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CONFORMABLE FRACTIONAL HYPER GEOMETRIC EQUATION

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Abstract. In this paper we study the fractional power series solution around the regular singular point $x = 0$ of the conformable fractional hyper geometric differential equation. Then, we compare such solutions with that of the corresponding ordinary differential equation.

Keywords: conformable fractional equation; conformable fractional hyper geometric equation.

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1. Introduction

The subject of fractional derivative is as old as calculus. In 1659, L'Hospital asked if the expression $\frac{d^{0.5}}{dt^{0.5}}f$ has any meaning. Since then, many researchers have been trying to generalize the concept of the usual derivative to fractional derivatives. Various definitions of non-integer order integral or derivative was given by many mathematicians. Most of these definitions use an integral form. The most popular definitions are:

1. $D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(x)}{(t-x)^{(a-n+1)}}dx$
2. $D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(x)}{(t-x)^{(a-n+1)}}dx$

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Now, all these definitions are attempted to satisfy the usual properties of the standard derivative. The only property inherited by all definitions of fractional derivative is the linearity property. However, the following are the setbacks of one definition or another:

(1) The Riemann-Liouville derivative does not satisfy $D_a^{\alpha}(1) = 0$ ($D_a^{\alpha}(1) = 0$) for the Caputo derivative), if α is not a natural number.
(2) All fractional derivatives don't satisfy the known product rule: $D_a^{\alpha}(fg) = fD_a^{\alpha}(g) + gD_a^{\alpha}(f)$
(3) All fractional derivatives don't satisfy the known quotient rule: $D_a^{\alpha}(\frac{f}{g}) = \frac{gD_a^{\alpha}(f) - fD_a^{\alpha}(g)}{g^2}$
(4) All fractional derivatives don't satisfy the chain rule: $D_a^{\alpha}(f \circ g) = f^{\alpha}g(t)g^{\alpha}(t)$
(5) All fractional derivatives don't satisfy: $D^{\alpha}D^{\beta}f = D^{\alpha+\beta}f$ in general
(6) Caputo definition assumes that the function f is differentiable.

In [2], a new definition called conformable fractional derivative was introduced. Let $0 < \alpha \leq 1$. Then,

$D^{\alpha}(f)(t) = \lim\limits_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$

$D^{\alpha}(f)(t)$ is called conformable fractional derivative of f of order α. We shall write $f^{\alpha}(t)$ for $D^{\alpha}(f)(t)$.

The new definition satisfies:

(1) $D^{\alpha}(af + bg) = aD^{\alpha}(f) + bD^{\alpha}(g)$,for all $a, b \in R$.
(2) $D^{\alpha}(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
(3) $D\alpha(fg) = fD^{\alpha}(g) + gD^{\alpha}(f)$.

Further, for $\alpha \in (0, 1]$ and f, g be αdifferentiable at a point t , with $g(t) \neq 0$, then

4. $D^{\alpha}(\frac{f}{g}) = \frac{gD^{\alpha}(f) - fD^{\alpha}(g)}{g^2}$

We list here the fractional derivatives of certain functions, for the purpose of comparing the results of the new definition with the usual definition of the derivative:

(1) $D^{\alpha}(t^p) = pt^{p-1}$
(2) $D^{\alpha}(sin\frac{1}{\alpha}t^{\alpha}) = cos\frac{1}{\alpha}t^{\alpha}$
(3) $D^{\alpha}(cos\frac{1}{\alpha}t^{\alpha}) = -sin\frac{1}{\alpha}t^{\alpha}$
(4) $D^{\alpha}(e^{\frac{1}{\alpha}t^{\alpha}}) = e^{\frac{1}{\alpha}t^{\alpha}}$

On letting $\alpha = 1$ in these derivatives, we get the corresponding ordinary derivatives. Recently, [1], used the conformable definition of fractional derivative to introduce fractional Laplace transform, and fractional Taylor expansion. For more on fractional derivative and fractional differential We refer to [4],[5],[6],[7] and [8].

In this paper we solve the well known fractional hyper geometric equation using fractional power series solutions developed in [8]

Throughout this paper, we let $D^{\alpha}y$ denoted the conformable fractional derivative of y, where $\alpha \in (0,1]$. The second αderivative of y will be denoted by $D^{2\alpha}y$.

2. The main result

The classical hyper geometric equation is:

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$$

The point $x = 0$ is a regular singular point for the equation. The corresponding fractional hyper geometric equation is

$$x^{\alpha}(1-x^{\alpha})D^{2\alpha}y + \alpha[c-(a+b+1)x^{\alpha}]D^{\alpha}y - \alpha^2 aby = 0$$

We will solve this equation around $x = 0$ which is a regular singular point.

Definition 1. *A series is called a fractional Frobenius series if it can be written in the*

form $\sum_{n=0}^{\infty} a_n x^{(n+r)\alpha}$,for $\alpha \in (0,1]$.

Now let us start the procedure of solving

$$x^{\alpha}(1-x^{\alpha})D^{2\alpha}y + \alpha[c-(a+b+1)x^{\alpha}]D^{\alpha}y - \alpha^2 aby = 0 \qquad (*)$$

where $\alpha \in (0,1]$, a,b and c are constants. Clearly, if $\alpha = 1$, then equation $(*)$ is just the classical hyper geometric equation.

Now, $x = 0$ is a regular singular point for the equation. Using the fractional Frobenius series expansion, and $x > 0$, we let

$y=\sum_{n=0}^{\infty} a_n x^{(n+r)\alpha}, a_0 \neq 0$

Then

$D^{\alpha} y=\sum_{n=0}^{\infty} \alpha(n+r) a_n x^{(n+r-1)\alpha}$

$D^{2\alpha} y=\sum_{n=0}^{\infty} \alpha^2(n+r)(n+r-1) a_n x^{(n+r-2)\alpha}$

Substitute these in equation $(*)$ we get

$x^{\alpha} \sum_{n=0}^{\infty} \alpha^2(n+r)(n+r-1) a_n x^{(n+r-2)\alpha}$

$-x^{2\alpha} \sum_{n=0}^{\infty} \alpha^2(n+r)(n+r-1) a_n x^{(n+r-2)\alpha}+\alpha c \sum_{n=0}^{\infty} \alpha(n+r) a_n x^{(n+r-1)\alpha}$

$-\alpha(a+b+1) x^{\alpha} \sum_{n=0}^{\infty} \alpha(n+r) a_n x^{(n+r-1)\alpha}-\alpha^2 ab \sum_{n=0}^{\infty} a_n x^{(n+r)\alpha}=0$

This can be rewritten as:

$$\sum_{n=0}^{\infty} \alpha^2[(n+r)(n+r-1)+c(n+r)] a_n x^{(n+r-1)\alpha}$$
$$-\sum_{n=0}^{\infty} \alpha^2[(n+r)(n+r-1)+(a+b+1)(n+r)+ab] a_n x^{(n+r)\alpha}=0 \quad (1)$$

After simplification we get the following equation:

$$\sum_{n=0}^{\infty} \alpha^2(n+r)(n+r-1+c) a_n x^{(n+r-1)\alpha}-\sum_{n=0}^{\infty} \alpha^2(n+r+a)(n+r+b) a_n x^{(n+r)\alpha}=0 \quad (2)$$

In the second term in (2), replace n by $n-1$ the term becomes

$$-\sum_{n=1}^{\infty} \alpha^2(n+r+a-1)(n+r+b-1) a_{n-1} x^{(n+r-1)\alpha}$$

Now, unifying all summation to start from $n=1$ and put them in one summation to get:

$$\alpha^2 r(r-1+c) a_0 x^{(r-1)\alpha}$$
$$+\sum_{n=1}^{\infty}[\alpha^2(n+r)(n+r-1+c) a_n-\alpha^2(n+r+a-1)(n+r+b-1) a_{n-1}] x^{(n+r-1)\alpha}=0 \quad (3)$$

We now equate to zero the coefficient of the smallest power of x , namely, $(r-1)\alpha$, to get the indicial equation as

$r(r-1+c)=0,(a_0 \neq 0)$ which gives $r=0$ and $r=1-c$ (4)

We equate to zero the coefficient of $x^{(n+r-1)\alpha}$ (for recurrence relation) to get

$$\alpha^2(n+r)(n+r-1+c)a_n - \alpha^2(n+r+a-1)(n+r+b-1)a_{n-1} = 0$$

which gives

$$a_n = \frac{(n+r+a-1)(n+r+b-1)}{(n+r)(n+r-1+c)}a_{n-1} \tag{5}$$

When $r = 0$, substituting $n = 1,2,3,...$ successively in (5) , that is, in

$a_n = \frac{(n+a-1)(n+b-1)}{n(n-1+c)}a_{n-1}$ (6)

We obtain

$a_1 = \frac{ab}{1.c}a_0$
$a_2 = \frac{(a+1)(b+1)}{2.(c+1)}a_1 = \frac{a(a+1)b(b+1)}{1.2c(c+1)}a_0$
$a_3 = \frac{(a+2)(b+2)}{3.(c+2)}a_2 = \frac{a(a+1)(a+2)b(b+1)(b+2)}{1.2.3c(c+1)(c+2)}a_0$
$\vdots$

$$a_n = \frac{a(a+1)...(a+n-1)b(b+1)...(b+n-1)}{n!c(c+1)\ldots(c+n-1)}a_0 \tag{7}$$

Putting the a_i's in $y = \sum_{n=0}^{\infty} a_n x^{(n+r)\alpha}$ with $r = 0$, we get

$$y = a_0 + \sum_{n=1}^{\infty} \frac{a(a+1)...(a+n-1)b(b+1)...(b+n-1)}{n!c(c+1)\ldots(c+n-1)}a_0 x^{n\alpha} \tag{8}$$

$$y = a_0[1 + \sum_{n=1}^{\infty} \frac{a(a+1)...(a+n-1)b(b+1)...(b+n-1)}{n!c(c+1)\ldots(c+n-1)}x^{n\alpha}] = a_0 Y_1 \tag{9}$$

When $r = 1-c$, (5) reduces to to

$$b_n = \frac{(n-c+a)(n-c+b)}{(n-c+1)n}b_{n-1} \tag{10}$$

Substituting $n = 1,2,3,\ldots$successively in (9), we obtain

$b_1 = \frac{(1+a-c)(1+b-c)}{1.(2-c)}b_0$
$b_2 = \frac{(2+a-c)(2+b-c)}{2.(3-c)}b_1 = \frac{(1+a-c)(2+a-c)(1+b-c)(2+b-c)}{1.2.(2-c)(3-c)}b_0$
$b_3 = \frac{(3+a-c)(3+b-c)}{3.(4-c)}b_2 = \frac{(1+a-c)(2+a-c)(3+a-c)(1+b-c)(2+b-c)(3+b-c)}{1.2.3.(2-c)(3-c)(4-c)}b_0$

$\vdots$

$$b_n = \frac{(1+a-c)\dots(n+a-c)(1+b-c)\dots(n+b-c)}{n!(2-c)\dots(n-c)} b_0 \tag{11}$$

Putting the b_i's in $y = \sum_{n=0}^{\infty} b_n x^{(n+r)\alpha}$ with $r = 1-c$, we get

$$y = b_0 + \sum_{n=1}^{\infty} \frac{(1+a-c)\dots(n+a-c)(1+b-c)\dots(n+b-c)}{n!(2-c)\dots(n-c)} b_0 x^{(n+1-c)\alpha} \tag{12}$$

$$y = b_0[1 + \sum_{n=1}^{\infty} \frac{(1+a-c)\dots(n+a-c)(1+b-c)\dots(n+b-c)}{n!(2-c)\dots(n-c)} x^{(n+1-c)\alpha}] = b_0 Y_2 \tag{13}$$

Where Y_2 is another independent solution of $(*)$. Therefore, The general series solution of $(*)$ can be written as

$$y = a_0 Y_1 + b_0 Y_2 \tag{14}$$

Where a_0 and b_0 are constants.

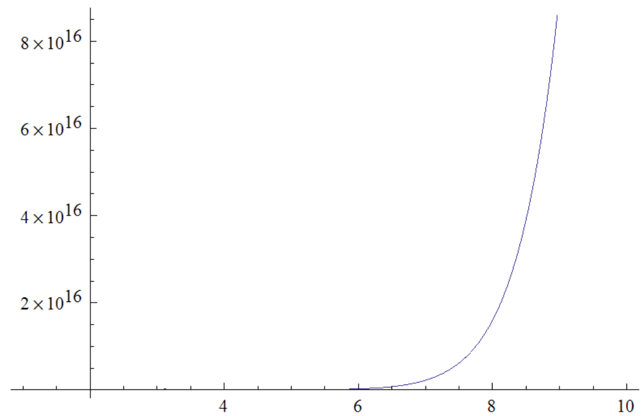
3. Some Applications

Example 1. *Let us consider the following example:*

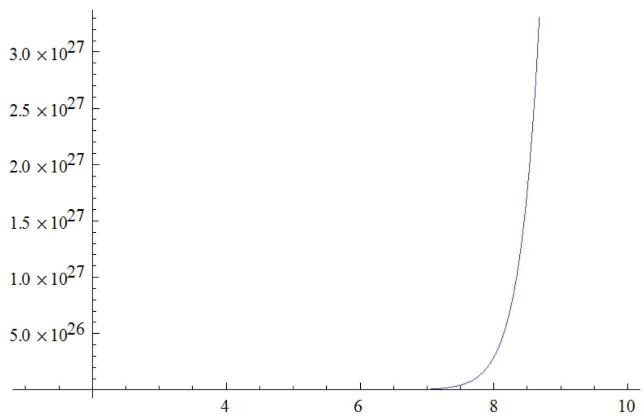
Let $r = 0$, $\alpha = 0.5, a = 1, b = 2, c = 0.5, n = 30$ and $a_0 = 1$ then figure (1) represents $y = a_0 Y_1, 0 \leq x \leq 5$

Example 2. *Let us consider the following example:*

Let $r = 0.5$, $\alpha = 0.5, a = 1, b = 2, c = 0.5, n = 30$ and $b_0 = 2$ then figure (2) represents $y = b_0 Y_2, 0 \leq x \leq 5$



$y = a_0 Y_1$



$y = b_0 Y_2$

Conflict of Interests

The authors declare that there is no conflict of interests.

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