarrow{\psi_i} & Frq_i = Frq_j\mathscr{C}_\alpha \\
\downarrow{\scriptstyle \varphi_\alpha} & \nearrow{\scriptstyle \psi_j * \mathscr{C}_\alpha} & \\
F_j\mathscr{C}_\alpha & &
\end{array}$$

we have the commutative diagram in $\mathscr{A}^{\mathscr{C}_j}$

$$\begin{array}{ccc}
\mathrm{Lan}^{\mathscr{C}_\alpha}F_i & \xrightarrow{\overline{\psi_i}} & Frq_j \\
{\scriptstyle \overline{\varphi_\alpha}}\downarrow & \nearrow{\scriptstyle \psi_j} & \\
F_j & &
\end{array}$$

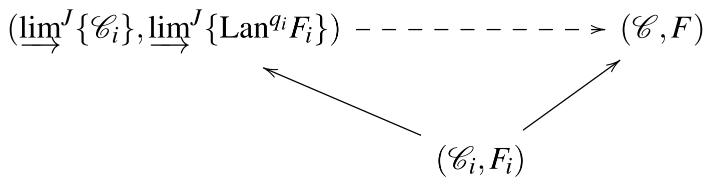
Applying Lan^{q_j}, we obtain the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Lan}^{q_j}\mathrm{Lan}^{\mathscr{C}_\alpha}F_i & \xrightarrow{\mathrm{Lan}^{q_j}\overline{\psi_i}} & \mathrm{Lan}^{q_j}Frq_j & \xrightarrow{(\varepsilon_j)_{Fr}} & Fr \\
{\scriptstyle \mathrm{Lan}^{q_j}\overline{\varphi_\alpha}}\downarrow & \nearrow{\scriptstyle \mathrm{Lan}^{q_j}(\psi_j)} & & & \\
\mathrm{Lan}^{q_j}F_j & & & &
\end{array}$$

which leads us to the (direct) cone over the diagram $\{\mathrm{Lan}^{q_i}F_i\}_{i\in J}$

$$\begin{array}{ccc}
\mathrm{Lan}^{q_i}F_i & \longrightarrow & Fr \\
\downarrow & \nearrow & \\
\mathrm{Lan}^{q_j}F_j & &
\end{array}$$

This cone gives the morphism $\varinjlim^J\{\mathrm{Lan}^{q_i}F_i\} \to Fr$ in $\mathscr{A}^{\varinjlim^J\{\mathscr{C}_i\}}$ which define a unique morphism in $(\mathfrak{C},\mathscr{A})$ making commutative triangles



Therefore, the diagram $(\varinjlim^J\{\mathscr{C}_i\},\varinjlim^J\{\mathrm{Lan}^{q_i}F_i\})$ in $\mathscr{A}$ is the colimit of the diagram $\{(\mathscr{C}_i,F_i)\}_{i\in J}$ in the category $(\mathfrak{C},\mathscr{A})$. This completes the proof.

4. Category of pointed polygons on trace monoids

We apply the result of Section 3 to the case when $\mathscr{A} = \mathrm{Set}_*$, $\mathfrak{C} = FPCM$ and $\mathfrak{C} = FPCM^{\|}$. Then we discuss the relationship between categories of right $M(E,I)$-sets and a certain category of asynchronous systems with polygonal morphisms.

Category of state spaces.

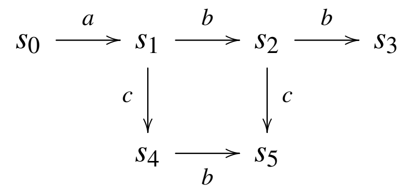
Definition 4.1. A *state space* $\mathscr{S} = (S,E,I,\mathrm{Tran})$ consists of a set S which elements called *states*, a *subset Tran* $\subseteq S\times E\times S$ of *transitions*, a set E of *events*, an irreflective symmetric relation $I\subseteq E\times E$ of *independence*, satisfying the conditions

(1) If $(s,a,s')\in\mathrm{Tran}$ & $(s,a,s'')\in\mathrm{Tran}$, then $s'=s''$ (determinism).
(2) If $(a,b)\in I$ & $(s,a,s')\in\mathrm{Tran}$ & $(s',b,s'')\in\mathrm{Tran}$, then there exists $s_1\in S$ such that $(s,b,s_1)\in\mathrm{Tran}$ & $(s_1,a,s'')\in\mathrm{Tran}$.

We write $s\xrightarrow{a}s'$ to indicate that $(s,a,s')\in\mathrm{Tran}$. It allows to represent any state space by a diagram.

Example 4.2 Consider the state space consisting of $S=\{s_0,s_1,s_2,s_3,s_4,s_5\}$, $E=\{a,b,c\}$, $I=\{(b,c),(c,b)\}$, and $\mathrm{Tran}=\{(s_0,a,s_1),(s_1,b,s_2),(s_2,b,s_3),(s_4,b,s_5),(s_1,c,s_4),(s_2,c,s_5)\}$.

It can be given by the following diagram:



It follows from (1) that for every $a \in E$, the set of pairs (s,s') for which $(s,a,s') \in Tran$ defines some partial map $\rho_a : S \rightharpoonup S$ by $\rho_a(s) = s'$.

We consider an each partial map $f : S \rightharpoonup S'$, as a pointed map $f_* : S_* \to S'_*$ such that

$$f_*(s) = \begin{cases} f(s), & \text{if } f(s) \text{ is defined;} \\ *, & \text{otherwise.} \end{cases}$$

So, we have the map $\rho : E \to End(S_*)$, $a \mapsto \rho_a$, where $End(S_*)$ is the monoid of all pointed map $S_* \to S_*$.

Lemma 4.3. *Let $\varphi : E \to End(S_*)$ be an arbitrary map. Denote by $\varphi_e : S_* \to S_*$ its values on $e \in E$. Then φ can be extended to a homomorphism $\widetilde{\varphi} : M(E,I)^{op} \to End(S_*)$ if and only if all $s \in S$ and $(a,b) \in I$ satisfy $\varphi_b(\varphi_a(s)) = \varphi_a(\varphi_b(s))$.*

It follows from the condition (2) and Lemma 4.3 the following:

Proposition 4.4. *Every state space (S,E,I,Tran) has a unique homomorphism $\widetilde{\rho} : M(E,I)^{op} \to End(S_*)$ extended the map $\rho : E \to End(S_*)$ and defined as $\widetilde{\rho}([a_1 \cdots a_n]) = \rho_{a_n} \cdots \rho_{a_1}$ for all $[a_1 \cdots a_n] \in M(E,I)$. And, conversely, every homomorphism $\widetilde{\rho} : M(E,I)^{op} \to End(S_*)$ can assigned state space (S,E,I,Tran), where $\rho = \widetilde{\rho}|_E$ and $\mathrm{Tran} = \{(s,a,\rho_a(s)) \mid s \in S, a \in E, \rho_a(s) \in S\}$.*

Consequently, we can consider any state space (S,E,I,Tran) as a trace monoid $M(E,I)$ with right action of the pointed set $X = S_*$ with the operation $\cdot : X \times M(E,I) \to X$, $(x,w) \mapsto x \cdot w$ for $x \in X$, $w \in M(E,I)$. Since the monoid is a category with a unique object, we can consider the state space as a functor $X : M(E,I)^{op} \to Set_*$ sending the unique object to the pointed set X and morphisms $w \in M(E,I)$ to maps $X(w) : X \to X$ given as $X(w)(x) = x \cdot w$. Here we denote by X the pointed set on which the monoid acts as well as functor defined by this action.

Definition 4.5. A *morphism of state spaces* $(M(E,I),X) \to (M(E',I'),X')$ is a pair (η,σ) where $\eta : M(E,I) \to M(E',I')$ is a basic homomorphism and $\sigma : X \to X' \circ \eta^{op}$ is a natural transformation.

A morphism of state spaces is possible to represent by means of the diagram

$$\begin{array}{ccc} M(E,I)^{op} & \xrightarrow{\eta^{op}} & M(E',I')^{op} \\ & {\scriptstyle X}\searrow \quad {\scriptstyle \sigma\nearrow} \quad \swarrow{\scriptstyle X'} & \\ & Set_* & \end{array}$$

Since we consider the objects of $FPCM$ as one object categories, then $FPCM$ is a subcategory of Cat. In the notation of Section 3 the category of state spaces is isomorphic to $(FPCM, Set_*)$.

By Proposition 3.1 if a subcategory $\mathfrak{C} \subseteq Cat$ has J-limits, then $(\mathfrak{C}, \mathrm{Set}_*)$ has J-limits. For $\mathfrak{C} = FPCM$ and for discrete category J with $\mathrm{Ob}\,(J) = \{1,2\}$, from Proposition 3.1 it follows that:

Proposition 4.6. *Let $(M(E_1,I_1),X_1)$ and $(M(E_2,I_2),X_2)$ be state spaces. Their product in $(FPCM, Set_*)$ is a state space*

$$(M(E_1,I_1)\prod M(E_2,I_2), X_1\circ\pi_1^{op}\times X_2\circ\pi_2^{op})$$

where $\pi_i : M(E_1,I_1)\prod M(E_2,I_2) \to M(E_i,I_i)$ are the projections of the product in the category $FPCM$ for $i \in \{1,2\}$, and

$$X_1\circ\pi_1^{op}\times X_2\circ\pi_2^{op} : (M(E_1,I_1)\prod M(E_2,I_2))^{op} \to \mathrm{Set}_*$$

is the state space considered as the functor. Here $\times$ is denoted the product in the category of functors $(M(E_1,I_1)\prod M(E_2,I_2))^{op} \to \mathrm{Set}_*$.

Definition 4.7. A morphism $(\eta,\sigma) : (M(E,I),X) \to (M(E',I'),X')$ of state spaces is independence preserving if $\eta : M(E,I) \to M(E',I')$ is independence preserving.

Let $(FPCM^{\parallel}, Set_*) \subset (FPCM, Set_*)$ be the subcategory of all state spaces and independence preserving morphisms.

Theorem 4.8. *The categories $(FPCM, Set_*)$ and $(FPCM^{\|}, Set_*)$ are both complete and cocomplete.*

Proof. The category $FPCM$ is complete by Theorem 2.7 and $FPCM^{\|}$ is complete by Theorem 2.16. Proposition 3.1 gives completeness of $(FPCM, Set_*)$ and $(FPCM^{\|}, Set_*)$. The cocompleteness of $(FPCM, Set_*)$ follows from Theorem 2.9 and Proposition 3.2 applied to $\mathfrak{C} = FPCM$ and $\mathscr{A} = Set_*$. The cocompleteness of $(FPCM^{\|}, Set_*)$ follows from Theorem 2.17 and Proposition 3.2. This completes the proof.

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Category of weak asynchronous system and polygonal morphisms.

Definition 4.9. *The weak asynchronous system* $\mathscr{A} = (S, s_0, E, I, \mathrm{Tran})$ is a state space (S, E, I, Tran) with an arbitrary element $s_0 \in S_*$ called *an initial state*.

If we add to Definition 4.9 the conditions $s_0 \in S$ and $S \neq \emptyset$, then we obtain asynchronous systems in the sense of M. Bednarczyk [2]. If more than that, we require the condition $(\forall e \in E)(\exists s, s' \in S)\ (s, e, s') \in \mathrm{Tran}$, then we get an *asynchronous transition system* [26].

Proposition 4.4 with the notation

$$s \cdot e = h_e(s) = \begin{cases} s', & \text{if } (s, e, s') \in \mathrm{Tran}; \\ *, & \text{if } s = * \text{ or there is no } s' \text{ such that } (s, e, s') \in \mathrm{Tran}. \end{cases}$$

leads to the following

Proposition 4.10. *An every weak asynchronous system $(S, s_0, E, I, \mathrm{Tran})$ gives a state space $(M(E, I), S_*)$ with a distinguished element $s_0 \in S_*$ wherein the action is defined by*

$$(s, [e_1 \cdots e_n]) \mapsto (\ldots((s \cdot e_1) \cdot e_2) \ldots \cdot e_n),$$

for all $s \in S_*$ and $e_1, \ldots, e_n \in E$. This correspondence is one-to-one. The inverse map takes any state space $(M(E, I), S_*)$ and $s_0 \in S_*$ to an asyncronous system $(S, s_0, E, I, \mathrm{Tran})$ where $\mathrm{Tran} = \{(s, e, s \cdot e) \mid s \in S \ \&\ s \cdot e \in S\}$.

In other words, any weak asynchronous system and hence an asynchronous system can be viewed as an action of a trace monoid on a pointed set $(M(E, I), S_*)$ with distinguished $s_0 \in S_*$.

Definition 4.11 A *morphism* of weak asynchronous systems $(f,\sigma):\mathscr{A}\to\mathscr{A}'$ consists of partial maps $f:E\rightharpoonup E'$ and $\sigma:S\rightharpoonup S'$ satifying the following conditions

(1) $\sigma(s_0)=s'_0$;

(2) for any triple $(s_1,e,s_2)\in \mathrm{Tran}$, there is an alternative

$$\begin{cases} (\sigma(s_1),f(e),\sigma(s_2))\in \mathrm{Tran}', & \text{if } f(e) \text{ is defined},\\ \sigma(s_1)=\sigma(s_2), & \text{if } f(e) \text{ is undefined},\end{cases}$$

(3) for each pair $(e_1,e_2)\in I$ such that $f(e_1)$ and $f(e_2)$ are defined, the pair $(f(e_1),f(e_2))$ must belong to I'.

Let AS^w be a category of weak asynchronous systems and morphisms.

If $s_0\neq *$, $s'_0\neq *$ and $\sigma:S\to S'$ is defined on the whole S, then these conditions gives a *morphism of asynchronous systems* in the sense of [2]. Following [2] denote by AS the category of asynchronous systems.

The category AS is a subcategory of AS^w.

A partial map $\sigma:S\rightharpoonup S'$ can be considered as pointed map $\sigma_*:S_*\to S'_*$ where

$$\sigma_*(s)=\begin{cases}\sigma(s) & \text{if } \sigma(s) \text{ is defined},\\ *, & \text{otherwise}\ .\end{cases}$$

For a set of generators E of $M(E,I)$, we take $*$ equal to the identity $1\in M(E,I)$. Hence, for a map $f:E\rightharpoonup E'$, we let $f_*(e)=f(e)$ if $f(e)$ is defined, and $f_*(e)=1$ otherwise. For a map $f:E\rightharpoonup E'$ satisfying the condition (3) of Definition 4.11, let $\widetilde{f}:M(E,I)\to M(E',I')$ be a homomorphism defined as $\widetilde{f}([a_1\cdots a_n])=[f_*(a_1)]\cdots[f_*(a_n)]$ for all $[a_1\cdots a_n]\in M(E,I)$.

Definition 4.12. A morphism of weak asynchronous systems $(f,\sigma):\mathscr{A}\to\mathscr{A}'$ is *polygonal* if $(\widetilde{f},\sigma_*)$ defines the independence preserving morphism of the corresponding state spaces. This means that a homomorphism $\widetilde{f}:M(E,I)\to M(E',I')$ is independence preserving and

$$(\forall s\in S_*)(\forall \mu\in M(E,I))\ \sigma(s\cdot\mu)=\sigma(s)\cdot\widetilde{f}(\mu).$$

Let $\mathscr{A}=(S,s_0,E,I,Tran)$ be an asynchronous system. Denote by $Tran_*=Tran\cup\{(s,1,s)\mid s\in S\}$. A morphism of asynchronous systems $(f,\sigma):\mathscr{A}\to\mathscr{A}'$ consists of a partial map $f:E\rightharpoonup E'$

and a (total) map $\sigma : S \to S'$ satisfying the conditions $(1)-(3)$ of Definition 4.11. The condition (2) can be formulated as follows

$$(\forall s_1, s_2 \in S)(\forall e \in E)\ (s_1, e, s_2) \in Tran \Rightarrow (\sigma(s_1), f_*(e), \sigma(s_2)) \in Tran'_*.$$

It is equivalent to the following property

$$(\forall s \in S)(\forall a \in E)\ s \cdot a \in S \Rightarrow \sigma(s) \cdot f_*(a) = \sigma(s \cdot a). \tag{4.5}$$

Hence, we have the following

Lemma 4.13. *A morphism (f, σ) of asynchronous systems is polygonal if and only if it satisfies the following implication*

$$(\forall s \in S)(\forall a \in E)\ s \cdot a = * \ \Rightarrow\ \sigma(s) \cdot f_*(a) = *. \tag{4.6}$$

Proof. Let (f, σ) be a morphism of asynchronous systems. We have the following implication by the condition (3) of Definition 4.11:

$$\text{if } (e_1, e_2) \in I, \text{ then } f_*(e_1) = 1 \text{ or } f_*(e_2) = 1 \text{ or } (f(e_1), f(e_2)) \in I'. \tag{4.7}$$

The conditions of Definition 4.12 are satisfied if and only if

$$(\forall s \in S_*)(\forall a \in E)\ \sigma_*(s \cdot a) = \sigma_*(s) \cdot f_*(a). \tag{4.8}$$

We have to prove the formula (4.8). This is equivalent to proving the following three formulas.

(1) $(\forall a \in E)\ \sigma_*(* \cdot a) = \sigma_*(*) \cdot f_*(a)$,
(2) $(\forall s \in S)(\forall a \in E)\ s \cdot a \in S \Rightarrow \sigma(s \cdot a) = \sigma(s) \cdot f_*(a)$,
(3) $(\forall s \in S)(\forall a \in E)\ \ s \cdot a = * \Rightarrow \sigma_*(s \cdot a) = \sigma_*(s) \cdot f_*(a)$.

Formula (1) is a consequence of $\sigma_*(*) = *$ and $* \cdot a = *$. Formula (2) coincides with (4.5). Formula (3) derives from $\sigma_*(s \cdot a) = *$ and (4.6). Hence (f, σ) is the polygonal morphism.

The converse is obvious: If (f, σ) is polygonal morphism, then (4.8) holds. Hence, for all $s \in S$ and $a \in E$, in the case of $s \cdot a = *$, we have

$$\sigma(s) \cdot f_*(a) = \sigma_*(s) \cdot f_*(a) = \sigma_*(s \cdot a) = \sigma_*(*) = *.$$

This completes the proof.

Denote by $\mathrm{AS}^\flat$ the category of weak asynchronous systems and polygonal morphisms. It is a subcategory of AS^w.

We have the following inclusions of the categories

$$\mathrm{AS}^\flat \hookrightarrow \mathrm{AS}^w \hookleftarrow \mathrm{AS}$$

where $\mathrm{AS} \subseteq \mathrm{AS}^w$ is the full subcategory. Every object of AS is an object of $\mathrm{AS}^\flat$ and $\mathrm{Ob}\,\mathrm{AS}^\flat = \mathrm{Ob}\,\mathrm{AS}^w$. But, the category AS is not a subcategory of $\mathrm{AS}^\flat$. Following statement characterizes polygonal morphisms of the category AS.

Proposition 4.14. *A morphism $(\eta,\sigma) : (S, s_0, E, I, \mathrm{Tran}) \to (S', s'_0, E', I', \mathrm{Tran}')$ in the category* AS *is polygonal if and only if for any $s_1 \in S$, $e \in E$, $s'_2 \in S'$ the following implication holds* $(\sigma(s_1), \eta_*(e), s'_2) \in \mathrm{Tran}'_* \;\Rightarrow\; (\exists s_2 \in S)(s_1, e, s_2) \in \mathrm{Tran}$.

Proof. It follows by Lemma 4.13 that a morphism (η,σ) of asynchronous systems is polygonal if and only if for all $s_1 \in S$ and $e \in E$ the following implication holds $s_1 \cdot e = * \Rightarrow \sigma(s_1) \cdot \eta_*(e) = *$. By the law of contraposition, we obtain for all $s_1 \in S$ and $e \in E$ that

$$(\exists s'_2 \in S')(\sigma(s_1), \eta_*(e), s'_2) \in \mathrm{Tran}'_* \Rightarrow (\exists s_2 \in S)(s_1, e, s_2) \in \mathrm{Tran}. \qquad (4.9)$$

Taking out from the formula (4.9) the variable s'_2 with the quantifier, we get

$$(\forall s'_2 \in S')\left((\sigma(s_1), \eta_*(e), s'_2) \in \mathrm{Tran}'_* \Rightarrow (\exists s_2 \in S)(s_1, e, s_2) \in \mathrm{Tran}\right).$$

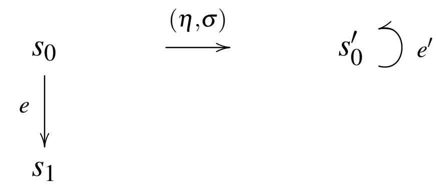
Adding to the formula the quantifiers $(\forall s_1 \in S)(\forall e \in E)$, we obtain the required assertion. This completes the proof.

Example 4.15. Consider the morphism (η,σ) of asynchronous systems:

$$(S, s_0, E, I, Tran) = (\{s_0, s_1\}, s_0, \{e\}, \emptyset, \{(s_0, e, s_1)\}),$$

$$(S', s'_0, E', I', Tran') = (\{s'_0\}, s'_0, \{e'\}, \emptyset, \{(s'_0, e', s'_0)\}),$$

where $\eta(e) = e'$, $\sigma(s_0) = \sigma(s_1) = s'_0$. These asynchronous systems can be shown as follows:



If (η, σ) is polygonal, then for $s_1, e, s'_2 = s'_0$, we have

$$(\sigma(s_1), e', s'_0) \in \mathrm{Tran}'_* \ \Rightarrow \ (\exists s_2 \in S)(s_1, e, s_2) \in \mathrm{Tran}.$$

Since no such s_2, we can conclude that (η, σ) is not polygonal. This example shows that the category AS is not contained in the category $\mathrm{AS}^\flat$.

Let $\mathrm{pt}_* = \{p, *\}$ be a state space with the monoid $M(\emptyset, \emptyset) = \{1\}$. Associating with weak asynchronous system the morphism of state spaces $\mathrm{pt}_* \to (M(E,I), S_*)$ defined as $p \mapsto s_0$, we obtain

Proposition 4.16. *The category of weak asynchronous systems and polygonal morphisms* $\mathrm{AS}^\flat$ *is isomorphic to the comma category* $\mathrm{pt}_*/(FPCM^{\|}, \mathrm{Set}_*)$.

For any complete category $\mathscr{A}$ and object $A \in \mathrm{Ob}\mathscr{A}$, the comma-category $A/\mathscr{A}$ is complete. From Theorem 4.8 and Proposition 4.16 it follows that:

Theorem 4.17. *The category* $\mathrm{AS}^\flat$ *is complete.*

The completeness of AS is shown in [2]. From Theorem 4.8 and Proposition 4.16 it follows that:

Theorem 4.18. *The category* $\mathrm{AS}^\flat$ *is cocomplete.*

5. Conclusion

There are possible applications of the results related with building adjoint functors between the category of $\mathrm{AS}^\flat$ and the category of higher dimensional automata. *Unlabeled semiregular higher dimensional automation* [14] is a contravariant functor from the category of cubes into the category Set. Let Υ_{sr} be a category of unlabeled semiregular higher dimensional automata and natural transformations. By [11, Proposition II.1.3] for each functor from the category of cubes to the category $(FPCM^{\|}, \mathrm{Set})$, there exists a pair of adjoint functors between the categories Υ_{sr} and $(FPCM^{\|}, \mathrm{Set})$. We can take the functor assigning to n-dimensional cube the state space $(\mathbb{N}^n, h_{\mathbb{N}^n})$ where $h_{\mathbb{N}^{op}} : \mathbb{N}^{op} \to \mathrm{Set}$ is the contravariant functor of morphisms. So, we get left adjoint to the composition $(FPCM^{\|}, \mathrm{Set}_*) \to (FPCM^{\|}, \mathrm{Set}) \to \Upsilon_{sr}$. Taking initial point, we obtain adjoint functors between $\mathrm{AS}^{\flat}$ and the category of higher dimensional automata with the initial point.

Considering the comma categories, we can compare the labelled asynchronous systems with labelled higher dimensional automata. For details, we refer to the paper [17].

We have considered deterministic transition systems. More general notions where transitions are nondeterministic were studied as contravariant functors from small categories into the category of relations [1, 12, 25]. Nevertheless, since the category of relations is neither complete nor cocomplete, it is unknown how these models can be compared with higher dimensional automata.

Conflict of Interests

The authors declare that there is no conflict of interests.

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