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# L-SEQUENCES OF SATURATED NUMERICAL SEMIGROUPS WITH MULTIPLICITY $\leq 7$

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**Abstract:** In this paper, we will investigate Lipman sequences (L-sequences) of saturated numerical semigroups with multiplicity $\leq 7$ and conductor $C$. Also, we will give some results about Frobenius number, determine number and genus in these Lipman sequences.

**Keywords:** saturated numerical semigroups; Lipman sequences; genus.

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## 1. Introduction

A numerical semigroup is a subset $S$ of $\mathbb{N}$ ( Here $\mathbb{N}$ denotes the set of nonnegative integers) if $x+y \in S$, for all $x, y \in S$, $0 \in S$ and $S$ has finite complement in $\mathbb{N}$. If $S$ is a numerical semigroup, then the greatest integer that does not belong to $S$ is called the Frobenius number of $S$, denoted by $F(S)$. If $S =< s_1, s_2 >$, then $F(S) = s_1.s_2 - s_1 - s_2$ ( see, for instance [1], [6]). If $S$ is a numerical semigroup then $C$ is conductor of $S$

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if $C = F(S)+1$. Also, $n(S) = Card\ \{0,1,2,...,F(S)\} \cap S$ is called the number determine of $S$.

Given a numerical semigroup $S = < v_1, v_2, ..., v_r >$, we have

$$S = < v_1, v_2, ..., v_r > = \left\{ \sum_{i=1}^{r} k_i v_i : k_i \in \mathbb{N} \right\} .$$

In this case, $r$ and $\min\{x \in S : x > 0\}$ is called embedding dimension and multiplicity of $S$, denoted by $e(S)$ and $m(S)$, respectively. In general, $e(S) \leq m(S)$. If $e(S) = m(S)$ then $S$ is called maximal embedding dimension (see, [6] ).

If $S$ is a numerical semigroup such that $S = < v_1, v_2, ..., v_n >$, then we write that

$$S = < v_1, v_2, ..., v_n > = \{s_0 = 0, s_1, s_2, ..., s_{n-1}, s_n = F(S)+1, \rightarrow ...\} ,$$

where $s_i < s_{i+1}$, $n = n(S)$ and the arrow means that every integer greater than $F(S)+1$ belongs to $S$ for $i = 1, 2, ..., n = n(S)$. If $u \in \mathbb{N} \backslash S$ then $u$ is called gap of $S$ and we denote the set of gaps of $S$, by $H(S)$, i.e, $H(S) = \mathbb{N} \backslash S$. The cardinality of the set $H(S)$ is called the genus of $S$,

denoted by $G(S)$. It is known that $G(S) = F(S)+1-n(S)$ (see, for detail [1], [2], [6] ).

A numerical semigroup $S$ is called Arf if $a+b-c \in S$, for all $a, b, c \in S$ such that $a \geq b \geq c$. It is well known that any Arf numerical semigroup is maximal embedding dimension, but its inverse is not true. A numerical semigroup $S$ is called saturated if $s + n_1 s_1 + n_2 s_2 + ... + n_t s_t \in S$, where $s, s_i \in S$ and $n_i \in \mathbb{Z}$ such that $n_1 s_1 + n_2 s_2 + ... + n_t s_t \geq 0$ and $s_i \leq s$ for $i = 1, 2, ..., t$. A saturated numerical semigroup is Arf, but an Arf numerical semigroup need not be saturated (see, for instance [2], [3], [4], [5], [6] ).

Let $S$ be a numerical semigroup with the maximal ideal $T = S \backslash \{0\}$. For each $k \geq 1$, we define $B(S)=T-T=\{x\in\mathbb{N}: x+T\subseteq T\}$ and $kT-kT=\{a\in\mathbb{N}: a+kT\subseteq kT\}$. We note that $B(S)$ and $kT-kT$ are numerical semigroups. In this case, $L(S)=\bigcup_{k\geq 1}(kT-kT)$ is a numerical semigroup containing $S$. Evidently, $B(S)\subseteq L(S)$ and $S$ is maximal embedding dimension if and only if $B(S)=L(S)$. If $S$ is a numerical semigroup then we have the following chains,

$$B_0(S)=S\subseteq B_1(S)=B(S)\subseteq B_2(S)=B(B_1(S))\subseteq\cdots\subseteq B_{r+1}(S)=B(B_r(S))\subseteq\cdots$$

and

$$L_0=S\subseteq L_1=L(S)\subseteq L_2=L(L_1(S))\subseteq\cdots\subseteq L_k=L(L_{k-1}(S))\subseteq\cdots.$$

The sequence $L_i(S)$ is called the Lipman sequence of semigroups of $S$. If there exists $\lambda$ such that $L_\lambda(S)=\mathbb{N}$ then $\lambda$ is called as Lipman index of $S$. If $S$ is an Arf numerical semigroup then $B_i(S)=L_i(S)$ for each $i\geq 0$ (see, for detail [2], [11] ).

In this paper , we find Lipman sequences of saturated numerical semigroups with multiplicity $\leq 7$ and conductor $C=mk$, for $k\geq 1$, $k\in\mathbb{Z}$ and $m=m(S)=2,3,4,5,6,7$. Also, we write formulas about Fobenius number, determine number and genus in these Lipman sequences and we obtain some results for these numerical semigroups.

## 2. Main results

**Proposition 2.1.** ([11]) *Let* $S=<a_1,a_2,...,a_p>$ *be a numerical numerical semigroup and* $F(S)$ *be its Frobenius number. Then we have*

***1.*** $F(B_1(S))=F(S)-a_1$,

***2.*** $L(S)=<a_1,a_2-a_1,...,a_p-a_1>$,

***3.*** *If* $S$ *is symmetric, then* $B_1(S) = < a_1, a_2, ..., a_p, F(S) >$,

***4.*** $S$ *is maximal embedding dimension if and only if* $B_1(S) = L_1(S)$.

**Theorem 2.2.** *For* $k \geq 1$, $k \in \mathbb{Z}$ *and* $i = 0, 1, 2, ...,.$, *we have the following statement:*

***1.*** *The Lipman semigroups sequence of* $S_k = < 2, 2k+1 >$ *saturated numerical semigroup*

*is* $L_i(S_k) = < 2, 2k - 2i + 1 >$.

***2.*** *The Lipman semigroups sequence of* $S_k = < 3, 3k+1, 3k+2 >$ *saturated numerical semigroup is* $L_i(S_k) = < 3, 3k - 3i + 1, 3k - 3i + 2 >$.

***3.*** *The Lipman semigroups sequence of* $S_k = < 4, 4k+1, 4k+2, 4k+3 >$ *saturated numerical semigroup is*

$$L_i(S_k) = < 4, 4k - 4i + 1, 4k - 4i + 2, 4k - 4i + 3 >.$$

***4.*** *The Lipman semigroups sequence of* $S_k = < 5, 5k+1, 5k+2, 5k+3, 5k+4 >$ *saturated numerical semigroup is*

$$L_i(S_k) = < 5, 5k - 5i + 1, 5k - 5i + 2, 5k - 5i + 3, 5k - 5i + 4 >.$$

***5.*** *The Lipman semigroups sequence of* $S_k = < 6, 6k+1, 6k+2, 6k+3, 6k+4, 6k+5 >$ *saturated numerical semigroup is*

$$L_i(S_k) = < 6, 6k - 6i + 1, 6k - 6i + 2, 6k - 6i + 3, 6k - 6i + 4, 6k - 6i + 5 >.$$

***6.*** *The Lipman semigroups sequence of*

$$S_k = < 7, 7k+1, 7k+2, 7k+3, 7k+4, 7k+5, 7k+6 >$$

*saturated numerical semigroup is*

$$L_i(S_k) = < 7, 7k - 7i + 1, 7k - 7i + 2, 7k - 7i + 3, 7k - 7i + 4, 7k - 7i + 5, 7k - 7i + 6 >.$$

**Proof.** For $k \geq 1$, $k \in \mathbb{Z}$ and $i = 0$, it is clear that

**1.** Let $S_k = < 2, 2k+1 >$ be saturated numerical semigroup. So, we proof by induction on $i$:

If $i=1$ then we have $L_i(S_k)=L(S)=L(<2,2k+1>)=<2,2k-1>$. We assume that this expression is true for $i=r$, i.e. $L_r(S_k)=<2,2k-2r+1>$. Now, let be $i=r+1$. Then we have

$$L_{r+1}(S_k)=L(L_r(S_k))=L(<2,2k-2r+1>)=<2,2k-2r-1>=<2,2k-2(r+1)+1>.$$

Thus, the proof is completed.

**2.** Let $S_k=<3,3k+1,3k+2>$ be saturated numerical semigroup, then we proof by induction on $i$:

If $i=1$ then we have

$$L_i(S_k)=L(S)=L(<3,3k+1,3k+2>)=<3,3k-2,3k-1>.$$

We assume that this expression is true for

$$i=r \text{ , so, } L_r(S_k)=<3,3k-3r+1,3k-3r+2>.$$

If $i=r+1$ then we have

$$\begin{aligned}L_{r+1}(S_k)&=L(L_r(S_k))=L(<3,3k-3r+1,3k-3r+2>)\\&=<3,3k-3r-2,3k-3r-1>\\&=<3,3k-3(r+1)+1,3k-3(r+1)+2>.\end{aligned}$$

Thus, the proof is completed. So, we can proof same way expression of (3), (4), (5) and (6).

**Corollary 2.3** *If we take $i=1$ in Theorem 2.2, then we obtain following Lipman semigroups of $S_k$ saturated numerical semigroups which given by Theorem 2.2, respectively:*

1. $L(S_k)=<2,2k-1>$,
2. $L(S_k)=<3,3k-2,3k-1>$
3. $L(S_k)=<4,4k-3,4k-2,4k-1>$
4. $L(S_k)=<5,5k-4,5k-3,5k-2,5k-1>$
5. $L(S_k)=<6,6k-5,6k-4,6k-3,6k-2,6k-1>$

**6.** $L(S_k) = < 7, 7k-6, 7k-5, 7k-4, 7k-3, 7k-2, 7k-1 >$.

**Corollary 2.4** *The Arf index of each saturated numerical semigroup which given by Theorem 2.2 is* $\lambda = k$.

**Theorem 2.5** *If* $S_k = < m, mk+1, mk+2, ..., mk+m-1 >$ *is saturated numerical semigroups, then* $F(L(S_k)) = mk-(m+1)$, *where* $k \geq 1$, $k \in \mathbb{Z}$ *and* $m(S) = m = 2,3,4,5,6,7$.

**Proof.**

If $m = 2$ then $S_k = < 2, 2k+1 >$ and $L(S_k) = < 2, 2k-1 >$. Thus, we have

$$F(L(S_k)) = 2.(2k-1) - 2 - 2k + 1 = 2k-3.$$

If $m = 3$ then $S_k = < 3, 3k+1, 3k+2 >$ and $L(S) = < 3, 3k-2, 3k-1 >$. So, we obtain $F(L(S_k) = F(S_k) - 3 = 3k-1-3 = 3k-4$ from Proposition 2.1. and Theorem 3 in [9] .

If $m = 4$ then $S_k = < 4, 4k+1, 4k+2, 4k+3 >$ and

$$L(S) = < 4, 4k-3, 4k-2, 4k-1 >.$$

So, we obtain $F(L(S_k) = F(S_k) - 4 = 4k-1-4 = 4k-5$ from Proposition 2.1 and [3].

If $m = 5$ then $S_k = < 5, 5k+1, 5k+2, 5k+3, 5k+4 >$ and

$$L(S) = < 5, 5k-4, 5k-3, 5k-2, 5k-1 >.$$

So, we obtain

$$F(L(S_k) = F(S_k) - 5 = 5k-1-5 = 5k-6$$

from Proposition 2.1 and Theorem 5 in [9].

If $m = 6$ then $S_k = < 6, 6k+1, 6k+2, 6k+3, 6k+4, 6k+5 >$ and

$$L(S) = < 6, 6k-5, 6k-4, 6k-3, 6k-2, 6k-1 >.$$

So, we obtain $F(L(S_k) = F(S_k) - 6 = 6k-1-6 = 6k-7$ from Proposition 2.1 and [12].

If $m = 7$ then $S_k = < 7, 7k+1, 7k+2, 7k+3, 7k+4, 7k+5, 7k+6 >$ and

$$L(S) = <7, 7k-6, 7k-5, 7k-4, 7k-3, 7k-2, 7k-1> \; .$$

So, we obtain $F(L(S_k) = F(S_k) - 7 = 7k - 1 - 7 = 7k - 8$ from Proposition 2.1 and [10].

**Theorem 2.6** *If* $S_k = <m, mk+1, mk+2, ..., mk+m-1>$ *is saturated numerical semigroups, then we have*

**a.** $n(L(S_k)) = k - 1$

**b.** $G(L(S_k)) = (m-1)(k-1)$

*where* $k \geq 1,\ k \in \mathbb{Z}$ *and* $m(S) = m = 2, 3, 4, 5, 6, 7$.

**Proof.**

**a.** We write $n(L(S_k)) = n(S_k) - 1$ and $n(S_k) = k$ for $k \geq 1,\ k \in \mathbb{Z}$. Thus we have

$$n(L(S_k)) = n(S_k) - 1 = k - 1$$

**b.** We know $G(L(S_k)) = G(S_k) - (m-1)$ and $G(S_k) = (m-1)k$ for $k \geq 1,\ k \in \mathbb{Z}$.

Thus we have $G(L(S_k)) = G(S_k) - (m-1) = (m-1)k - (m-1) = (m-1)(k-1)$ .

**Corollary 2.7** *If* $S_k = <m, mk+1, mk+2, ..., mk+m-1>$ *is saturated numerical semigroups, then we have* $G(L(S_k)) = (m-1)\, n(L(S_k))$ , *where* $k \geq 1$, $k \in \mathbb{Z}$ *and* $m(S) = m = 2, 3, 4, 5, 6, 7$.

**Example 2.8** We take $k = 4$ and $m = 3$. Then, we write saturated numerical semigroup $S = S_4 = <3,13,14> = \; 0, 3, 6, 9, 12, \rightarrow ... \;$, $F(S) = 11$ and $n(S) = 4$ . So, $H(S) = \; 1, 2, 4, 5, 7, 8, 10, 11$ and the genus of $S$ is $G(S) = \#(\, H(S)\,) = 8$ . The Lipman semigroup of $S$ is $L(S) = L(<3,13,14>) = <3,10,11> = \; 0, 3, 6, 9 \rightarrow ... \;$ . In this case, we obtain following Lipman semigroups chain of $S$ :

$$L_0(S) = S \;, L_1(S) = L(S) = <3,10,11>,\ L_2(S) = L(L_1(S)) = <3,7,8>,$$
$$L_3(S) = L(L_2(S)) = <3,4,5>, L_4(S) = L(L_3(S)) = <3,1,2> = <1> = \mathbb{N}.$$

Thus, the Arf index of $S$ is $\lambda = k = 4$ since $L_4(S) = \mathbb{N}$. On the other hand, we obtain

$$F(L(S)) = mk - (m+1) = 3.4 - (3+1) = 8,$$

$$n(L(S)) = k - 1 = 4 - 1 = 3,$$

$$G(L(S)) = (m-1)(k-1) = (3-1)(4-1) = 6,$$

so

$$G(L(S)) = (k-1)\, n(L(S)) = 2.3 = 6.$$

## Conflict of Interests

The authors declare that there is no conflict of interests.


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