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ON SOME SATURATED NUMERICAL SEMIGROUPS WITH MULTIPLICITY EIGHT

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Abstract: In this paper, we will investigate saturated numerical semigroups with multiplicity 8 and conductor C. Also, we will give formulas for Frobenius number, determiner number and genus of these semigroups.

Keywords: saturated numerical semigroups; Frobenius number; genus.

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1. Introduction

We consider that $\mathbb{N}=\{0,1,2,...,n,...\}$. Let $\mathbb{Z}$ be integer set. The subset $S\subseteq\mathbb{N}$ is a numerical semigroup if

i. $x+y\in S$, for $x,y\in S$

ii. $\gcd(S)=1$

iii. $0\in S$

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(Here, $\gcd(S)=$ greatest common divisor the elements of S).

A numerical semigroup S can be written that

$$S=<x_1,x_2,...,x_n>=\left\{\sum_{i=1}^{n}a_ix_i : a_i\in\mathbb{N}\right\}.$$

$T\subset\mathbb{N}$ is minimal system of generators of S if $<T>=S$ and there isn't any subset $M\subset T$ such that $<M>=S$. Also, $\mu(S)=\min\{x\in S: x>0\}$ is called as multiplicity of S (see [3]).

Let S be a numerical semigroup, then $F(S)=\max(\mathbb{Z}\backslash S)$ is called as Frobenius number of S. Also, C is conductor of S if $C=F(S)+1$, and $n(S)=Card\left(\{0,1,2,...,F(S)\}\cap S\right)$ is called as the determiner number of S.

If S is a numerical semigroup such that $S=<x_1,x_2,...,x_n>$, then we observe that

$$S=<x_1,x_2,...,x_n>=\{s_0=0,s_1,s_2,...,s_{n-1},s_n=F(S)+1,\rightarrow...\},$$

where $s_i<s_{i+1}$, $n=n(S)$ and the arrow means that every integer greater than $F(S)+1$ belongs to S for $i=1,2,...,n=n(S)$.

If $y\in\mathbb{N}$ and $y\notin S$, then y is called gap of S. We denote the set of gaps of S, by $H(S)$, i.e, $H(S)=\mathbb{N}\backslash S$. The $G(S)=\#(H(S))$ is called the genus of S. It known that $G(S)=F(S)+1-n(S)$ (see [3]).

A numerical semigroup S is Arf if $x_1+x_2-x_3\in S$, for all $x_1,x_2,x_3\in S$ such that $x_1\geq x_2\geq x_3$. Also, a numerical semigroup S is saturated if $s+d_1s_1+d_2s_2+...+d_ms_m\in S$, where $s,s_i\in S$ and $d_i\in\mathbb{Z}$ such that $d_1s_1+d_2s_2+...+d_ms_m\geq 0$ and $s_i\leq s$ for $i=1,2,...,m$. A saturated numerical is Arf, but an Arf numerical semigroup need not be saturated. For example, $S=\langle 8,13,17,18,19,20,22,23\rangle=\{0,8,13,16,\rightarrow...\}$ is Arf numerical semigroup but it is not saturated. Many researchs have studied on saturated numerical semigroups

(see, [2], [3], [9]). Especialy, saturated numerical semigroups with multiplicity 3, 4, 5, 6 and 7 have studied by Ilhan et al. (for details, see [1], [4], [5], [6], [7], [8]).
In this paper, we will give some saturated numerical semigroups multiplicity 8 and conductor C. Also, we will obtain formulas for Frobenius number, determiner number and genus of these saturated numerical semigroups.

2. Main results

Proposition 2.1. *([3]) Let S be a numerical semigroup. Then following conditions are equivalent:*

1) *S is a saturated numerical semigroup.*
2) *$y + d_S(y) \in S$ for all $y \in S$, $y > 0$ where $d_S(y) = \gcd\{ x \in S : x \leq y \}$.*
3) *$y + md_S(y) \in S$ for all $y \in S$, $y > 0$ and $m \in \mathbb{N}$.*

Now, we give our first result in the following theorem.

Theorem 2.2. *Let $C \neq 8q + 1$ ($q \in \mathbb{N}$, $q \geq 1$) be an integer and S a numerical semigroup with multiplicity 8 and conductor $C \geq 8$. Then*

1) *The semigroup $S = < 8, C+1, C+2, C+3, C+4, C+5, C+6, C+7 >$ is saturated numerical semigroup, where $C \equiv 0 (\bmod 8)$,*
2) *The semigroup $S = < 8, C, C+1, C+2, C+3, C+4, C+5, C+7 >$ is saturated numerical semigroup, where $C \equiv 2 (\bmod 8)$,*
3) *The semigroup $S = < 8, C, C+1, C+2, C+3, C+4, C+6, C+7 >$ is saturated numerical semigroup, where $C \equiv 3 (\bmod 8)$,*
4) *The semigroup $S = < 8, C, C+1, C+2, C+3, C+5, C+6, C+7 >$ is saturated numerical semigroup, where $C \equiv 4 (\bmod 8)$,*

5) The semigroup $S =< 8, C, C+1, C+2, C+4, C+5, C+6, C+7 >$ *is saturated numerical semigroup, where* $C \equiv 5 (\bmod 8)$,

6) The semigroup $S =< 8, C, C+1, C+3, C+4, C+5, C+6, C+7 >$ *is saturated numerical semigroup, where* $C \equiv 6 (\bmod 8)$,

7) The semigroup $S =< 8, C, C+2, C+3, C+4, C+5, C+6, C+7 >$ *is saturated numerical semigroup, where* $C \equiv 7 (\bmod 8)$.

Proof. We will prove only one case. Other cases can be proved in a similar way.

Let prove case (1).

Let $C = 8q$ ($q \in \mathbb{N}$, $q \geq 1$) be an integer. Then we have

$$\begin{aligned} S &=< 8, C+1, C+2, C+3, C+4, C+5, C+6, C+7 > \\ &=< 8, 8q+1, 8q+2, 8q+3, 8q+4, 8q+5, 8q+6, 8q+7 >. \\ &= \{0, 8, 16, 24, ..., 8(q-1), 8q, \rightarrow ...\}. \end{aligned}$$

In this case,

i. if $s > C$ then $s + d_S(s) = s + 1 \in S$, since $d_S(s) = 1$ and $s \in S, s > 0$. Thus, we obtain that S is saturated numerical semigroup by Proposition 2.1.

ii. if $s \leq C$ then $s + d_S(s) = s + 8 \in S$, since $d_S(s) = 8$ and $s \in S, s > 0$. Thus, we obtain that S is saturated numerical semigroup by Proposition 2.1.

Theorem 2.3. *Let* $C = 8q$ ($q \in \mathbb{N}$, $q \geq 1$) *be an integer and* $S =< 8, C+1, C+2, C+3, C+4, C+5, C+6, C+7 >$ *is saturated numerical semigroup with multiplicity 8 and conductor* C. *Then, we have*

a) $F(S) = 8q - 1$,

b) $n(S) = q$,

c) $G(S) = 7q$.

Proof. Let $C = 8q$ ($q \in \mathbb{N}$, $q \geq 1$) be an integer and

$S = < 8, C+1, C+2, C+3, C+4, C+5, C+6, C+7 >$ is saturated numerical semigroup with multiplicity 8 and conductor C. Then we write that

a) $F(S) = 8q - 1$ since $C = F(S) + 1 = 8q$.

b) Since $C = 8q$ ($q \in \mathbb{N}$, $q \geq 1$), S is

$$\begin{aligned} S &= < 8, C+1, C+2, C+3, C+4, C+5, C+6, C+7 > \\ &= < 8, 8q+1, 8q+2, 8q+3, 8q+4, 8q+5, 8q+6, 8q+7 > \\ &= \{0, 8, 16, 24, ..., 8(q-1), 8q, \rightarrow ...\}. \end{aligned}$$

So, we have

$n(S) = \#(\{0, 1, 2, ..., 8q-8, ..., 8q-2, 8q-1\} \cap S) = \#(\{0, 8, 16, 24, ..., 8(q-1)\}) = q$.

c) $G(S) = F(S) + 1 - n(S) = 8q - 1 + 1 - q = 7q$.

Theorem 2.4. *Let* $C = 8q + 2$ ($q \in \mathbb{N}$, $q \geq 1$) *be an integer and*

$S = < 8, C, C+1, C+2, C+3, C+4, C+5, C+7 >$ *is saturated numerical semigroup with multiplicity 8 and conductor* C. *Then, we have*

a) $F(S) = 8q + 1$,

b) $n(S) = q + 1$,

c) $G(S) = 7q + 1$.

Proof. Let $C = 8q + 2$ ($q \in \mathbb{N}$, $q \geq 1$) be an integer and

$S = < 8, C, C+1, C+2, C+3, C+4, C+5, C+7 >$ is saturated numerical semigroup with multiplicity 8 and conductor C. Then,

a) It is trivial $F(S) = 8q + 1$ from $C = F(S) + 1$.

b) If $S = < 8, C, C+1, C+2, C+3, C+4, C+5, C+7 >$ is saturated numerical semigroup with multiplicity 8 and conductor C. Then we write

$$S =< 8, C, C+1, C+2, C+3, C+4, C+5, C+7>$$
$$=< 8, 8q+2, 8q+3, 8q+4, 8q+5, 8q+6, 8q+7, 8q+9>$$
$$= \{0,8,16,24,...,8(q-1),8q,8q+2,\rightarrow ...\}.$$

In this case, $n(S) = \#(\{0,1,2,...,8q-8,...,8q-2,8q-1,8q,8q+1,8q+2\} \cap S)$

$$= \#(\{0,8,16,24,...,8(q-1),8q\}) = q+1.$$

c) $G(S) = F(S)+1-n(S) = 8q+1+1-(q+1) = 7q+1$.

The following theorems will be given without their proofs. Anyone can be proved by similar ways in Theorem 2.3 and Theorem 2.4.

Theorem 2.5. *Let* $C = 8q+3$ $(q \in \mathbb{N},\ q \geq 1)$ *be an integer and* $S =< 8, C, C+1, C+2, C+3, C+4, C+6, C+7>$ *is saturated numerical semigroup with multiplicity 8 and conductor* C. *Then, we have*

a) $F(S) = 8q+2$,

b) $n(S) = q+1$,

c) $G(S) = 7q+2$.

Theorem 2.6. *Let* $C = 8q+4$ $(q \in \mathbb{N},\ q \geq 1)$ *be an integer and* $S =< 8, C, C+1, C+2, C+3, C+5, C+6, C+7>$ *is saturated numerical semigroup with multiplicity 8 and conductor* C. *Then, we have*

a) $F(S) = 8q+3$,

b) $n(S) = q+1$,

c) $G(S) = 7q+3$.

Theorem 2.7. *Let* $C = 8q+5$ $(q \in \mathbb{N},\ q \geq 1)$ *be an integer and* $S =< 8, C, C+1, C+2, C+4, C+5, C+6, C+7>$ *is saturated numerical semigroup with multiplicity 8 and conductor* C. *Then, we have* *a)* $F(S) = 8q + 4$,

b) $n(S) = q + 1$,

c) $G(S) = 7q + 4$.

Theorem 2.8. *Let* $C = 8q + 6\,(q \in \mathbb{N},\ q \geq 1)$ *be an integer and* $S =< 8,\ C,\ C+1,\ C+3,\ C+4,\ C+5,\ C+6,\ C+7>$ *is saturated numerical semigroup with multiplicity 8 and conductor* C. *Then, we have*

a) $F(S) = 8q + 5$,

b) $n(S) = q + 1$,

c) $G(S) = 7q + 5$.

Theorem 2.9. *Let* $C = 8q + 7\,(q \in \mathbb{N},\ q \geq 1)$ *be an integer and* $S =< 8,\ C,\ C+2,\ C+3,\ C+4,\ C+5,\ C+6,\ C+7>$ *is saturated numerical semigroup with multiplicity 8 and conductor* C. *Then, we have*

a) $F(S) = 8q + 6$,

b) $n(S) = q + 1$,

c) $G(S) = 7q + 6$.

Example 2.10. If we take $C = 15$ (*for* $q = 1$) in Theorem 2.9, then we write

$$\begin{aligned} S &=< 8,\ C,\ C+2,\ C+3,\ C+4,\ C+5,\ C+6,\ C+7> \\ &=<8,15,17,18,19,20,21,22>= \{0,8,15,\rightarrow ...\}. \end{aligned}$$

In this case, we find that $F(S) = 8q + 6 = 14$, $n(S) = q + 1 = 2$ and $G(S) = 7q + 6 = 13$.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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