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A NOTE ON E-UNITARY INVERSE ω-SEMIGROUPS

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Abstract. In this paper, we investigate the Bruck-Reilly extension of monoids and the Clifford semigroups and then determine when they are E-unitary. This motivates us to characterize E-unitary bisimple inverse ω-semigroups and E-unitary simple inverse ω-semigroups respectively.

Keywords: Bruck-Reilly extensions; Clifford semigroups; E-unitary semigroups; ω-semigroups.

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1. Introduction and Preliminaries

E-unitary inverse semigroups form one of the most important classes of inverse semigroups. Indeed McAlister [2], [3] proved two remarkable theorems concerning these semigroups: (1) every E-unitary inverse semigroup admits a faithful representation as a P-semigroup (which is reminiscent of a semidirect product of a semilattice and a group), and (2) every inverse semigroup is an idempotent separating homomorphic image of an E-unitary inverse semigroup. Munn and Reilly [4] devised a different proof of both of these theorems.

In this paper, we study E-unitary inverse ω-semigroups using the Bruck-Reilly extension of monoids and the Clifford semigroups. In particular, we prove that the Bruck-Reilly extension of monoids is an E-unitary bisimple inverse ω-semigroup while the Clifford semigroup is an E-unitary simple inverse ω-semigroup.

Now we recall some definitions which will be useful in the sequel.

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We refer the reader to [1] for more detailed knowledge.

Definition 1.1. Let S be an inverse semigroup and let $a, b \in S$. Then $a \leq b$ if there exists an idempotent e in S such that $a = be$.

Definition 1.2. An inverse semigroup is E-unitary if $e \leq a$ (where e is an idempotent) implies $a^2 = a$.

The class of E-unitary inverse semigroups are important as many inverse semigroups are E-unitary.

Example 1.3. Groups are E-unitary inverse semigroups.

Example 1.4. Let $B = \mathbb{N}^0 \times \mathbb{N}^0$ ($\mathbb{N}^0$ is the set of non negative integers) and for $(m,n), (p,q) \in B$,

$(m,n)(p,q) = (m - n + t, q - p + t)$, where $t = \max\ (n,p)$.

Then B is a semigroup and is known as the bicyclic semigroup. It can be shown that B is an inverse semigroup and the set of its idempotents $E(B) = \{(m,m) : m \in \mathbb{N}^0\}$.

Let $(m,n) \in B$ and let $(r,r) \in E(B)$. Now suppose $(r,r) \leq (m,n)$. Then

$(m,n)(r,r) = (m - n + t, r - r + t) = (m - n + t, t)$,

where $t = \max\ (n,r)$. Since $(m,n)(r,r)$ is an idempotent, it must be equal to (u,u) for some $u \in \mathbb{N}^0$. This implies that $m - n + t = u$ and $t = u$, so $m - n + t = t$. Hence $m = n$ and therefore $(m,n) \in E(B)$. So B is E-unitary.

Definition 1.5. Let S be a semigroup and $E(S)$ be its sets of idempotents. Then S is an ω-semigroup if and only if there exists a one-to-one mapping θ of $E(S)$ onto $\mathbb{N}^0$ such that for any elements $e, f \in E(S)$, $e\theta \leq f\theta$ if and only if $f \leq e$. Thus, if S is an ω-semigroup, then we write

$E(S) = \{e_m : m \in \mathbb{N}^0\}$ where $e_m \leq e_n \Leftrightarrow m \geq n$ or $E(S) = \{e_i : i = 0,1,2,\ldots\}$ such that $e_0 > e_1 > e_2 \ldots \ldots$

Definition 1.6. Given a map $\theta: S \to P$ we define

$$Ker\ \theta = \{(x,y) \in S \times S\ :\ x\theta = y\theta\}$$

and call this the kernel of the map.

Definition 1.7. Let S be an inverse semigroup with semilattice of idempotents $E(S)$. Then for all $a, b \in S$, $a\ \sigma\ b \Leftrightarrow ae = be$ for some $e \in E(S)$.

Definition 1.8. Let S be a semigroup and let $a, b \in S$. We define the following relations on S

$a\,\mathcal{L}\,b \Leftrightarrow S^1a = S^1b$, $a\,\mathcal{R}\,b \Leftrightarrow aS^1 = bS^1$, $a\,\mathcal{J}\,b \Leftrightarrow S^1aS^1 = S^1bS^1$,

$a\,\mathcal{H}\,b \Leftrightarrow a\,\mathcal{L}\,b$ and $a\,\mathcal{R}\,b$ i.e $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, $a\,\mathcal{D}\,b \Leftrightarrow (\exists\ c \in S)$ such that $a\,\mathcal{L}\,c$ and $c\,\mathcal{R}\,b$.

The relations $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ are called Green's relations.

Remark 1.9. i) It is more or less easy to see that all Green's relations are equivalences.

ii) In any commutative semigroup, $\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J} = G \times G$.

Since for any elements $a \in G$ we have $G^1a = G$ and $aG^1 = G$.

iii) $\mathcal{D}$ is defined such that it is the smallest equivalence containing $\mathcal{L}$ and $\mathcal{R}$.

iv) If we regard relations as subsets of $S\times S$ we have the inclusions $\mathcal{H} \subseteq \mathcal{L}, \mathcal{R} \subseteq \mathcal{D} \subseteq \mathcal{J}$.

The last inclusion follows from iii) and the fact that $\mathcal{L}, \mathcal{R} \subseteq \mathcal{J}$.

A proof of the above statements can be found in [1].

Definition 1.10. A (left, right) proper ideal I of a semigroup S is an (left, respectively, right) ideal such that $I \neq S$. That is, such that $I \subseteq S$ and $I \neq S$. A semigroup S is called right simple if it contains no proper right ideals, dually a semigroup S is called left simple if contains no proper left ideals, and a semigroup S is called simple if it has no two-sided ideals.

It is easy to see that a semigroup S is right (left) simple if and only if $\mathcal{R} = S\times S$ $(\mathcal{L} = S\times S)$, and simple if and only if $\mathcal{J} = S\times S$. A semigroup is called bisimple if $\mathcal{D} = S\times S$.

Since in a group G we have that $\mathcal{L} = \mathcal{R} = G\times G$, we conclude that groups are left and right simple. Thus G is simple.

The following example shows that the bicyclic semigroup is simple as well as bisimple.

Example 1.11. Let B be a bicyclic semigroup. Let $I \subseteq B$ be an ideal, and $(m,n) \epsilon I$. Then we have $(0,n) = (0,m)(m,n) \epsilon I$. Hence $(0,0) = (0,n)(n,0) \epsilon I$.

Take any arbitrary element $(a,b) \epsilon B$. Then$(a,b) = (a,b)(0,0) \epsilon I$, thus, $B \subseteq I$. Therefore $B = I$, so B is simple. Infact more is true: let $(m,n),(k,l) \epsilon B$. Then

$$(m,n)\ \mathcal{R}\ (m,l)\ \mathcal{L}\ (k,l),$$

So that $(m,n)\ \mathcal{D}\ (k,l)$. Hence B is bisimple.

2. Bisimple inverse ω-semigroups

Let M be a monoid with identity e and $\theta : M \to M$ be a morphism. Let θ^0 be the identity map on M and let$S$consist of set $S = \mathbb{N}^0\times M\times\mathbb{N}^0$ (where $\mathbb{N}^0$ denote the set of non-negative integers) with multiplication defined by the rule

$$(m,x,n)(p,y,q) = (m-n+t, x\theta^{t-n}y\theta^{t-p}, q-p+t),$$

where $t = \max(n,p)$, for $(m,x,n),(p,y,q) \epsilon S$.

Under this operation,$S = \mathbb{N}^0\times M\times\mathbb{N}^0$ is a semigroup and it is the one we refer to as the Bruck-Reilly extensions of the monoid M. This semigroup is usually denoted by $BR(M,\theta)$.

The following useful results are proved in [1].

Lemma 2.1. Let $S = BR(M,\theta)$ be the Bruck-Reilly extension of a monoid M. Suppose that (m,x,n) and (p,y,q) are elements of S. Then

i) S is a simple semigroup with identity $(0,e,0)$.

ii) $(m,x,n)\ \mathcal{D}(p,y,q)$ if and only if $x\ \mathcal{D}(M)y$.

iii) The element (m,x,n) is an idempotent in S if and only if $m = n$ and $x^2 = x \epsilon M$.

iv) S is inverse if and only if M is inverse.

v) $(m,x,n) \geq (p,y,q)$ if and only if $m+t=p,\ n+t=q$ for some $t \in \mathbb{N}^0$ and for some $e \in E(M)$.

If we consider the special case of the Bruck-Reilly extension where M is a group (with identity e). By (ii) and (iv), $BR(M,\theta)$ then becomes a bisimple inverse semigroup with identity $(0,e,0)$ and θ an endomorphism of M. From (v), we know that

$$(0,e,0) > (1,e,1) > (2,e,2) > \cdots$$

Since a group morphism maps the identity element to the identity element.

Hence $BR(M,\theta)$ is a bisimple inverse ω-semigroup. The converse of this theorem also holds.

Theorem 2.2 (structure theorem). Let M be a group and let θ be an endomorphism of M. Let $S = BR(M,\theta)$ be the Bruck-Reilly extension of M determined by θ. Then S is a bisimple inverse ω-semigroup. Conversely, every bisimple inverse ω-semigroup is isomorphic to one of this type.

In the next Proposition, we now establish a connection between E-unitariness of M and $BR(M,\theta)$.

Proposition 2.3. Let M be an inverse monoid and let θ be an endomorphism into the group of units of M. Then $S = BR(M,\theta)$ is E-unitary if and only if M is E-unitary and $\sigma = \ker\theta$.

Proof. Let $BR(M,\theta)$ be E-unitary. From Lemma 2.1(iii), we know that the idempotents of S are of the form (m,e,m), where $e \in E(M)$. Let $e, ae \in E(M)$. Then $(0,e,0), (0,ae,0) \in E(S)$. But we have that $(0,ae,0) = (0,a,0)(0,e,0) \leq (0,a,0)$ and so $(0,a,0) \in E(S)$ by assumption. This implies in particular that $a \in E(M)$. Hence M is E-unitary.

Let $x, y \in$ M such that $x\,\sigma\,y$. Then $xe = yf$ for $e, f \in E(M)$. Since idempotents are mapped to idempotents and since the only idempotent in the group of units is the identity element, we get

$$x\theta = x\theta e\theta = (xe)\theta$$
$$= (yf)\theta = y\theta f\theta = y\theta.$$

Hence $\sigma \subseteq \ker\theta$. The reverse inclusion and the converse of the proof is clear.

As an application of Proposition 2.3, we can now characterize E-unitary bisimple inverse ω-semigroups with the help of Theorem 2.2.

Theorem 2.4. A bisimple inverse ω-semigroup is E-unitary if and only if θ is one -to-one.

Proof. It is clear from Proposition 2.3 since every group is E-unitary with the σ-relation being the equality relation.

3. Simple inverse ω-semigroups

In this section, we obtain a result analogous to Theorem 2.4. But first we have the following useful definitions.

Definition 3.1. An element $a \in S$ is called central if $ax = xa$ for all $a \in S$.

Definition 3.2. We call a semigroup a Clifford semigroup if it is regular and its idempotents are central. Obviously, a Clifford semigroup is inverse, since in particular its idempotents commute. Its structural characterization is given below.

3.3 The Structural Characterization [1]. Let $E(S)$ be a semilattice and let $\{G_e : e \in E(S)\}$ be a family of disjoin groups indexed by the elements of $E(S)$. We denote the identity element of G_e by 1_e. For each pair $e, f \in E(S)$ such that $e \geq f$ let $\varphi_{e,f} : G_e \to G_f$ be a group morphism such that the following conditions hold:

i) $\varphi_{e,e}$ is the identity morphism on G_e

ii) if $e \geq f \geq g$ then $\varphi_{f,g}\varphi_{e,f} = \varphi_{e,g}$

We endow the set $\bigcup_{e \in E(S)} G_e$ with a product defined by

$x \circ y = (x\varphi_{e,ef})(y\varphi_{f,ef})(x \in G_e,\ y \in G_f)$.

It is shown in [1] that $(\bigcup_{e \in E(S)} G_e, \circ)$ is a Clifford semigroup. Infact this semigroup is called a strong semilattice of groups and it is denoted by $S(E; G_e; \varphi_{e,f})$.

We know from Theorem 2.2 that the Bruck-Reilly extension $BR(M, \theta)$ of a monoid M is a bisimple inverse ω-semigroup. To find a structure theorem for simple inverse ω-semigroups, we examine a particular type of Clifford semigroups of the Bruck-Reilly extension.

We now introduce this construction.

3.4 Construction. Let $Y = \{0,1, \dots, d-1\}$ be a chain with the reversed usual order. To simplify the notation we shall use the convention to denote by $\leq$ the usual order of the natural numbers, whereas by $\wedge$ we refer to the order of the chain, for example $4 \leq 5$ but $4 \wedge 5 = 5$. For every $i \in Y$ let G_i denote a group such that all the groups G_i are disjoint. Put $M \coloneqq \bigcup_{i=0,1,\dots,d-1} G_i$. For every $0 \leq i \leq d-2$ choose and fix a morphism $\alpha_{i,j}: G_i \to G_{i+1}$. Moreover, we define for every $0 \leq i < j \leq d-1$
a new morphism $\alpha_{i,j} : G_i \to G_j$ by the rule

$\alpha_{i,j} = \gamma_{j-1} \circ \gamma_{j-2} \circ \dots \circ \gamma_i$.

Putting the identity of G_i as $\alpha_{i,i}$ we have

$$\alpha_{j,k} \circ \alpha_{i,j} = \alpha_{i,k} (i \leq j \leq k).$$

From 3.3, we know that the strong semilattice of groups $(M, \circ)$ is a Clifford semigroup. We can say that the semilattice is a chain isomorphic to Y. The idempotents of M are the identity elements of the groups G_i denoted by $e_0, e_1, \dots, e_{d-1}$. Recall that identity elements are mapped to identity elements by group morphisms and notice that e_0 is the identity element of the monoid M :

$$\forall\, i\, \forall\, x \in G_i :\ e_0 x = \alpha_{0,0\wedge i}(e_0)\alpha_{i,0\wedge i}(x) = \alpha_{0,i}(e_0)\alpha_{i,i}(x)$$
$$= e_i x = x\ .$$

A similar argument shows that $xe_0 = x$ for all $x \epsilon M$. Furthermore, a straightforward calculation yields $e_0 > e_1 > \cdots > e_{d-1}$.

We shall refer to M as a finite chain of groups of length d .

Let M be a finite chain of groups of length d . Notice that the group of units of M is G_0 because a product in which an element $x \epsilon G_i$ is involved does necessarily lie in G_j for some $j \geq i$. Now let $S = BR(M, \theta)$, where θ is a morphism from M to G_0 . By Lemma 2.1 (i), S is a simple inverse semigroup since M is inverse. Also by Lemma 2.1 (ii) the $\mathcal{D}$-classes of S are the subsets $\mathbb{N}^0 \times G_i \times \mathbb{N}^0 (i = 0,1, \dots, d-1)$.

Lemma 3.5. S is an ω-semigroup.

Proof. Let$(m, e_i, m), (n, e_j, n)$ be two idempotents. We assume without loss of generality that $m \geq n$ and distinguish between two cases :

Case i. For $m = n$ we have

$$(m, e_i, m) \leq (m, e_j, m) \Leftrightarrow (m, e_i, m)(m, e_j, m) = (m, e_i, m).$$

Having in mind that $(m, e_i, m)(m, e_j, m) = (m, e_i e_j, m)$, this is the case if and only if $e_i \leq e_j$ in M, i.e. if and only if $i \wedge j = i$.

Case ii. For $m < n$ we have $\theta^{m-n}(e_j) = e_0$, the identity of M. Hence

$$(m, e_i, m)(n, e_j, n) = (m, e_i \theta^{m-n}(e_j), m) = (m, e_i, m)$$

and so $(m, e_i, m) < (n, e_j, n)$ regardless of the values of i and j. In effect, the idempotents of S form a chain

$$\begin{aligned}
&(0, e_0, 0) > (0, e_1, 0) > \cdots > (0, e_{d-1}, 0) > \\
&(1, e_0, 1) > (1, e_1, 1) > \cdots > (1, e_{d-1}, 1) > \\
&. \\
&. \\
&. \\
&(d-1, e_0, d-1) > (d-1, e_1, d-1) > \cdots > (d-1, e_{d-1}, d-1).
\end{aligned}$$

Thus $S = BR(M, \theta)$ is a simple inverseω-semigroup. The converse of this also holds.

Theorem 3.6 [1]. Let M be a finite chain of groups of length d (≥ 1). If θ is a morphism from M into the group of units of M, then the Bruck-Reilly extension $BR(M, \theta)$of M determined by θ is a simple inverse ω-semigroup with d $\mathcal{D}$-classes. Conversely, every simple inverse ω-semigroup is isomorphic to one of this type.

Our next task is to characterize the E-unitary simple inverseω-semigroups. From Theorem 3.6, we know that within M the multiplication is defined via morphism $\gamma_i : G_i \to G_{i+1} (i = 0, \dots, d-2)$. From Proposition 2.3, we can say exactly when $BR(M, \theta)$ is E-unitary, namely when M is E-unitary and $\sigma_M =$

Ker θ. In order to obtain a more elegant criterion we formulate a Lemma that enables us to know when a Clifford semigroup is E-unitary and examine the σ-relation on the finite chain of groups.

Lemma 3.7 [5]. Let $S = S(E, G_e, \varphi_{e,f})$ be a Clifford semigroup. Then S is E-unitary if and only if the connecting morphisms $\varphi_{e,f}$ are one-one.

Lemma 3.8. Let $S = S(E, G_e, \varphi_{e,f})$ be a Clifford semigroup. Then $a\ \sigma\ b$ if and only if there exists $l \epsilon E$: $a\varphi_{e,l} = b\varphi_{f,l}\big(a \epsilon G_e,\ b \epsilon G_f\big)$.

Proof. The proof is clear.

Theorem 3.9. With the notation used in 3.4, a simple inverse ω-semigroup $BR(M, \theta)$ is E-unitary if and only if γ_i is one-to-one for all $i \epsilon \{0, \dots, d-2\}$and $a\theta = b\theta$ if and only if $a\alpha_{j,k} = b\big(a \epsilon G_j$, $b \epsilon G_k,\ \ j \leq k\big)$.

Proof. We know from Proposition 2.3 that $S = BR(M, \theta)$ is E-unitary if and only if M is E-unitary and $\sigma_M =$ Ker θ. From Lemma 3.7 and Lemma 3.8 it follows that this is the case exactly when all connecting morphisms are one-to-one and $a\theta = b\theta$ if and only if there exists $l \geq j, k : a\alpha_{j,l} = b\alpha_{k,l}\big(a \epsilon G_j$, $b \epsilon G_k\big)$. But $S = BR(M, \theta)$ is not just any Clifford semigroup. It is a finite chain of groups. The rest of the Proof is clear.

Conflict of Interests

The authors declare that there is no conflict of interests.

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