Available online at http://scik.org
J. Semigroup Theory Appl. 2018, 2018:5
https://doi.org/10.28919/jsta/3562
ISSN: 2051-2937

ON SOME SEMIGROUPS GENERATED FROM CAYLEY FUNCTIONS

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Abstract. Any transformation on a set S is called a Cayley function on S if there exists a semigroup operation on S such that β is an inner-translation. In this paper we describe a method to generate a semigroup with k number of idempotents, study some properties of such semigroups like greens relations and bi-ordered sets.

Keywords: semigroups; idempotents; cayley functions; greens relation.

2010 AMS Subject Classification: 20M99.

1. Introduction

Let α be a transformation on a set S. Following [4] we say that α is a Cayley function on S if there is a semigroup with universe S such that α is an inner translation of the semigroup S. A section of group theory has developed historically through the characterisation of inner translations as regular permutations. The problem of characterising inner translations of semigroups was raised by Schein [7] and solved by Goralcik and Hedrlin [6].In 1972 Zupnik characterised

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Received November 14, 2017

all Cayley functions algebraically(in powers of β) [4]. In 2016, Araoujo et all characterised the Cayley functions using functional digraphs[5]. In this paper we use a Cayley permutation to generate a semigroup with n elements and $k \leq n$ idempotents and discuss some of its properties.

In the sequel β will denote a function mapping a non-empty set S onto itself. For any positive integer n, β^n denotes the n^{th} iterate of β. By β^0 we mean the identity function on S, so $\beta^0(x) = x$. Let S be a set, then $T(S)$ denotes the set of all transformations from S to S.

2. Preliminaries

A semigroup is a non empty set S along with a binary operation $*$ on S such that $(S,*)$ is associative. An idempotent element ε in S is an element such that $\varepsilon^2 = \varepsilon * \varepsilon = \varepsilon$. The set of all idempotents in S is denoted by $E(S)$

If a is an element of a semigroup S, the smallest left ideal containing a is $Sa \cup \{a\}$ or S^1a the principal left ideal generated by a. The equivalence relation $\mathscr{L}$ on S is defined on S by $a \mathscr{L} b$ if and only if $S^1a = S^1b$. Similarly we say that $a \mathscr{R} b$ if and only if $aS^1 = bS^1$. The following is due to J.A. Green.

Lemma 2.1. Let a, b be elements of a semigroup S. Then

- $a \mathscr{L} b$ if and only if $\exists\, x, y \in S^1$ such that $xa = b$ and $yb = a$
- $a \mathscr{R} b$ if and only if $\exists\, x, y \in S^1$ such that $ax = b$ and $by = a$

The following lemma is lemma 2.1 of [3]

Lemma 2.2. The relations $\mathscr{L}$ and $\mathscr{R}$ commute and so the relation $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$ is the smallest equivalence relation $\mathscr{L} \vee \mathscr{R}$ containing both $\mathscr{L}$ and $\mathscr{R}$. We define $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$

Let S be a semigroup. For a fixed $a \in S$, the mapping $\lambda_a : S \to S$ $[\rho_a : S \to S]$ defined by $\lambda_a(x) = ax$ $[\rho_a(x) = xa]$ is called a left [right] inner translation of S.

Definition 2.1. Let β be a transformation on a set S. We say that β is a Cayley function on S if there is a semigroup with universe S such that β is an inner translation of the semigroup S.

Note that β is a left inner translation of a semigroup $(S,*)$ if and only if β is a right inner translation of the semigroup $(S,.)$, where for all $a,b \in S$, $a*b = b.a$.

Definition 2.2. The stabilizer of $\beta \in T(S)$ is the smallest integer $s \geq 0$ such that $img(\beta^s) = img(\beta^{s+1})$. If such an s does not exist, we say that β has no stabilizer.

The following definition though originally by Zupnik was modified by Araujo in [5]

Definition 2.3. Suppose $\beta \in T(S)$ has the stabilizer s. If $s > 0$, we define the subset Ω_β of S by:

$$\Omega_\beta = \{a \in S : \beta^n(a) \in Ran(\beta^s) \text{ if and only if } n \geq s-1\}$$

If $s = 0$, we define Ω_β to be S.

Theorem 2.1. [4] Let $\beta \in T(S)$. Then β is a Cayley function if and only if exactly one of the following conditions holds:

- **a:** has no stabilizer and there exists $a \in S$ such that $\beta^n(a) \notin img(\beta^{n+1})$ for every $n \geq 0$;
- **b:** has the stabilizer s such that $\beta|img(\beta^s)$ is one-to-one and there exists $a \in \Omega_\beta$ such that $\beta^m(a) = \beta^n(a)$ implies $\beta^m = \beta^n$ for all $m,n \geq 0$; or
- **c:** has the stabilizer s such that $\beta|img(\beta^s)$ is not one-to-one and there exists $a \in \Omega_\beta$ such that:
  1. $\beta^m(a) = \beta^n(a)$ implies $m = n$ for all $m,n \geq 0$;and
  2. For every $n > s$, there are pairwise distinct elements $y_1, y_2, ...$of S such that $\beta(y_1) = \beta^n(a)$, $\beta(y_k) = y_{k-1}$ for every $k \geq 2$, and if $n > 0$ then $y_1 \neq \beta^{n-1}(a)$.

A Cayley function that is also a permutation is called a Cayley permutation. Similarly a Cayley function that also an idempotent is called a Cayley Idempotent.

3. Some Class of Semigroup from Cayley Functions

In this section we construct a semigroup S_β from a Cayley permutation β on a finite set and study some of its properties. Let S be a set with n elements, and $a \in S$ be a fixed element. For any a_1, in S we consider set $\{r : \beta^r(a) = a_1\}$ of all non-negative integers such that $\beta^r(a) = a_1$ in case the set is non empty we define $\delta_{a_i} = min\{r : \beta^r(a) = a_i\}$.

Theorem 3.1. Let S be a set with n elements for $0 < k \leq n$ then it is possible to construct a semigroup with k idempotents using a Cayley permutation.

Proof. Let $0 < k < n$, and a_1 be a fixed element of S. Let β be the permutation that such that $\beta = (a_1 a_2 a_3 ....a_{n-k} a_{n-k+1})$ an $n-k+1$ cycle in S_n that permutes $n-k+1$ terms and fixes the rest of the $k-1$ terms, it is a Cayley function by theorem 1 above. Now consider the binary operation on S given by

$$a_i * a_j = \begin{cases} \beta^{\delta_{a_i}+1}(a_j) & \text{if } a_i \text{ is not a fixed element of } \beta \\ a_i & \text{if } a_i \text{ is a fixed element of } \beta \end{cases}.$$

where $\delta_{a_i} = min\{r : \beta^r(a_1) = a_i\}$. For $k = n$ consider the identity permutation with the same construction. We can see that $*$ is well defined binary operation and that $*$ is associative. So $(S, *)$ is a semigroup. By the choice of the permutation β and the definition of $*$, we can see that a_{n-k+1} is an idempotent as

$$a_{n-k+1} * a_{n-k+1} = \beta^{n-k+1}(a_{n-k+1}) = a_{n-k+1}$$

and all the $k-1$ fixed elements of β are also idempotents. □

Example 1.1. Let $S = \{abcde\}$ and let $k = 3$ then we chose $\beta = \begin{pmatrix} a & b & c & d & e \\ b & c & a & d & e \end{pmatrix}$ now following the construction as in the above theorem we have the following Cayley table on S

*	a	b	c	d	e
a	b	c	a	d	e
b	c	a	b	d	e
c	a	b	c	d	e
d	d	d	d	d	d
e	e	e	e	e	e

For the rest of the paper we denote the semigroup generated in the above theorem as S_β.

Lemma1.1. Let $a, b \in S_\beta$ then

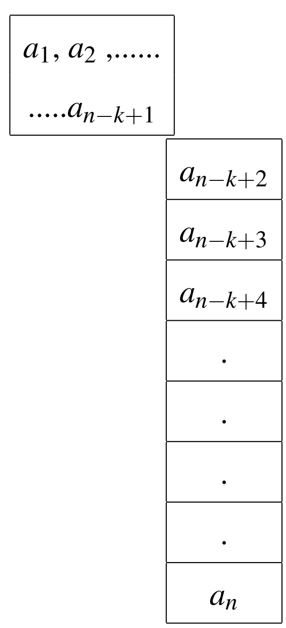
(1) If a, b are both non-fixed elements of β , then $a \mathscr{R} b$. If a is a fixed element of β, then a is $\mathscr{R}$ related only to itself.
(2) If a, b are both non-fixed elements of β, then $a \mathscr{L} b$. If a, b are both fixed elements of β then $a \mathscr{L} b$
(3) S_β contains two $\mathscr{D}$classes.
(4) S_β contains k number of $\mathscr{H}$classes.

Proof. (1) If a is not a fixed element of β then $aS_\beta = S_\beta$, if a is a fixed element of β then $aS_\beta = \{a\}$. Hence (1). In correspondence to lemma1 we have for non-fixed elements a_i , a_j of S_β $a_q * a_i = a_j$ where $q + i = j mod(n-k+1)$ for fixed elements $a_i * a_i = a_i$

(2) If a is not a fixed element of β then $S_\beta a = S_\beta$, if a is a fixed element of β then $S_\beta a = \{b : b$ is a fixed element of $\}$. And in correspondence to lemma1
(3) from lemma 2 $\mathscr{D} = \mathscr{L} \circ \mathscr{R} = \mathscr{R} \circ \mathscr{L}$. Hence from 1 and 2 we get two $\mathscr{D}$classes
(4) from lemma 2 $\mathscr{H} = \mathscr{L} \cap \mathscr{R}$ and hence k $\mathscr{H}$classes

□

Generally the egg box picture of the semigroup S_β is as follows.



Remark 3.1. Let β be an $n-k+1$ cycle (a Cayley Permutation) and S_β be the semigroup constructed as in theorem 3.1, then it is easy to observe the following properties.

- $\{a_1, a_2, ...a_{n-k+1}\}$ forms a subgroup of S_β.
- a_{n-k+1} is the identity element of S_β.
- k-1 idempotents act as left zeros (absorbing elements).
- Idempotents do not commute.
- S_β is a regular semigroup.
- In fact S_β is a completely regular semigroup
- S_β is a not an inverse semigroup.
- S_β is a union of a group and a band.
- $E(S_\beta)$ forms a sub-semigroup of S_β(i. e, $S_\beta$is an orthodox semigroup.)

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgement

The first authors was financially supported by KSCSTE, Thiruvananthapuram.

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