Available online at http://scik.org
J. Semigroup Theory Appl. 2019, 2019:3
https://doi.org/10.28919/jsta/3816
ISSN: 2051-2937

ON PAIRWISE C-CLOSED SPACE IN BITOPOLOGICAL SPACE

NOSAIBA M. OMER*, HASAN Z. HDEIB

Department of Mathematics, University of Jordan, Amman, Jordan

Abstract. In this paper, we will obtain several results concerning the properties of pairwise C-closed spaces and to study the relations of pairwise C-closed spaces with some related pairwise topological properties like pairwise compactness, sequential spaces, pairwise quasi-k spaces and pairwise C-sequential spaces.

Keywords: pairwise C-closed spaces; pairwice k-spaces; pairwise quasi-k-spaces; pairwise tightness; sequential; pairwise C-sequential.

2010 AMS Subject Classification: 54A05.

1. Introduction

The study of bitopological spaces was first initiated by J. C. Kelly [1] in 1963 and thereafter a large number of papers have been done to generalize the topological concepts to bitopological setting. In thiseaper, we study the notion of pairwise C-closed spaces in bitopological spaces and their relation with other bitopological concepts. we will show that pairwise countably compact C- closed space has countable tightness and we will introduce characterization of pairwice sequential compact hausdorff spaces . We use R to denote the set of all real and P- to denote pairwise, Cl to denote the closure of a set, and t (X) to denote the tightness of X.

*Corresponding author

E-mail address: nosaiba1984@gmail.com

Received July 27, 2018

2. Pairwise C-Closed Space

Definition 2.1: [6] A cover V of the bitoplogical space $(X, \tau_{1,} \tau_2)$ is called pairwise open cover if $V \in \tau_1 \vee \tau_2$.

Definition 2.2: . A bitopological space $(x, \tau_{1,} \tau_2)$ is said to be pairwise countably compact if every countably pairwise open cover of X has finite subcover.

Definition 2.3: [1] A bitobological space $(x, \tau_{1,} \tau_2)$ is called pairwise hausdorff if for any tow distinct points x,y $\in$ X , there exist disjoint $V_1 \in \tau_1$ and $V_2 \in \tau_2$ with x $\in V_1$ and y $\in V_2$

Definition 2.4: [4] In a space (x, τ_1, τ_2), τ_1 is said to be regular with respect to τ_2 if, for each point x$\in X$ and each τ_1- closed subset F s.t x$\notin$ F, there are τ_1-open set U and τ_2-open set V s.t x$\in U$ and F$\subset V$ and $U \cap V = \emptyset$. (x, τ_1, τ_2) is p- regular if τ_1 regular with respect to τ_2 and vise versa.

Reilly [5] proves the following proposition:

Proposition 2.5: If (X,τ_1,τ_2) is a bitopological space, the following are equivalent:

a) τ_1 is regular with respect to τ_2

b) For each point $x \in X$ and τ_1-open set U containing X, there is a τ_1-open set V such that X $\in V \subset \tau_2$-cl $V \subset U$

Definition 2.6: A bitopological space $(x, \tau_{1,} \tau_2)$ is called pairwise C- closed if every τ_1-countably compact subset of X is τ_{2}- closed in X and every τ_2- countably compact subset of X is τ_1- closed in X.

Definition 2.7: Let $(x, \tau_{1,} \tau_2)$ be bitoplogical space, $A \subset X$, we say that x $\in X$ is a τ_i- cluster point for A, if for every τ_i-open set U containing x, $U \cap A$ / $\{x\} \neq \emptyset$ i=1,2.

Definition 2.8: A bitoplogical space $(X, \tau_{1,} \tau_2)$ is called pairwise C-closed if every non τ_1-closed subset A of X contains a sequence which has no τ_2-cluster point in A,and every non τ_2-closed subset B of X contains a sequence which has no τ_1-cluster point in B.

From definition of pairwise C-closed we have:

Corollary 2.9: Every subspace of pairwise c-closed is pairwise C-closed.

Definition 2.10: A bitopological space $(x, \tau_{1,} \tau_2)$ is said to be sequential if both (x, τ_1) and (x, τ_2) sequantial, i.e every non τ_1- closed subset A of X contains a sequance converting to a point in X \ A and every non τ_2- closed subset B of X contains a sequance converting to a point in X\ B

Theorem 2.11: Let $(X,\tau_{1,}\tau_2)$ be pairwise Hausdorff space, let (x_n) be a convergent sequnce in X, then (x_n) has exactly one limit point.

Proof: Suppose the contrary. Then $X_n \to x$ and $X_n \to y$ for some $x \neq y$, there exist disjoint $U \in \tau_1$ and $V \in \tau_2$ with x $\in U$ and y $\in V$. Therefore, there exist $N_U \in \mathbb{N}$ such that $x_n \in U$ for every $n > N_U$ and $N_V \in \mathbb{N}$ such that $x_n \in V$ for every $n > N_V$,choose $N = \max\{N_U, N_V\}$. Thus, there exist $N \in \mathbb{N}$ such that $x_n \in U, x_n \in V$ for every $n > N$.

But $U \cap V = \phi$, which is the contradiction.

Proposition 2.12: Every pairwise Hausdorff sequantial space is pairwise C-closed.

Proof: let A be non τ_1-closed subset of X,since X is sequantial, there exist a sequnce (x_n) converting to apoint in $X \setminus A$ say x, By uniquness of limit point of the sequnce in pairwice hausdorff space, we conclude that (X_n) has no τ_2-cluster point in A,similarly we can proof that every non τ_2-closed subset B of X contain sequnce has no τ_1-cluster point in B.

Hence the result.

Proposition 2.13: If X is pairwise Hausdroff and every pairwise countably compact subset of X is sequantial then X is pairwise C- closed.

Proof: let A be τ_1- countably compact subset of X and suppose that A is not τ_2- closed in X, then there exist x $\in \tau_2$-Cl A \ A, let B = A $\cup \{x\}$, then B is also τ_1- countably compact, now A is not τ_2- closed in B, Since B is sequantial then there exist sequance x$_n$ in A s.t x$_n \to$ B\ A = $\{x\}$. Therefore there exist seq x$_n$ in A has no τ_1- cluster point in A, this is contradiction.

Note that every pairwise countably compact subset of a bitopological space X may be sequantial and X may still be not sequantial. such is, the follwing example:

Example 2.14: The space of all continous real valued function on the interval [0,1] and generalize example [7] by letting $\tau_1 = \tau_2 =$ the point wise convergence topology.

Definition 2.15: [2] A map f: X$\longrightarrow$ Y from bitopological space (X,τ_1,τ_2) to another bitopological space (Y,σ_1,σ_2) is called pairwise continous if f is continous both as a map from (X,τ_1) to (Y,σ_1) and as a map from (X,τ_2) to (Y,σ_2).

Proposition 2.16: Let f: $(X,\tau_{1,}\tau_2) \longrightarrow (Y,\sigma_{1,}\sigma_2)$ be a pairwise continous one - to - one function, if (X,τ_1,τ_2) is pairwise hausdroff space and (Y,σ_1,σ_2) is pairwise C-closed, then(X,τ_1,τ_2) is pairwise C-closed.

Proof: Let f: $(X,\tau_{1,}\tau_2) \longrightarrow (Y,\sigma_{1,}\sigma_2)$ be pairwise continous and one- to - one map, then f: $(X,\tau_1) \longrightarrow (Y,\sigma_1)$ and f: $(X,\tau_2) \longrightarrow (Y,\sigma_2)$ are both continous, let A be τ_1-countably compact subset of X, then $f(A)$ is σ_1-countably compact subset of Y, but Y is pairwise C-closed, thus $f(A)$ is σ_2-closed subset of Y, since f is pairwise continous and one- to - one map, we get $f^{-1}(f(A)) = A$ is τ_2- closed subset of X. similarly we can prove that if A is τ_2-countably compact subset of X then A is τ_1-closed subset of X, this completes the proof.

Corollary 2.17: In a bitobological space $(x,\tau_{1,}\tau_2)$, if X has a weaker bitoplogical space which is pairwise C-closed, then X is is pairwise C-closed.

Proposition 2.18: Let X be a pairwise regular space and every point has a pairwise C-closed neighbourhood, then X is pairwice C-closed.

Proof: Let A be τ_1-countably coompact subset of X and x$\in$ τ_2-cl (A), wont to show that x$\in A$, let U be a τ_2- open set containing x and U is pairwise C-closed,then by p- regularity there is a τ_2-open set V such that x$\in V \subset \tau_1$-cl $(V) \subset U$. since A is τ_1- countably compact, then τ_1-cl $(V)\cap A$ is also τ_1-countably compact subset of U,hence it is τ_2-closed subset of U. But x$\in$ τ_2-cl(τ_1-cl$(V)\cap A$)= τ_1-cl($V)\cap A$, hence x$\in A$,therefore A is τ_2-closed subset of of X. similarly, we can prove that if A is τ_2-countably compact subset of X, then A is τ_1-closed subset of A,this complete the proof.

Definition 2.18: [3] The tightness of x $t(X)$ denoted by the smallest cardinal numbers Γ s.that whenever $A \subset X$ and x$\in \overline{A}$,then there is a subset B of A so that $|B| \leq \Gamma$ and $x \in \overline{B}$.

Definition 2.19: A bitopological space $(x,\tau_{1,}\tau_2)$ is said to have a pairwise countable tightness property if it has τ_1-countable tightness and τ_2-countable tightness property.

Definition 2.20: A subset A of bitopological space $(x,\tau_{1,}\tau_2)$ is called pairwise k- closed if for every pairwise compact subset K of X, A $\cap$ K is τ_1- closed (τ_2 - closed) in K.

Definition 2.21: A subset A of bitopological space is called pairwise quasi k- closed if for every pairwise countably compact subset K of X, A $\cap$ K is τ_1- closed(τ_2 - closed) in K.

Definition 2. 22: A bitopological space $(x,\tau_{1,}\tau_2)$ is said to be pairwise k- space if every τ_1-k-closed (τ_2- k-closed) subset of X is τ_1-closed (τ_2- closed) in X.

Example 2.23: Consider (R, $\tau_{1,}\tau_2$) where τ_1 is the discrete topology and τ_2= $\{U \subset R : 0 \notin R\}$ $\cup$ $\{R\}$, then (R, $\tau_{1,}\tau_2$) is a a pairwise-k space.

Definition 2.24: A bitopological space $(x,\tau_{1,}\tau_2)$ is said to be pairwise quasi- k- space if every τ_1-quasi- k-closed (τ_2-quasi- k-closed) subset of X is τ_1-closed (τ_2- closed) in X.

Proposition 2.25: If X is a pairwise hausdorff, pairwise quasi -k and (in particular pairwise countably compact or pairwise k) and pairwise C-closed space, then t$(X) \leq w_0$.

Proof: Let

$A \subset X$ $Y = \cup\{\tau_1\text{-cl}(B) : B \subset A \text{ and } |B| \leq \omega_0\}$ and $Z = \cup\{\tau_2\text{-cl}(F) : F \subset A \text{ and } |F| \leq \omega_0\}$. wont to show that τ_1-cl$(A) = Y$ and τ_2-cl$(B) = Z$. now $A \subseteq Y \subseteq \tau_1$-cl$(A)$ and $A \subseteq Z \subseteq \tau_2$-cl$(A)$, we need to show that Y is τ_1-closed in X and Z is is τ_2-closed in X. assume the contrary that Y is not τ_1-closed in X or Z is not τ_2-closed in X. if Y is not τ_1-closed in X, then Y is not quasi-k-closed in X, i.e there is pairwise coutably compact subset K of X s.that K $\cap Y$ is not τ_1-closed in K. since K is pairwise C-closed, then K $\cap Y$ is not τ_2- countably compact, i.e there is a sequnce x$_n$ in K $\cap Y$ which has no cluster point in K $\cap Y$,but K is pairwise countably compact, hence x$_n$ must have cluster point in K say x, therefore $x \notin Y$. now for every n choose $B_n \subseteq A$ s.that B_n is countable and x$_n \in \tau_1$-cl (B_n) and let $B = \bigcup_{n=1}^{\infty} B_n$, then x $\in \tau_1$-cl(B), but τ_1-cl$(B) \subseteq Y$, thus x$\in y$, this is a contraduction.

The assumption of quasi-k space in the above propsition is very importent to get the result, the following example shows this:

Example 2.26: Let $(X,\tau_{1,}\tau_2)$ be topological space, where X = $Y \cup$ $\{x\}$, where τ_1 consist of Y which is discrete space of cardinality ω_1 and x has countable neighborhoods and τ_2 has discrete topology, then every τ_1- countably compact subset of X is finite, therefore it is τ_2-closed, and every τ_2-countably compact subset of X is finite and hence it is τ_1-closed subset of X, therefore X is pairwise C-closed space, but t(X) = ω_1

Definition 2.27: Let $(x,\tau_{1,}\tau_2)$ be a bitopological space, let $A \subset X$, then x is called τ_i-isolated point of A if there exist open set $U \in \tau_i$ s.that $U \cap A = \{x\}$, i = 1,2.

Definition 2.28: A bitopological space $(x,\tau_{1,}\tau_2)$ is said to be (C-sequantial) if for every τ_1-closed (τ_2-closed) subset A of X and for every non τ_1-isolated (non τ_2-isolated) point x of A, there is a sequnce x_n in A / $\{x\}$ converting to x.

Proposition 2.29: If X is pairwise Hausdorff, pairwise quasi- k and pairwise C-closed, then X is pairwise C-sequantial.

Proof: since every P– closed subset of X is pairwise quasi-K and pairwise C-closed, it is enough to show that if x is not τ_1-isolated(not τ_2-isolated) point in X, then there is a sequnce in X / $\{x\}$ converting to x. if x is not τ_1-isolated point of X, then $U\cap X\neq\{x\}$ for every $U\in\tau_1$ and hence X / $\{x\}$ is not τ_1-closed in X. similarly, if x is not τ_2- isolated point in X, then $V\cap A\neq\{x\}$ for every $V\in\tau_2$ and hence X / $\{x\}$ is not τ_2-closed in X. If X / $\{x\}$ is not τ_1-closed in X, then there is τ_1- countably compact subset K of X S.that K / $\{x\}$ is not τ_1-closed in K. since K is C-closed, K / $\{x\}$ is not τ_1-closed in K, then there is a sequnce x_n in K / $\{x\}$ which has no τ_2-cluster point in K / $\{x\}$, Therefore $x_n\to$ x. similarly, if X / $\{x\}$ is not τ_2-closed in X, we get $x_n\longrightarrow$ x. This complete the proof.

Definition 2.30: A bitoplogical space $(x,\tau_{1,}\tau_2)$ is said to be sequantially compact with respect to τ_i if every infinite sequence has convergent subsequance with respect to τ_i,i.e for every sequnce$\{x_n:n\in w\}$ and for every τi-open *nhd* U of x s.that $x_n\in U$ whenever n $\geq m$ for some m, there exist subsequnce $\left\{x_{n_k}:k\in w\right\}$of x_n s. that $x_{n_k}\in U$ whenever k $\geq m$. i= 1,2.

Definition 2.31: A bitoplogical space space $(x,\tau_{1,}\tau_2)$ is said to be pairwise sequantially compact if it is sequantially compact with respect to τ_1 and sequantially compact with respect to τ_2.

Proposition 2.32: A pairwise sequentially compact Hausdorff space X is pairwise sequential iff it is pairwise C-closed.

Proof: ($\Rightarrow$) it is obvious from Corollary 2. 12 ($\Leftarrow$) let A be non τ_1-closed subset of X, then there is a sequance x_n in A which has no τ_2-cluster point in A, but X is pairwise seuantially compact, thus x_n has convergent subsequnce x_{n_K} with respect to τ_2 say to x$\in X$, since x_{n_K} has no τ_2-cluster point in A, then x$\in X$ / A, therefore there is a sequance in A converting to a point in X/A. we get (X,τ_1) is sequantial. (1) similarly, if B is non τ_2-closed subset of X, then there is a sequance x_m in B which has no τ_1-cluster point in B, since X is pairwise seuantially compact, x_m has convergent subsequnce x_{m_L} with respect to τ_1 say to y $\in X$,but x_{m_L} has no τ_1- cluster point in A,hence y$\in X\ /B$ and (X,τ_2) is sequantial. (2) from 1 and 2, we get X is pairwise sequantial.

Conflict of Interests

The authors declare that there is no conflict of interests.

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