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EPIMORPHICALLY PRESERVED SEMIGROUP IDENTITIES

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Abstract. In this paper, it is shown that a particular classes of heterotypical identities whose both sides contain repeated variables and are preserved under epis in conjunction with all seminormal identities.

Keywords: Epimorphism, saturated semigroup, saturated variety, closed under epis, preserved under epis, heterotypical identity, semicommutative and seminormal identities.

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1. Introduction

An identity of the form

$$x_1x_2\cdots x_n = x_{i_1}x_{i_2}\cdots x_{i_n}\ (n \geq 2) \tag{1}$$

is called a permutation identity, where i is any permutation of the set $\{1,2,3,\ldots,n\}$ and i_k $(1 \leq k \leq n)$ is the image of k under the permutation i. A permutation identity of the form (1) is said to be nontrivial if the permutation i is different from the identity permutation. Further a nontrivial permutation identity $x_1x_2\cdots x_n = x_{i_1}x_{i_2}\cdots x_{i_n}$ is called *left semicommutative* if $i_1 \neq 1$, *right semicommutative* if $i_n \neq n$ and *seminormal* if $i_1 = 1$ and $i_n = n$. Clearly, every nontrivial permutation identity is either *left semicommutative*, *right semicommutative*, or *seminormal*. A

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semigroup S satisfying a nontrivial permutation identity is said to be permutative, and a variety $\mathscr{V}$ of semigroups is said to be permutative if it admits a nontrivial permutation identity. Commutativity $[xy = yx]$, left normality $[x_1x_2x_3 = x_1x_3x_2]$, right normality $[x_1x_2x_3 = x_2x_1x_3]$, and normality $[x_1x_2x_3x_4 = x_1x_3x_2x_4]$ are some of the well known permutation identities.

For any word u, the *content* of u (necessarily finite) is the set of all variables appearing in u and is denoted by $C(u)$. An identity $u = v$ is said to be *heterotypical* if $C(u) \neq C(v)$; otherwise *homotypical*. A variety $\mathscr{V}$ of semigroups is said to be *heterotypical* if it admits a heterotypical identity.

Let U and S be any semigroups with U a subsemigroup of S. Following Isbell [8], we say that U dominates an element d of S if for every semigroup T and for all homomorphisms $\alpha, \beta : S \to T$, $u\alpha = u\beta$ for all $u \in U$ implies $d\alpha = d\beta$. The set of all elements of S dominated by U is called the *dominion* of U in S, and we denote it by $Dom(U,S)$. It may easily be seen that $Dom(U,S)$ is a subsemigroup of S containing U. A semigroup U is said to be *saturated* if $Dom(U,S) \neq S$ for every properly containing semigroup S, and *epimorphically embedded* or *dense* in S if $Dom(U,S) = S$.

A morphism $\alpha : S \to T$ in the category of all semigroups is called an *epimorphism* (*epi* for short) if for all morphisms β, γ, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$. Every onto morphism is epi, but the converse is not true in general. It may easily be checked that $\alpha : S \to T$ is epi if and only if the inclusion map $i : S\alpha \to T$ is epi and the inclusion map $i : U \to S$ is epi if and only if $Dom(U,S) = S$. A variety $\mathscr{V}$ of semigroups is said to be *saturated* if all its members are saturated and *epimorphically closed* or *closed under epis* if whenever $S \in \mathscr{V}$ and $\varphi : S \to T$ is epi in the category of all semigroups, then $T \in \mathscr{V}$ or equivalently whenever $U \in \mathscr{V}$ and $Dom(U,S) = S$, then $S \in \mathscr{V}$.

An identity μ is said to be preserved under epis in conjunction with an identity τ if whenever S satisfies τ and μ, and $\varphi : S \to T$ is an epimorphism in the category of all semigroups, then T also satisfies τ and μ; or equivalently, whenever U satisfies τ and μ and $Dom(U,S) = S$, then S also satisfies τ and μ.

In [10], Khan had shown that all identities are preserved under epis in conjunction with commutativity. In [11], Khan gave a sufficient condition for a heterotypical variety to be saturated. He showed that if a semigroup variety $\mathscr{V}$ admits a heterotypical identity of which atleast one side has no repeated variable, then $\mathscr{V}$ is saturated, and, hence, all heterotypical identities whose atleast one side has no repeated variable are preserved under epis in conjunction with all non-trivial permutation identities. Khan [13] had further shown that all identities are preserved under epis in conjunction with left [right] semicommutativity. However Higgins [5] had shown that the identity $xyx = yxy$ is not preserved under epis in conjunction with the normality identity $x_1x_2x_3x_4 = x_1x_3x_2x_4$.

Therefore, it is natural to find those semigroup identities whose both sides contain repeated variables and preserved under epis in conjunction with any seminormal identity.

In the present paper, we obtain a result about heterotypical identity (Theorem 3.6) towards this goal, by establishing some sufficient conditions for such identities to lie in this class and thus, extending [1, Theorem 3.4]. However, a full determination of all such identities to be preserved under epis in conjunction with all seminormal permutation identities still remains an open problem.

2. Preliminaries

Now we state some results to be used in the rest of the paper. Our notation will be standard and, for any unexplained symbols and terminology, we refer the reader to Cliford and Preston [4] and Howie [7]. Further in whatever follows, we will often speak of a semigroup *satisfying (1)* to mean that the semigroup in question *satisfies an identity of that type*.

Result 2.1 [13, Theorem 3.1]. *All permutation identities are preserved under epis.*

A most useful characterization of semigroup dominions is provided by Isbell's Zigzag Theorem.

Result 2.2 ([9, Theorem 2.3]).*Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U,S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:* $$d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m} \qquad (2)$$
where $m \geq 1,\ a_i \in U\ (i = 0,1,\ldots,2m)$, $y_i,t_i \in S\ (i = 1,2,\ldots,m)$, and
$a_0 = y_1a_1,\ a_{2m-1}t_m = a_{2m},\ a_{2i-1}t_i = a_{2i}t_{i+1},\ y_ia_{2i} = y_{i+1}a_{2i+1}\ (1 \leq i \leq m-1)$.

Such a series of factorization is called a *zigzag* in S over U with value d, length m and spine

$a_0, a_1, \ldots, a_{2m}$. We refer to the equations in Result 2.2 as *the zigzag equations*.

Result 2.3 [12, Result 3].*Let U be any subsemigroup of a semigroup S and let d in $Dom(U,S) \setminus U$. If (2) is a zigzag of minimal length m over U with value d, then $y_j, t_j \in S \setminus U$ for all $j = 1, 2, \ldots, m$.*

In the following results, let U and S be any semigroups with U dense in S.

Result 2.4 [12, Result 4].*For any $d \in S \setminus U$ and k any positive integer, if (2) is a zigzag of minimal length over U with value d, then there exist $b_1, b_2, \ldots, b_k \in U$ and $d_k \in S \setminus U$ such that $d = b_1 b_2 \cdots b_k d_k$.*

Result 2.5 [12, Corollary 4.2].*If U be permutative, then $sx_1x_2 \cdots x_k t = sx_{j_1}x_{j_2} \cdots x_{j_k} t$, for all $x_1, x_2, \ldots, x_k \in S$, $s, t \in S \setminus U$ and any permutation j of the set $\{1, 2, \ldots, k\}$.*

The symmetrical statement in the following result is in addition to the original statement.

Result 2.6 [13, Proposition 4.6].*Assume that U is permutative. If $d \in S \setminus U$ and (2) be a zigzag of length m over U with value d such that $y_1 \in S \setminus U$, then $d^k = a_0^k t_1^k$ for each positive integer k; in particular, the conclusion holds if (2) is of minimal length. Symmetrically, if $d \in S \setminus U$ and (2) be a zigzag of length m over U with value d such that $t_m \in S \setminus U$, then $d^k = y_m^k a_{2m}^k$ for each positive integer k; in particular, the conclusion holds if (2) is of minimal length.*

3. Main results

Proposition 3.1:*Let U be a permutative semigroup satisfying a seminormal permutation identity of a semigroup S such that $Dom(U,S) = S$. If U satisfies the semigroup identity*

$$x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} = w_1^{t_1} w_2^{t_2} \cdots w_n^{t_n} \tag{3}$$

then (3) also holds for all $x_1, x_2, \ldots, x_r \in S$ and $y_1, y_2, \ldots, y_s, w_1, w_2, \ldots, w_n$ in U, where $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s, t_1, t_2, \ldots, t_n$ are any non negative integers such that: $(r, s, n \geq 1)$; $p_1 \leq p_2 \leq \cdots \leq p_{r-1} \leq p_r$; $q_s \leq q_{s-1} \cdots \leq q_2 \leq q_1$ and $t_1 \leq t_2 \leq \cdots \leq t_n$.

Proof. Assume that U satisfies the identity (3). Therefore

$$u_1^{p_1} u_2^{p_2} \cdots u_r^{p_r} v_1^{q_1} v_2^{q_2} \cdots v_s^{q_s} = a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}$$

for all $u_1,u_2,\ldots,u_r,v_1,v_2,\ldots,v_s,a_1,a_2,\ldots,a_n \in U$.

We show that (3) is true for all $x_1,\ldots,x_r \in S$ and $y_1,\ldots,y_s,w_1,\ldots,w_n \in U$. For $k=1,2,3,\ldots,r$; consider the word $x_1^{p_1}x_2^{p_2}\cdots x_k^{p_k}$ of length $p_1+p_2+\cdots p_k$. We shall show that (3) is satisfied by induction on k, assuming that the remaining elements $x_{k+1},x_{k+2},\ldots,x_r \in U$. First for $k=0$, the equation (3) is vacuously satisfied. So assume next that (3) is true for all $x_1,x_2,\ldots,x_{k-1} \in S$ and all $x_k,x_{k+1},\ldots,x_r \in U$. Then we shall show that (3) is true for all $x_1,x_2,\ldots,x_{k-1},x_k \in S$ and all $x_{k+1},\ldots,x_r$ in U. If $x_k \in U$, then (3) is satisfied by inductive hypothesis. So assume that $x_k \in S\setminus U$. As $x_k \in S\setminus U$ and $Dom(U,S)=S$, by Result 2.2, let (2) be a zigzag of minimal length m over U with value x_k. So assume that $1 \le k < r$.

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

$$= x_1^{p_1}x_2^{p_2}\cdots x_{k-1}^{p_{k-1}}y_m^{p_k}a_{2m}^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

(by zigzag equations and Result 2.6)

$$= zy_m^{(m)\,p_k}b_1^{(m)\,p_k}\cdots b_{k-1}^{(m)\;p_k}a_{2m}^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

(by Results 2.4 and 2.5 for some b$_1^{(m)},\ldots,b_{k-1}^{(m)} \in U$ and $y_m^{(m)} \in S\backslash U$ as $y_m \in S\backslash U$ and $a_{2m}=a_{2m-1}t_m$ with $t_m \in S\backslash U$ and where $z=x_1^{p_1}x_2^{p_2}\cdots x_{k-1}^{p_{k-1}}$)

$$= zy_m^{(m)\,p_k}v^{(m)}b_1^{(m)\,p_1}\cdots b_{k-1}^{(m)\;p_{k-1}}a_{2m}^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

(by Result 2.5 as $y_m^{(m)},\ t_m \ \in\ S\backslash U$ and where $v^{(m)}=b_1^{(m)\,p_k-p_1}\cdots b_{k-1}^{(m)\;p_k-p_{k-1}}$)

$$= zy_m^{(m)\,p_k}v^{(m)}b_1^{(m)\,p_1}\cdots b_{k-1}^{(m)\;p_{k-1}}(a_{2m-1}^2t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}\cdots y_s^{q_s}$$

(as $a_{2m-1}^2t_m=a_{2m-1}a_{2m-1}t_m=a_{2m-1}a_{2m} \in U, y_m^{(m)}, t_m \in S\setminus U$ and U satisfies (3))

$$= zy_m^{(m)\,p_k}v^{(m)}b_1^{(m)\,p_1}\cdots b_{k-1}^{(m)\;p_{k-1}}a_{2m-1}^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

(by Result 2.5 as $y_m^{(m)},t_m \in S\backslash U$)

$$= zy_m^{(m)^{p_k}} b_1^{(m)^{p_k}} \cdots b_{k-1}^{(m)\ p_k} a_{2m-1}^{p_k} (a_{2m-1}t_m)^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$$
(by Result 2.5 as $y_m^{(m)}, t_m \in S\backslash U$ and as $v^{(m)} = b_1^{(m)^{p_k-p_1}} \cdots b_{k-1}^{(m)\ p_k-p_{k-1}}$)

$$= zy_m^{p_k} a_{2m-1}^{p_k} (a_{2m-1}t_m)^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \text{(by Result 2.5 as } y_m^{(m)}, t_m \in S\backslash U)$$

$$= z(y_m a_{2m-1})^{p_k} (a_{2m-1}t_m)^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \text{(by Result 2.5 as } y_m, t_m \in S \setminus U)$$

$$= z(y_{m-1} a_{2m-2})^{p_k} (a_{2m-1}t_m)^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \text{ (by zigzag equations)}$$

$$= zy_{m-1}^{p_k} a_{2m-2}^{p_k} (a_{2m-1}t_m)^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \text{(by Result2.5 as } y_{m-1}, t_m \in S\backslash U)$$

$$= zy_{m-1}^{(m-1)^{p_k}} b_1^{(m-1)^{p_k}} \cdots b_{k-1}^{(m-1)^{p_k}} a_{2m-2}^{p_k} (a_{2m-1}t_m)^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$$
(by Results 2.4 and 2.5 for some $b_1^{(m-1)}, \ldots, b_{k-1}^{(m-1)} \in U$ and $y_{m-1}^{(m-1)} \in S\backslash U$ as $y_{m-1}, t_m \in S\backslash U$)

$$= zy_{m-1}^{(m-1)^{p_k}} v^{(m-1)} b_1^{(m-1)^{p_1}} \cdots b_{k-1}^{(m-1)^{p_{k-1}}} (a_{2m-2}a_{2m-1}t_m)^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$$
(by Result 2.5 as $y_{m-1}^{(m-1)}, t_m \in S\backslash U$ and where $v^{(m-1)} = b_1^{(m-1)^{p_k-p_1}} \cdots b_{k-1}^{(m-1)^{p_k-p_{k-1}}}$)

$$= zy_{m-1}^{(m-1)^{p_k}} v^{(m-1)} b_1^{(m-1)^{p_1}} \cdots b_{k-1}^{(m-1)^{p_{k-1}}} (a_{2m-3}a_{2m-1}t_m)^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$$
(as $a_{2m-3}a_{2m-1}t_m = a_{2m-3}a_{2m} \in U$ and U satisfies (3))

$$= zy_{m-1}^{(m-1)^{p_k}} v^{(m-1)} b_1^{(m-1)^{p_1}} \cdots b_{k-1}^{(m-1)^{p_{k-1}}} a_{2m-3}^{p_k} (a_{2m-1}t_m)^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$$
(by Result 2.5 as $y_{m-1}^{(m-1)}, t_m \in S\backslash U$)

$$= zy_{m-1}^{(m-1)^{p_k}} b_1^{(m-1)^{p_k}} \cdots b_{k-1}^{(m-1)^{p_k}} a_{2m-3}^{p_k} (a_{2m-1}t_m)^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$$
(by Result 2.5 as $v^{(m-1)} = b_1^{(m-1)^{p_k-p_1}} \cdots b_{k-1}^{(m-1)^{p_k-p_{k-1}}}$ and $y_{m-1}^{(m-1)}, t_m \in S\backslash U$)

$$= zy_{m-1}^{p_k} a_{2m-3}^{p_k} (a_{2m-1}t_m)^{p_k} x_{k+1}^{p_{k+1}} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$$
(by Result 2.5 as $y_{m-1}^{(m-1)}, t_m \in S\backslash U$)

$$= \quad z(y_{m-1}a_{2m-3})^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
(by Result 2.5 as $y_{m-1}, t_m \in S\backslash U$)

$$= \quad z(y_{m-2}a_{2m-4})^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$ (by zigzag equations)

$$= \quad zy_{m-2}^{p_k}a_{2m-4}^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
(by Result 2.5 as $y_{m-2}, t_m \in S\backslash U$)

$\vdots$

$$= \quad zy_2^{p_k}a_4^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

$$= \quad zy_2^{(2)^{p_k}}b_1^{(2)^{p_k}}\cdots b_{k-1}^{(2)\ p_k}a_4^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$ (by Results 2.4 and 2.5 for some $b_1^{(2)},\ldots,b_{k-1}^{(2)} \in U$ and $y_2^{(2)} \in S\backslash U$ as $y_2, t_m \in S\backslash U$)

$$= \quad zy_2^{(2)^{p_k}}v^{(2)}b_1^{(2)^{p_1}}\cdots b_{k-1}^{(2)\ p_{k-1}}a_4^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$ (by Result 2.5 as $y_2^{(2)}, t_m \in S\backslash U$ and where $v^{(2)} = b_1^{(2)^{p_k-p_1}}\cdots b_{k-1}^{(2)\ p_k-p_{k-1}}$)

$$= \quad zy_2^{(2)^{p_k}}v^{(2)}b_1^{(2)^{p_1}}\cdots b_{k-1}^{(2)\ p_{k-1}}(a_4a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
(by Result 2.5 as $y_2^{(2)}, t_m \in S\backslash U$)

$$= \quad zy_2^{(2)^{p_k}}v^{(2)}b_1^{(2)^{p_1}}\cdots b_{k-1}^{(2)\ p_{k-1}}(a_3a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
(as $a_3a_{2m-1}t_m = a_3a_{2m} \in U$ and U satisfies (3))

$$= \quad zy_2^{(2)^{p_k}}v^{(2)}b_1^{(2)^{p_1}}\cdots b_{k-1}^{(2)\ p_{k-1}}a_3^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
(by Result 2.5 as $y_2^{(2)}, t_m \in S\backslash U$)

$$= \quad zy_2^{(2)^{p_k}}b_1^{(2)^{p_k}}\cdots b_{k-1}^{(2)\ p_k}a_3^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
(by Result 2.5 as $y_2^{(2)}, t_m \in S\backslash U$ and $v^{(2)} = b_1^{(2)^{p_k-p_1}}\cdots b_{k-1}^{(2)\ p_k-p_{k-1}}$)

$$= \quad zy_2^{p_k}a_3^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$ (by Result 2.5 as $y_2^{(2)}, t_m \in S\backslash U$)

$$= z(y_2a_3)^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} \text{ (by Result 2.5 as } y_2,t_m\in S\backslash U)$$

$$= z(y_1a_2)^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} \text{ (by zigzag equations)}$$

$$= zy_1^{p_k}a_2^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} \text{ (by Result 2.5 as } y_1,t_m\in S\backslash U)$$

$$= zy_1^{(1)^{p_k}}b_1^{(1)^{p_k}}\cdots b_{k-1}^{(1)\ p_k}a_2^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} \text{ (by Results 2.4 and 2.5 for some } b_1^{(1)},\dots,b_{k-1}^{(1)}\in U \text{ and } y_1^{(1)}\in S\backslash U \text{ as } y_1,t_m\in S\backslash U)$$

$$= zy_1^{(1)^{p_k}}v^{(1)}b_1^{(1)^{p_1}}\cdots b_{k-1}^{(1)\ p_{k-1}}a_2^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
$$\text{(by Result 2.5 as } y_1^{(1)},t_m\in S\backslash U \text{ and where } v^{(1)}=b_1^{(1)^{p_k-p_1}}\cdots b_{k-1}^{(1)\ p_k-p_{k-1}})$$

$$= zy_1^{(1)^{p_k}}v^{(1)}b_1^{(1)^{p_1}}\cdots b_{k-1}^{(1)\ p_{k-1}}(a_2a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
$$\text{(by Result 2.5 as } y_1^{(1)},t_m\in S\backslash U)$$

$$= zy_1^{(1)^{p_k}}v^{(1)}b_1^{(1)^{p_1}}\cdots b_{k-1}^{(1)\ p_{k-1}}(a_1a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
$$\text{(as } a_2a_{2m-1}t_m=a_2a_{2m}\in U \text{ and U satisfies (3))}$$

$$= zy_1^{(1)^{p_k}}v^{(1)}b_1^{(1)^{p_1}}\cdots b_{k-1}^{(1)\ p_{k-1}}a_1^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
$$\text{(by Result 2.5 as } y_1^{(1)},t_m\in S\backslash U)$$

$$= zy_1^{(1)^{p_k}}b_1^{(1)^{p_k}}\cdots b_{k-1}^{(1)\ p_k}a_1^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
$$\text{(by Result 2.5 as } y_1^{(1)},t_m\in S\backslash U \text{ and } v^{(1)}=b_1^{(1)^{p_k-p_1}}\cdots b_{k-1}^{(1)\ p_k-p_{k-1}})$$

$$= zy_1^{p_k}a_1^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} \text{(by Result 2.5 as } y_1^{(1)},t_m\in S\backslash U)$$

$$= x_1^{p_1}x_2^{p_2}\cdots x_{k-1}^{p_{k-1}}y_1^{p_k}a_1^{p_k}(a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} \text{ (as } z=x_1^{p_1}x_2^{p_2}\cdots x_{k-1}^{p_{k-1}})$$

$$= x_1^{p_1}x_2^{p_2}\cdots x_{k-1}^{p_{k-1}}(y_1a_1a_{2m-1}t_m)^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
$$\text{(by Result 2.5 as } y_1,t_m \text{ in } S\backslash U)$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_{k-1}^{p_{k-1}}(a_0a_{2m})^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} \text{ (by the zigzag equations)}$$

$$= \quad w_1^{t_1}w_2^{t_2}\cdots w_n^{t_n}$$
$$\text{(by inductive hypothesis as } a_0a_{2m}\in U)$$

as required. □

Proposition 3.2: *Let U be a permutative semigroup satisfying a seminormal permutation identity of a semigroup S such that $Dom(U,S)=S$. If U satisfies the semigroup identity (3), then (3) also holds for all $x_1,\ldots,x_r,y_1,\ldots,y_s\in S$ and $w_1,\ldots,w_n$ in U, where $p_1,p_2,\ldots,p_r;q_1,q_2,\ldots,q_s;$ $t_1,t_2,\ldots,t_n$, are any non negative integers such that: $(r,s,n\geq 1)$; $p_1\leq p_2\leq\cdots\leq p_{r-1}\leq p_r$, $q_s\leq q_{s-1}\cdots\leq q_2\leq q_1$ and $t_1\leq t_2\leq\cdots\leq t_n$.*

Proof: We show that (3) is true for all $x_1,x_2,\ldots,x_r,y_1,y_2,\ldots,y_s\in S$ and $w_1,w_2,\ldots,w_n\in U$. For $k=1,2,3,\ldots,s$; consider the word $y_1^{q_1}y_2^{q_2}\cdots y_k^{q_k}$ of length $q_1+q_2+\cdots+q_k$. We shall show that (3) is satisfied by induction on k assuming that the remaining elements $y_{k+1},y_{k+2},...,y_s$ in U. For $k=0$, (3) trivially holds. So assume that (3) is true for all $x_1,x_2,\ldots,x_r,y_1,y_2,\ldots,y_{k-1}\in S$ and for all $y_k,y_{k+1},\ldots,y_s\in U$. Then we shall show that (3) is true for all $x_1,x_2,\ldots,x_r,y_1,y_2,\ldots,y_k\in S$ and $y_{k+1},\ldots,y_s\in U$. If $y_k\in U$, then (3) holds by inductive hypothesis. So assume that $y_k\in S\backslash U$. As $y_k\in S\backslash U$ and $Dom(U,S)=S$, by Result 2.2, let (2) be a zigzag of minimal length m over U with value y_k. So assume that $1\leq k<r$. As the equalities (4) and (5) follow by Results 2.4 and 2.5 for some $b_{k+1}^{(1)},\ldots,b_r^{(1)}\in U$ and $t_1^{(1)}$ in $S\setminus U$ as $y_1,\ t_1\in S\setminus U$ and where $z=y_{k+1}^{q_{k+1}}\cdots y_s^{q_s}$ and $w^{(1)}=b_{k+1}^{(1)\ q_k-q_{k+1}}\cdots b_r^{(1)q_k-q_s}$, we have

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}t_1^{q_k}y_{k+1}^{q_{k+1}}\cdots y_s^{q_s} \text{ (by zigzag equations and Result 2.6)}$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}b_{k+1}^{(1)\ q_k}\cdots b_s^{(1)q_k}t_1^{(1)q_k}y_{k+1}^{q_{k+1}}\cdots y_s^{q_s} \tag{4}$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}b_{k+1}^{(1)\ q_{k+1}}\cdots b_s^{(1)q_s}w^{(1)}t_1^{(1)q_k}z \tag{5}$$

$$\begin{aligned}
&= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1^2)^{q_k}{b_{k+1}^{(1)}}^{q_{k+1}}\cdots {b_s^{(1)}}^{q_s}w^{(1)}{t_1^{(1)}}^{q_k}z \\
&\quad \text{(by inductive hypothesis as } y_1a_1^2=y_1a_1a_1=a_0a_1\in U) \\
&= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1)^{q_k}a_1^{q_k}{b_{k+1}^{(1)}}^{q_{k+1}}\cdots {b_s^{(1)}}^{q_s}w^{(1)}{t_1^{(1)}}^{q_k}z \\
&\quad \text{(by Result 2.5 as } y_1,t_1^{(1)}\in S\backslash U) \\
&= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1)^{q_k}a_1^{q_k}{b_{k+1}^{(1)}}^{q_k}\cdots {b_s^{(1)}}^{q_k}{t_1^{(1)}}^{q_k}z \\
&\quad \text{(by Result 2.5 and definition of } w^{(1)}) \\
&= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1)^{q_k}a_1^{q_k}t_1^{q_k}z \\
&\quad \text{(by Result 2.5 as } {b_{k+1}^{(1)}}^{q_k}\cdots {b_s^{(1)}}^{q_k}{t_1^{(1)}}^{q_k}=t_1^{q_k}) \\
&= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1)^{q_k}(a_1t_1)^{q_k}z \text{ (by Result 2.5 as } y_1,t_1\in S\backslash U) \\
&= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1)^{q_k}(a_2t_2)^{q_k}z \text{ (by zigzag equations)} \\
&= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1)^{q_k}a_2^{q_k}t_2^{q_k}z \text{ (by Result 2.5 as } y_1,t_2\in S\backslash U) \\
&= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1a_2)^{q_k}t_2^{q_k}z \text{ (by Result 2.5 as } y_1,t_2\in S\backslash U) \\
&= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1a_2)^{q_k}{b_{k+1}^{(2)}}^{q_k}\cdots {b_s^{(2)}}^{q_k}{t_2^{(2)}}^{q_k}z
\end{aligned}$$

where the last equality follows by Results 2.4 and 2.5 for some $b_{k+1}^{(1)},\ldots,b_s^{(1)}$ in U, $t_2^{(2)}$ in $S\setminus U$ as $y_1,\ t_2\ \in\ S\setminus U$.

As the equalities (6), (7) and (8) follow by letting $w^{(2)}={b_{k+1}^{(2)}}^{q_k-q_{k+1}}\cdots {b_s^{(2)}}^{q_k-q_s}$ and by Result 2.5 as $y_1,t_2^{(2)}\in S\setminus U$; by Result 2.5 as $y_1,t_2^{(2)}\in S\setminus U$; and by Result 2.5 and the definition of $w^{(2)}$ respectively, we have

$$\begin{aligned}
&x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} \\
&= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1a_2)^{q_k}{b_{k+1}^{(2)}}^{q_k}\cdots {b_s^{(2)}}^{q_k}{t_2^{(2)}}^{q_k}z
\end{aligned}$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1a_2)^{q_k}b_{k+1}^{(2)\ q_{k+1}}\cdots b_s^{(2)q_s}w^{(2)}t_2^{(2)q_s}z \qquad (6)$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1a_3)^{q_k}b_{k+1}^{(2)\ q_{k+1}}\cdots b_s^{(2)q_s}w^{(2)}t_2^{(2)q_k}z$$
(by inductive hypothesis as $y_1a_1a_3 = a_0a_3 \in U$)

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1)^{q_k}a_3^{q_k}b_{k+1}^{(2)\ q_{k+1}}\cdots b_s^{(2)q_s}w^{(2)}t_2^{(2)q_k}z \qquad (7)$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1)^{q_k}a_3^{q_k}b_{k+1}^{(2)\ q_k}\cdots b_s^{(2)q_k}t_2^{(2)q_k}z \qquad (8)$$

$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1)^{q_k}a_3^{q_k}t_2^{q_k}z$ (by Result 2.5 as$b_{k+1}^{(2)\ q_k}\cdots b_s^{(2)q_k}t_2^{(2)q_k} = t_2^{q_k}$)

$\vdots$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1)^{q_k}a_{2m-3}^{q_k}t_{m-1}^{q_k}z$$

$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1)^{q_k}(a_{2m-3}t_{m-1})^{q_k}z$ (by Result 2.5 as y_1, t_{m-1} in $S\setminus U$)

$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1)^{q_k}(a_{2m-1}t_m)^{q_k}z$ (by zigzag equations) ‘

$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1)^{q_k}a_{2m-2}^{q_k}t_m^{q_k}z$ (by Result 2.5 as $y_1, t_m \in S\setminus U$)

$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1a_{2m-2})^{q_k}t_m^{q_k}z$ (by Result 2.5 as $y_1, t_m \in S\setminus U$)

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1a_{2m-2})^{q_k}b_{k+1}^{(m)\ q_k}\cdots b_s^{(m)q_k}t_m^{(m)q_k}z$$

where the last equality follows by Results 2.4 and 2.5 for some $b_{k+1}^{(m)},\ldots,b_s^{(m)} \in U$ and $t_m^{(m)} \in S\setminus U$ as $y_1, t_m \in S\setminus U$. As the equality (9) follows by Result 2.5 as $y_1, t_m^{(m)} \in S\setminus U$ and where $w^{(m)} = b_{k+1}^{(m)\ q_k-q_{k+1}}\cdots b_s^{(m)q_k-q_r}$, we have

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1a_{2m-2})^{q_k}b_{k+1}^{(m)\ q_{k+1}}\cdots b_s^{(m)q_s}w^{(m)}t_m^{(m)q_k}z \qquad (9)$$

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1a_{2m-1})^{q_k}b_{k+1}^{(m)\ q_{k+1}}\cdots b_s^{(m)q_s}w^{(m)}t_m^{(m)q_k}z$$

(by inductive hypothesis as $y_1a_1a_{2m-1} = a_0a_{2m-1} \in U$)

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1a_{2m-1})^{q_k}b_{k+1}^{(m)\ q_k}\cdots b_s^{(m)q_k}t_m^{(m)q_k}z \tag{10}$$

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1a_{2m-1})^{q_k}t_m^{q_k}z$$

(by Result 2.5 as $b_{k+1}^{(m)\ q_k}\cdots b_s^{(m)q_k}t_m^{(m)q_k} = t_m^{q_k}$)

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(y_1a_1a_{2m-1}t_m)^{q_k}y_{k+1}{}^{q_{k+1}}\cdots y_s{}^{q_s}$$

(by Result 2.5 and the definition of z)

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}(a_0a_{2m})^{q_k}y_{k+1}{}^{q_{k+1}}\cdots y_s{}^{q_s} \text{ (by zigzag equations)}$$

$$= w_1^{t_1}w_2^{t_2}\cdots w_n^{t_n} \text{ (by inductive hypothesis as } a_0a_{2m} \in U),$$

where equality (10) follows by Result 2.5 as $y_1, t_m^{(m)} \in S\backslash U$ and the definition of $w^{(m)}$.
Therefore

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} = w_1^{t_1}w_2^{t_2}\cdots w_n^{t_n}$$

holds for all $x_1, x_2, \ldots x_r, y_1, y_2, \ldots, y_s \in S$ and $w_1, w_2, \ldots, w_n \in U$. $\square$

Proposition 3.3: *Let U be a permutative semigroup satisfying a seminormal permutation identity of a semigroup S such that $Dom(U,S) = S$. If U satisfies the semigroup identity (3), then (3) also holds for all $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s \in U$ and $w_1, \ldots, w_n$ in S, where $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s, t_1, t_2, \ldots, t_n$ are any non negative integers $(r,s,n \geq 1)$; $p_1 \leq p_2 \leq \cdots \leq p_{r-1} \leq p_r$, $q_s \leq q_{s-1} \cdots \leq q_2 \leq q_1$ and $t_1 \leq t_2 \leq \cdots \leq t_n$.*

Proof: We show that (3) is true for all $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s \in U$ and $w_1, w_2, \ldots, w_n \in S$. For $k = 1, 2, 3, \ldots, r$; consider the word $w_1^{t_1}w_2^{t_2}\cdots w_k^{t_k}$ of length $t_1 + t_2 + \cdots + t_k$. We shall show that (3) holds for all $w_1, w_2, \ldots, w_n \in S$ and $x_1, \ldots, x_r, y_1, \ldots, y_s$ in U by induction on k assuming that the remaining elements $w_{k+1}, w_{k+2}, ..., w_r$ in U. For $k = 0$, (3) trivially holds. So assume that (3) is true for all $w_1, w_2, \ldots, w_{k-1} \in S$ and for all $w_k, w_{k+1}, \ldots, w_n \in U$. Then we shall show that (3) is true for all $w_1, w_2, \ldots, w_k \in S$ and $w_{k+1}, \ldots, w_n \in U$. If $w_k \in U$, then (3) holds by inductive hypothesis. So assume that $w_k \in S \backslash U$. As $w_k \in S \backslash U$ and $Dom(U,S) = S$, by Result 2.2, let (2) be a zigzag of minimal length m over U with value w_k. Now for $1 \leq k < r$

$w_1^{t_1} w_2^{t_2} \cdots w_n^{t_n}$

$= w_1^{t_1} w_2^{t_2} \cdots w_{k-1}^{t_{k-1}} y_m^{t_k} a_{2m}^{a} t_k w_{k+1}^{t_{k+1}} \cdots w_n^{t_n}$ (by zigzag equations and Result 2.6)

$= z {y_m^{(m)}}^{t_k} {b_1^{(m)}}^{t_k} \cdots {b_{k-1}^{(m)}}^{t_k} a_{2m}^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n}$
(by Results 2.4 and 2.5 for some $b_1^{(m)}, \ldots\ldots, b_{k-1}^{(m)} \in U$
and $y_m^{(m)} \in S \backslash U$ as $y_m \in S \backslash U$ and $a_{2m} = a_{2m-1} t_m$
with $t_m \in S \backslash U$ and where $z = w_1^{t_1} w_2^{t_2} \cdots w_{k-1}^{t_{k-1}}$)

$= z {y_m^{(m)}}^{t_k} v^{(m)} {b_1^{(m)}}^{t_1} \cdots {b_{k-1}^{(m)}}^{t_{k-1}} a_{2m}^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n}$
(by Result 2.5 as $y_m^{(m)}$, $t_m \in S \backslash U$ and where
$v^{(m)} = {b_1^{(m)}}^{t_k - t_1} \cdots {b_{k-1}^{(m)}}^{t_k - t_{k-1}}$)

$= z {y_m^{(m)}}^{t_k} v^{(m)} {b_1^{(m)}}^{t_1} \cdots {b_{k-1}^{(m)}}^{t_{k-1}} (a_{2m-1}^2 t_m)^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n}$
(as $a_{2m-1}^2 t_m = a_{2m-1} a_{2m-1} t_m = a_{2m-1} a_{2m} \in U$,
$y_m^{(m)}, t_m \in S \setminus U$ and U satisfies (3))

$= z {y_m^{(m)}}^{t_k} v^{(m)} {b_1^{(m)}}^{t_1} \cdots {b_{k-1}^{(m)}}^{t_{k-1}} a_{2m-1}^{t_k} (a_{2m-1} t_m)^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n}$
(by Result 2.5 as $y_m^{(m)}, t_m \in S \backslash U$)

$= z {y_m^{(m)}}^{t_k} {b_1^{(m)}}^{t_k} \cdots {b_{k-1}^{(m)}}^{t_k} a_{2m-1}^{t_k} (a_{2m-1} t_m)^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n}$
(by Result 2.5 as $y_m^{(m)}, t_m \in S \backslash U$ and as
$v^{(m)} = {b_1^{(m)}}^{t_k - t_1} \cdots {b_{k-1}^{(m)}}^{t_k - t_{k-1}}$)

$$\begin{aligned}
&= zy_m^{t_k} a_{2m-1}^{t_k} (a_{2m-1} t_m)^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n} \\
&\quad \text{(by Result 2.5 as } y_m^{(m)}, t_m \in S\backslash U \text{ and as } y_m^{(m)^{t_k}} b_1^{(m)^{t_k}} \cdots b_{k-1}^{(m)\ t_k} = y_m^{t_k}) \\[2ex]
&= z(y_m a_{2m-1})^{t_k} (a_{2m-1} t_m)^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n} \\
&\quad \text{(by Result 2.5 as } y_m, t_m \in S \setminus U) \\[2ex]
&= z(y_{m-1} a_{2m-2})^{t_k} (a_{2m-1} t_m)^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n} \text{ (by zigzag equations)} \\[2ex]
&= zy_{m-1}^{t_k} a_{2m-2}^{t_k} (a_{2m-1} t_m)^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n} \\
&\quad \text{(by Result2.5 as } y_{m-1}, t_m \in S\backslash U) \\[2ex]
&= zy_{m-1}^{(m-1)^{t_k}} b_1^{(m-1)^{t_k}} \cdots b_{k-1}^{(m-1)^{t_k}} a_{2m-2}^{t_k} (a_{2m-1} t_m)^{t_k} x_{k+1}^{t_{k+1}} \cdots w_n^{t_n} \\
&\quad \text{(by Results 2.4 and 2.5 for some } b_1^{(m-1)}, \ldots, b_{k-1}^{(m-1)} \in U \\
&\quad \text{and } y_{m-1}^{(m-1)} \in S\backslash U \text{ as } y_{m-1}, t_m \in S\backslash U) \\[2ex]
&= zy_{m-1}^{(m-1)^{t_k}} v^{(m-1)} b_1^{(m-1)^{t_1}} \cdots b_{k-1}^{(m-1)^{t_{k-1}}} (a_{2m-2} a_{2m-1} t_m)^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n} \\
&\quad \text{(by Result 2.5 as } y_{m-1}, t_m \in S\backslash U \text{ and where} \\
&\quad v^{(m-1)} = b_1^{(m-1)^{t_k - t_1}} \cdots b_{k-1}^{(m-1)^{t_k - t_{k-1}}}) \\[2ex]
&= zy_{m-1}^{(m-1)^{t_k}} v^{(m-1)} b_1^{(m-1)^{t_1}} \cdots b_{k-1}^{(m-1)^{t_{k-1}}} (a_{2m-3} a_{2m-1} t_m)^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n} \\
&\quad \text{(as } a_{2m-3} a_{2m-1} t_m \ = \ a_{2m-3} a_{2m} \in U \text{ and U satisfies (3))} \\[2ex]
&= zy_{m-1}^{(m-1) t_k} v^{(m-1)} b_1^{(m-1)^{t_1}} \cdots b_{k-1}^{(m-1)^{t_{k-1}}} a_{2m-3}^{t_k} (a_{2m-1} t_m)^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n} \\
&\quad \text{(by Result 2.5 as } y_{m-1}^{(m-1)}, t_m \in S\backslash U) \\[2ex]
&= zy_{m-1}^{(m-1)^{t_k}} b_1^{(m-1)^{t_k}} \cdots b_{k-1}^{(m-1)^{t_k}} a_{2m-3}^{t_k} (a_{2m-1} t_m)^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n} \\
&\quad \text{(by Result 2.5 as } v^{(m-1)} = b_1^{(m-1)^{t_k - t_1}} \cdots b_{k-1}^{(m-1)^{t_k - t_{k-1}}} \\
&\quad \text{and } y_{m-1}^{(m-1)}, t_m \in S\backslash U)
\end{aligned}$$

$$= \quad zy_{m-1}^{t_k}a_{2m-3}^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n} \text{ (by Result 2.5 as } y_{m-1}^{(m-1)}, t_m \in S\backslash U \text{ and } y_{m-1}^{(m-1)^{t_k}} b_1^{(m-1)^{t_k}} \cdots b_{k-1}^{(m-1)^{t_k}} = y_{m-1}^{t_k})$$

$$= \quad z(y_{m-1}a_{2m-3})^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n} \text{ (by Result 2.5 as } y_{m-1}, t_m \in S\backslash U)$$

$$= \quad z(y_{m-2}a_{2m-4})^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n} \text{ (by zigzag equations)}$$

$$= \quad zy_{m-2}^{t_k}a_{2m-4}^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n} \text{ (by Result 2.5 as } y_{m-2}, t_m \in S\backslash U)$$

$\vdots$

$$= \quad zy_2^{t_k}a_4^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n}$$

$$= \quad zy_2^{(2)^{t_k}} b_1^{(2)^{t_k}} \cdots b_{k-1}^{(2)\ t_k} a_4^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n} \text{ (by Results 2.4 and 2.5 for some } b_1^{(2)},\dots,b_{k-1}^{(2)} \in U \text{ and } y_2^{(2)} \in S\backslash U \text{ as } y_2, t_m \in S\backslash U)$$

$$= \quad zy_2^{(2)^{t_k}} v^{(2)} b_1^{(2)^{t_1}} \cdots b_{k-1}^{(2)\ t_{k-1}} a_4^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n} \text{ (by Result 2.5 as } y_2^{(2)}, t_m \in S\backslash U \text{ and where } v^{(2)} = b_1^{(2)^{t_k-t_1}} \cdots b_{k-1}^{(2)\ t_k-t_{k-1}})$$

$$= \quad zy_2^{(2)^{t_k}} v^{(2)} b_1^{(2)^{t_1}} \cdots b_{k-1}^{(2)\ t_{k-1}} (a_4a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n} \text{ (by Result 2.5 as } y_2^{(2)}, t_m \in S\backslash U)$$

$$= \quad zy_2^{(2)^{t_k}} v^{(2)} b_1^{(2)^{t_1}} \cdots b_{k-1}^{(2)\ t_{k-1}} (a_3a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n} \text{ (as } a_3a_{2m-1}t_m = a_3a_{2m} \in U \text{ and U satisfies (3))}$$

$$= \quad zy_2^{(2)^{t_k}} v^{(2)} b_1^{(2)^{t_1}} \cdots b_{k-1}^{(2)\ t_{k-1}} a_3^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n} \text{ (by Result 2.5 as } y_2^{(2)}, t_m \in S\backslash U)$$

$$= \quad zy_2^{(2)^{t_k}} b_1^{(2)^{t_k}} \cdots b_{k-1}^{(2)\ t_k} a_3^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n} \text{ (by Result 2.5 as } y_2^{(2)}, t_m \in S\backslash U \text{ and } v^{(2)} = b_1^{(2)^{t_k-t_1}} \cdots b_{k-1}^{(2)\ t_k-t_{k-1}})$$

$$
\begin{aligned}
&= zy_2^{t_k}a_3^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n}\\
&\quad \text{(by Result 2.5 as } y_2^{(2)}, t_m \in S\backslash U \text{ and } y_2^{(2)t_k}b_1^{(2)t_k}\cdots b_{k-1}^{(2)t_k} = y_2^{t_k})\\[2ex]
&= z(y_2a_3)^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n} \text{ (by Result 2.5 as } y_2, t_m \in S\backslash U)\\[2ex]
&= z(y_1a_2)^{t_k}(a_{2m-1}t_m)^{p_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n} \text{ (by zigzag equations)}\\[2ex]
&= zy_1^{t_k}a_2^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n} \text{ (by Result 2.5 as } y_1, t_m \in S\backslash U)\\[2ex]
&= zy_1^{(1)^{t_k}}b_1^{(1)^{t_k}}\cdots b_{k-1}^{(1)\ t_k}a_2^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n} \text{ (by Results 2.4 and}\\
&\quad \text{2.5 for some } b_1^{(1)},\ldots,b_{k-1}^{(1)} \in U \text{ and } y_1^{(1)} \in S\backslash U \text{ as } y_1, t_m \in S\backslash U)\\[2ex]
&= zy_1^{(1)^{t_k}}v^{(1)}b_1^{(1)^{t_1}}\cdots b_{k-1}^{(1)\ t_{k-1}}a_2^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n}\\
&\quad \text{(by Result 2.5 as } y_1^{(1)}, t_m \in S\backslash U \text{ and where}\\
&\quad v^{(1)} = b_1^{(1)^{t_k-t_1}}\cdots b_{k-1}^{(1)\ t_k-t_{k-1}})\\[2ex]
&= zy_1^{(1)^{t_k}}v^{(1)}b_1^{(1)^{t_1}}\cdots b_{k-1}^{(1)\ t_{k-1}}(a_2a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n}\\
&\quad \text{(by Result 2.5 as } y_1^{(1)}, t_m \in S\backslash U)\\[2ex]
&= zy_1^{(1)^{t_k}}v^{(1)}b_1^{(1)^{t_1}}\cdots b_{k-1}^{(1)\ t_{k-1}}(a_1a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n}\\
&\quad \text{(as } a_2a_{2m-1}t_m = a_2a_{2m} \in U \text{ and U satisfies (3))}\\[2ex]
&= zy_1^{(1)^{t_k}}v^{(1)}b_1^{(1)^{t_1}}\cdots b_{k-1}^{(1)\ t_{k-1}}a_1^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n}\\
&\quad \text{(by Result 2.5 as } y_1^{(1)}, t_m \in S\backslash U)\\[2ex]
&= zy_1^{(1)^{t_k}}b_1^{(1)^{t_k}}\cdots b_{k-1}^{(1)\ t_k}a_1^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n}\\
&\quad \text{(by Result 2.5 as } y_1^{(1)}, t_m \in S\backslash U \text{ and } v^{(1)} = b_1^{(1)^{t_k-t_1}}\cdots b_{k-1}^{(1)\ t_k-t_{k-1}})\\[2ex]
&= zy_1^{t_k}a_1^{t_k}(a_{2m-1}t_m)^{t_k}w_{k+1}^{t_{k+1}}\cdots w_n^{t_n}\\
&\quad \text{(by Result 2.5 as } y_1^{(1)}, t_m \in S\backslash U \text{ and } y_1^{(1)t_k}b_1^{(1)t_k}\cdots b_{k-1}^{(1)t_k} = y_1^{t_k})
\end{aligned}
$$

$$= \; w_1^{t_1} w_2^{t_2} \cdots w_{k-1}^{t_{k-1}} (y_1 a_1 a_{2m-1} t_m)^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n} \text{ (by Result 2.5 as } y_1, t_m \text{ in } S\backslash U)$$

$$= \; w_1^{t_1} w_2^{t_2} \cdots w_{k-1}^{t_{k-1}} (a_0 a_{2m})^{t_k} w_{k+1}^{t_{k+1}} \cdots w_n^{t_n} \text{ (by the zigzag equations)}$$

$$= \; x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \text{ (by inductive hypothesis as } a_0 a_{2m} \in U).$$

Therefore

$$w_1^{t_1} w_2^{t_2} \cdots w_n^{t_n} \; = \; x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}.$$

Now using Propositions 3.2, 3.3 and 3.4, we have the following:

Theorem 3.4: *All semigroup identities of the form*

$$x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} = w_1^{t_1} w_2^{t_2} \cdots w_n^{t_n};$$

are preserved under epis in conjunction with all seminormal permutation identities for all non negative integers $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s, t_1, t_2, \ldots, t_n$ $(r, s, n \geq 1)$*;* $p_1 \leq p_2 \leq \cdots \leq p_{r-1} \leq p_r$*,* $q_s \leq q_{s-1} \cdots \leq q_2 \leq q_1$*,* $t_1 \leq t_2 \leq \cdots \leq t_n$*.*

Proof: Take any $x_1, \ldots, x_r, y_1, \ldots, y_s, w_1, \ldots, w_n \in S$. Then by proposition 3.2, for any $u_1, u_2, \ldots, u_n \in U$, we have

$$x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} = u_1^{t_1} u_2^{t_2} \cdots u_n^{t_n} \tag{11}$$

Again, by proposition 3.3, for any $v_1, v_2, \ldots, v_{r+s} \in U$, we have

$$w_1^{t_1} w_2^{t_2} \cdots w_n^{t_n} = v_1^{p_1} v_2^{p_2} \cdots v_r^{p_r} v_{r+1}^{q_1} \cdots v_{r+s}^{q_s} \tag{12}$$

Now,

$$x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$$

$$= \; u_1^{t_1} u_2^{t_2} \cdots u_n^{t_n} \text{ (by equality (11))}$$

$$= \; v_1^{p_1} v_2^{p_2} \cdots v_r^{p_r} v_{r+1}^{q_1} \cdots v_{r+s}^{q_s} \text{ ((as U satisfies (3))}$$

$$= \; w_1^{t_1} w_2^{t_2} \cdots w_n^{t_n} \text{ (by equality (12))}$$

as required. $\square$

Similarly, we can prove the following theorem:

Theorem 3.5: *All semigroup identities of the form*

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r} = w_1^{t_1}w_2^{t_2}\cdots w_n^{t_n}z_1^{l_1}z_2^{l_2}\cdots z_m^{l_m};$$

are preserved under epis in conjunction with all seminormal permutation identities for all non negative integers $p_1, p_2, \ldots, p_r, t_1, t_2, \ldots, t_n, l_1, l_2, \ldots, l_m (r,n,m \geq 1)$*;* $p_1 \leq p_2 \leq \cdots \leq p_{r-1} \leq p_r$*,* $t_1 \leq t_2 \leq \cdots \leq t_n$ *and* $l_m \leq l_{m-1} \cdots \leq l_2 \leq l_1$.

Now using Theorems 3.4 and 3.5, Finally, we have the following main theorem which is the extension of [1, Theorem 3.4]:

Theorem 3.6: *All semigroup identities of the form*

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} = w_1^{t_1}w_2^{t_2}\cdots w_n^{t_n}z_1^{l_1}z_2^{l_2}\cdots z_m^{l_m};$$

are preserved under epis in conjunction with all seminormal permutation identities for all non negative integers $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s, t_1, t_2, \ldots, t_n, l_1, l_2, \ldots, l_m$ $(r,s,n,m \geq 1)$*;* $p_1 \leq p_2 \leq \cdots \leq p_{r-1} \leq p_r$*,* $q_s \leq q_{s-1} \cdots \leq q_2 \leq q_1$*,* $t_1 \leq t_2 \leq \cdots \leq t_n$ *and* $l_m \leq l_{m-1} \cdots \leq l_2 \leq l_1$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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