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SOME CHARACTERIZATIONS OF AMENABLE SEMIGROUPS

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Abstract. In this paper, we give a survey of some known results on amenable semigroups with different characterizations using some classes of semigroups which include right stationary semigroups, cancellative semigroups, inverse semigroups, finitely generated semigroups, Clifford ω-semigroups and inverse ω- semigroups.

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1. Introduction

Many literature have been written on amenable semigroups and their corresponding Banach algebras using different classes of semigroups. The first author to use the word 'amenable' was Mahlon M. Day in 1949 while studying amenability of locally compact groups.
In 1955, Folner [8], introduced and characterized groups that have full Banach mean value and as well gave necessary and sufficient conditions for such groups to possess such characteristics. Day, in 1957 introduced amenable semigroups and showed that every amenable semigroup is

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strongly amenable [3]. In his work, he gave the necessary and sufficient conditions for amenable semigroups to have an invariant mean.

In 1972, the notion of amenability in Banach algebras was initiated by B.E Johnson when he stated that a locally compact group G is amenable if and only its group algebra $L^1(G)$ is amenable as a Banach algebra. Ever since this pioneer work on amenability in Banach algebra, amenability has become a fruitful area of research and various equivalent characterizations have emerged from the notions of amenability in Banach algebras, which has in turn led to the development of various other notions of amenability.

Day, in [3] investigated the existence and behaviour of invariant means on semigroups using some known results. Mitchell in [11] gave characterizations of left invariant means of the space of all bounded real-valued functions on a semigroup S on an arbitrary element of the said Banach algebra in terms of pointwise convergence of finite averages of right translates of the element to a constant function.

The theorem of Frey was generalized in [4] by giving sufficient condition for a subsemigroup of a cancellative left amenable semigroup to be left amenable. It was particularly shown in [4] that if S is a cancellative left amenable and T is its subsemigroup that does not contain a free semigroup on two generators then T is left amenable as well.

In the monograph of Paterson [14], the author showed that left reversibility is a pre-condition for left amenability while the authors in [15] showed the relationship between left thick subsets and left reversibility.

The geometric characterisation of amenable semigroup considered in this work is the Folner condition vis-a-vis growth of finitely generated semigroup. Every finitely generated semigroup of subexponential growth satisfies the Folner condition [15]. It was also proved in [15] that a finitely generated semigroup of subexponential growth satisfying the Klawe condition is left amenable and in turn satisfies the strong Folner condition.

Paterson in [13], characterized amenable Clifford semigroup in terms of the set of normalized positive definite functions arising from the left regular representation of the semigroup. The author also introduced a type of mean associated with this set, a condition considered to be weaker than left invariance.

The purpose of this note is to give an overview of what has been done on the amenability of some classes of a semigroup S using different characterizations.

2. Preliminaries

In this section, we give basic concepts and terms that are relevant to this work.

By a semigroup, we mean a set S equipped with a binary operation:

$S\times S\to S$; $(x,y)\mapsto xy$ which is associative i.e $(x_1,x_2)x_3 = x_1(x_2,x_3)\ \forall x_1,x_2,x_3\in S$.

Let S be a semigroup. For each $s\in S$, define $L_s(t) = st, \quad R_s(t) = ts \quad (t\in S)$. An element $s\in S$ is left (resp. right) cancellable if L_s(resp. R_s) is injective on S and s is cancellable if it is both left cancellable and right cancellable.

The semigroup S is left (resp. right) cancellative if each element in S is left (resp. right) cancellable, and cancellative if each element is cancellable.

S is *near left cancellative* if for every element s of S there is a left thick subset E on which the left translation map by s restricts to an injective map, that is, $sx\neq sy$ whenever $x,y\in E$ with $x\neq y$.

A semigroup S satisfies the *Klawe condition* if whever s,x and $y\in S$ are such that $sx = sy$, there exists $t\in S$ so that $xt = yt$.

Every near left cancellative semigroup satisfies the Klawe condition [15, Proposition 2.3].

A semigroup S with $E(S) = \omega$ is called an ω-semigroup where $\omega = (\mathbb{Z}_+,\vee)$ with $m\vee n = max\{m,n\}$ [6].

An inverse ω-semigroup Ω is a semigroup with $E_\Omega = \omega$ where $\omega = (\mathbb{Z}_+,\vee)$ and $E_\Omega = \{e_n : n = 0,1,2,...\}$ such that $e_0 > e_1 > e_2...$ and e_0 is the identity element of Ω.

For more details on inverse ω-semigroup, see [6].

Let S be a non empty set. A measure μ on S is an extended real-valued, non-negative and countably additive set function such that $\mu(0) = 0$.

A set with full measure is one whose complement is of measure zero.

An element μ of $l^\infty(S)^*$ is called left(right) invariant if $\mu(l_sx) = \mu x(\mu(r_sx) = \mu(x))$ for all $x\in l^\infty(S)$ and $s\in S$.

A semigroup S is left(right) amenable if there exists a left(right) invariant mean on $l^\infty(S)$.

The following are equivalent definitions of left and right amenable semigroups:

A semigroup S is left (right) amenable if and only if there exists a net $\{\varphi_n\}$ of finite means in S such that $\{\varphi_n\}$ converges weakly to left (right) invariance [3].

Von Neumann defined a semigroup S as left amenable if and only if there exists a finite additive probability measure: $\mu : 2^s \to [0,1]$ such that $\forall s \in S, \quad X \subseteq S, \quad \mu(X) = \mu(s^{-1}X)$.

Examples:(i) Finite groups(uniform measure).

(ii) Compact groups (Haar measure).

(iii) Semigroup with zero: $\mu(X) = 1$ if $0 \in X$ and 0 if otherwise.

(iv) Commutative semigroup (Day).

(v) Bicyclic monoids.

The following definition is due to Day in 1957.

A semigroup S is right amenable if there is a finitely additive probability measure μ on S such that:

$\mu(X) = \mu(Xs^{-1}) \quad \forall X \subseteq S$ and $s \in S$ (where Xs^{-1} denotes $t \in S$ such that $ts = X$).

A semigroup S is called left (right) strongly amenable if there exists a net $\{\varphi_n\}$ of finite means convergent in norm to left (right) invariance; that is for each $s \in S, \quad lim_n ||l_s^* \phi\varphi_n - \phi\varphi_n|| = 0 \quad (lim_n ||r_s^* \phi\varphi_n - \phi\varphi_n|| = 0)$.

A semigroup S is called amenable if there is a mean μ on $l^\infty(S)$ which is both left and right invariant.

Let S be a semigroup and let $l^\infty(S)$ be a space of all bounded real-valued functions on S. If S is a left amenable [right amenable] (amenable) semigroup, and $f_0 \in l^\infty(S)$, then f_0 is called left almost [right almost] (almost) convergent to $\alpha \in \mathbb{R}$, if all left invariant[right invariant] (invariant) means on f_0, have the same value, α.

A cancellative semigroup S is left amenable if and only if for each $\varepsilon \in (0,1)$ and for each finite, non empty subset $H \subseteq S$, there exists a finite non empty subset $E \subseteq S$ such that for each $h \in H, \frac{|hE \cap E|}{|E|} > \varepsilon$ [4, Theorem 3]. The set E above is called a Folner set of the semigroup S.

The following is the Weak Folner condition:

There exists a number $k, 0 < k < 1$, such that for any elements $s_1, ..., s_n$ of S (not necessarily distinct), there is a finite subset A of S satisfying: $\frac{1}{n}\sum_{i+1}^{n} |A \ s_iA| \leq k|A|$.

The Folner number $\varphi(S)$ for an arbitrary semigroup S is the infimum of all numbers $k \leq 1$ such that the Weak Folner Condition holds.

A function $f : S \to \mathbb{C}$ is said to be positive definite if $\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\alpha}_j f(s_i s_j^*) \geq 0$ whenever $n \in \mathbb{Z}^+, s_1, ..., s_n \in S$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}$.

3. Semigroup amenability and invariant mean

In this section, we give some existing results on analytic characterization of amenable semigroups in terms of translation invariant mean.

We recall that an element μ of $l^\infty(S)^*$ is called left(right) invariant if:

$\mu(l_s x) = \mu x(\mu(r_s x) = \mu(x))$ for all $x \in l^\infty(S)$ and $s \in S$.

If a semigroup possesses a left (right) invariant mean, it is therefore called a left (right) amenable semigroup. Day in [3] gave some properties of invariant means and how these properties characterize amenable semigroups.

Theorem 3.1. Let S be a left amenable semigroup and let T be a subsemigroup of S. Then T is left amenable if :

(i) $\exists$ a left invariant mean μ such that $\mu(\chi) > 0$ where χ is the characteristics function of T.

(ii) S is a cancellative semigroup with no free subsemigroup on two generators.

Proof. (i) This is [3, Theorem 2].

(ii) This is [4, Theorem 2].

The following Theorem by Day in [3] shows the relationship between finite semigroups and finite groups.

Theorem 3.2. A finite semigroup S has an invariant mean if and only if it has just one minimal left ideal and just one minimal right ideal. When the ideals coincide, the resulting two-sided ideal is the kernel of S which is a finite group G. The unique invariant mean on $l^\infty(S)$ is that of G; if n equals the number of elements of G, then $\mu(x) = n^{-1}\sum_{g\in G} x(g)$.

A finite semigroup is left amenable if and only if it contains precisely one minimal right ideal [13].

We recall that an element $\mu \in l^{\infty}(S)'$ is a mean on $l^{\infty}(S)$ if $||\mu|| \leq 1$ and $\mu(e) = 1$ where e denotes the constant function on S [11].

Theorem 3.3. [11, Theorem 3] Let S be a semigroup. Then the following are equivalent:
(a) For every $f \in l^{\infty}(S)$, there exists a net of finite averages of right translates of f which converges pointwise to a constant function.
(b) S is left amenable.
(c) There exists a net $\{T_{\delta}\}$ of finite averages of right translations such that for every $f \in l^{\infty}(S)$, $\{T_{\delta} f\}$ converges pointwise to a constant function .

Definition 3.4 We recall the following definitions given by Mitchell in [11].
Let L(R) be a set of left (right) translations of $l^{\infty}(S)$ by elements of S. Let $\wedge = CO(L)$ (Convex hull of the set of all left translations of $l^{\infty}(S)$, $\quad P = CO(R)$ (Convex hull of the set of all right translations of $l^{\infty}(S)$).
For $f \in l^{\infty}(S), \quad Z_R(f) \subseteq l^{\infty}(S) \quad [Z_L(f) \subseteq l^{\infty}(S)]$ is given by :

$$Z_R(f) = w^{*}CL(CO(Rf)) = w^{*}CL(Pf).$$

$$Z_L(f) = w^{*}CL(CO(lf)) = w^{*}CL(\wedge f)$$

$K_R(f) = \{\alpha e : \alpha e \in Z_R(f)\}$ and $K_L(f) = \{\alpha e : \alpha e \in Z_L(f)\}$ where α ranges over the real numbers.
A right stationary semigroup is one such that for each $f \in l^{\infty}(S)$, the set $K_R(f)$ is non empty. A left stationary semigroup is similarly defined.
For more detail on the properties of right stationary semigroups, see [11].
For a right stationary semigroup, we have the following results from [11].

Lemma 3.5 Let S be a right stationary semigroup, and let $f, g \in l^{\infty}(S)$. Then:
(a) The set $\{\alpha; \alpha e \in K_R(f)$, where α is real $\}$ is a closed interval, $[\alpha_1, \alpha_2]$, where $\alpha_1 \leq \alpha_2$.
(b) $K_R(f+g) \subseteq K_R(f) + K_R(g)$.

Theorem 3.6 Let S be a right stationary semigroup, for an arbitrary element of $l^{\infty}(S)$ and $\alpha \in \mathbb{R}$. If there exists a net of finite averages of right translates of f_0 which converges pointwise to the constant function αe, then there exists a left invariant mean μ on $l^{\infty}(S)$ such that $\mu(f_0) = \alpha$.

Proposition 3.7 Every stationary semigroup is amenable

Proof Let S be a stationary semigroup. By the analogue of result in [11, Corollary 2], if S is a left stationary semigroup, then S is right amenable. It also follows that if S is a right stationary semigroup, S is left amenable. Then it suffice to say that if S is a stationary semigroup, then S is amenable.

4. Amenability, left thickness and reversibility in semigroups.

In this section, the abstract and geometric characterization of semigroups are discussed.
Left thickness in semigroup theory is an abstract characterisation of those subsets capable of having ”full measure” [15].
Mitchell in [11] introduced and studied left thick sets, the property of subsets of left amenable semigroup that supports left invariant means. For details on left thick sets, see [11]. Paterson in [14] showed that a left amenable semigroup S is left thick if and only if it has measure 1 in some left invariant, finitely additive probability measure on S.
We recall that a subset E of a semigroup S is called left thick if for every finite subset F of S, there is an element t of S such that $Ft \subseteq E$.
Reversibility is another terminology used in the geometric characterization of amenable semigroups. Left reversibility is easily seen to be a necessary precondition for left amenability [15].
Every left amenable semigroup is left reversible [13, Proposition 1.2.3].
A left reversible semigroup S is left amenable if and only if $S/\approx$ is left amenable where $\approx$ is a congruence on S [14].
A semigroup S is left reversible if every pair of right ideals (or equivalently, every pair of principal right ideals) intersects.
Examples of left reversible semigroups are inverse semigroups, commutative semigroups and cancellative semigroups which embed in groups of left quotients.
The following theorem shows the equivalent relation between reversibility and left thickness in an arbitrary semigroup S.

Theorem 4.1.[15, Proposition 2.1] For any semigroup S, the following conditions are equivalent:

(i) S is left reversible,

(ii) every principal right ideal of S is left thick,

(iii) every right ideal of S is left thick.

Theorem 4.2 Let E be a subset of a left amenable finite semigroup S. Then E is left thick in S if and only if the following are equivalent:

(i) E contains a left ideal of S,

(ii) $\exists\ \mu \in L(S)$ (where $L(S)$ is the set of left invariant measures on S) such that $\mu(E) = 1$,

(iii) $\exists$ a characteristic function $f_0 \in E$ such that $\mu(f_0) = 1$,

(iv) E is left amenable.

Proof. (i) This is straightforward by definition.

(ii) This is [11, Proposition 1.21].

(iii) This is [11, Theorem 7].

(iv) This is [11, Theorem 9].

Let $\cong$ be a binary relation on an arbitrary semigroup S. If S is left reversible, then the quotient semigroup $S/\cong$ is a right cancellative semigroup [15]. The author in [14, Proposition 1.25] proved that a left reversible semigroup S is left amenable if and only if $S/\cong$ is left amenable. The following proposition shows the relationship between reversibility, Klawe condition and left cancellation in a semigroup S.

Definition 4.3 Let S be a semigroup. S is said to satisfy the Klawe condition if whenever $s, x, y \in S$, such that $sx = sy$, there exists $t \in S$ so that $xt = yt$.

Proposition 4.4 [15, Proposition 2.4] Let S be a left reversible semigroup. Then S satisfies the Klawe condition if and only if $S/\cong$ is left cancellative.

Theorem 2.6 [15] gives a sufficient condition for a semigroup satisfying Klawe condition to be left amenable.

5. Semigroup amenability and Folner conditions.

In this section, we consider geometric characterisation of amenable semigroups using Folner set as earlier defined. We give a survey of some results on amenable semigroups which satisfy Folner conditions. The authors in [15] studied the connection between amenability, Folner conditions and the geometry of finitely generated semigroups. They use the results of Klawe to show that within an extremely broad class of semigroups, left amenability coincides with the strong Folner condition. These same authors in [15] also gave a new chracterization of the strong Folner condition in terms of the existence of weak Folner sets satisfying a local injectivity condition on the relevant translation action of the semigroup.

In [8], the author introduced the necessary and sufficient condition for a group S to be left amenable. Frey in his thesis [9] showed that every left amenable semigroup satisfies FC (Folner conditions). Argabright and Wilde in [1] introduced the strong Folner Condition (SFC) and showed that any semigroup satisfying SFC is left amenable.

Yang in [16] studied some general properties of Folner numbers and Folner type conditions. In his work, the author showed that if the folner number of a semigroup is zero, then the semigroup satisfies strong Folner condition. Some general properties of the Folner number $\varphi(S)$, for all finite semigroups and cancellative semigroups were also investigated in [16]. We recall that a semigroup S satisfies Folner condition (FC) if for every finite subset H of S and every $\varepsilon > 0$, there is a finite subset F of S with $|sF \setminus F| \leq \varepsilon |F|$ for all $s \in H$.

A semigroup S satisfies the strong Folner condition (SFC) if for every finite subset H of S and every $\varepsilon > 0$, there is a finite subset F of S with $|F \setminus sF| \leq \varepsilon |F|$ for all $s \in H$.

In left cancellative semigroup, FC and SFC are trivially equivalent but without left cancellativity, SFC is strictly stronger because left translation by an element s can map many elements in a set F onto a few elements allowing $SF \setminus F$ to be small but $F \setminus sF$ large [15].

The following result due to [15] shows the equivalent relationship between left amenability, strong Folner and Klawe conditions.

Theorem 5.1 Let S be a left amenable semigroup. Then the following are equivalent.

(i) S is left reversible,

(ii) every right ideal of S is left thick,

(iii) S satisfies the Klawe condition,

(iv) the right cancellative quotient $S/\cong$ is left cancellative,

(v) S satisfies the strong Folner Condition.

Corollary 5.2 *Let S be a semigroup satisfying the Klawe condition. Then S is left amenable if and only if S satisfies SFC.*

6. Semigroup amenability and growth.

This section deals with an abstract characterization of amenable semigroups using some existing results. The author in [15] showed that for finitely generated semigroups satisfying the Klawe condition, sub-exponential growth is a sufficient condition for left amenability.

The concept of growth is an important characterization of amenable finitely generated semigroups in abstract semigroup theory.

The author in [2] showed that finitely linear semigroups over a field K has intermediate growth. This same author called the intermediate growth a sub-exponential growth.

Now we recall the following definitions.

Let $S = \langle g_1, ..., g_m \rangle$ be a finitely generated semigroup. The growth function $f_S : \mathbb{N} \to \mathbb{N}$ of semigroup S is obtained by defining $f_S(n)$ as the number of elements of S that can be presented as words of length not exceeding n in the generators $g_1, ..., g_m$. The semigroup S has a polynomial growth if its growth function is bounded above by a polynomial i.e The growth of f_S is polynomial if there exists a positive integer d such that $f_S(n) \leq n^d$ for almost all $n \in \mathbb{N}$.

Let $f : \mathbb{N} \to \mathbb{R}^+$ be a monotone increasing function where $\mathbb{R}^+$ is the set of positive real numbers. The growth of f is said to be exponential if there exists $c > 1$ such that $f(n) \geq c^n$ for almost all $n \in \mathbb{N}$.

S has sub-exponential growth if its growth function is eventually bounded above by every increasing exponential function.

The following Theorem is an analog to [15].

Theorem 6.1 Let S be a finitely generated semigroup of subexponential growth. Then the following conditions are equivalent:

(i) S is left cancellative,

(ii) $\varphi(S) = 0$,

(iii) S is left amenable.

Proof $(i) \Longrightarrow (ii)$ By [15, Theorem 4.3], S satisfies the Folner condition. If S is left cancellative, then it satisfies the strong Folner condition. Since Folner condition is equivalent to strong Folner condition in left cancellativity. By [16, Proposition 2.1], $\varphi(S) = 0$.

$(ii) \Longrightarrow (iii)$ If $\varphi(S) = 0$, then S satisfies Strong Folner condition [16, Proposition 2.1]. The result follows from [16, Corollary 2.4].

$(iii) \Longrightarrow (i)$ Suppose S is left amenable. By [11], S satisfies strong Folner condition. This implies that S is left cancellative.

The following conjecture in [15] states that not all finitely generated semigroup are amenable.

Conjecture 6.2 There is a left reversible finitely generated semigroup of polynomial growth which is not left amenable.

7. Semigroup amenability and Weak containment.

This section discusses characterization of amenable inverse semigroup using the concept of group theory. Group theory has been a useful tool in the study of amenability of semigroups. Although not all results in group theory translate well into semigroups. An example is given by Folner in [8] where the amenability of a group G is inherited by all of the subgroups of G. However, in general a subsemigroup of a left amenable need not be left amenable [4]. Using the concept of group theory, Duncan and Namioka in [5] proved that the inverse semigroup S is amenable if and only if its maximal group homomorphic image is amenable. Some examples of amenable semigroups were given by the authors in [5].

The concept of weak containment property was defined by Fell in [7]. The author in [6] stated that weak containment property is an appropriate notion of amenability for inverse semigroups by showing that the weak containment property is another notion of amenability motivated by group theory.

A locally compact group G is amenable if and only if $C^*(G) = C_l^*(G)$ [15, Theorem 4.21].

Let $\varphi : C^*(S) \to B(l^2(S))$ be an induced $*$-homomorphism whose image is called the *reduced* $C^* - algebra$ of S, denoted by $C_r^*(S)$. The inverse semigroup S has weak containment if and only if φ is an isomorphism.

The authors in [6] introduced and studied Clifford ω-semigroup and inverse ω-semigroups. They studied their amenability using weak containment property of the semigroups and characterized their respective C^*-algebras.

Given an inverse semigroup S and a homomorphism φ of S onto a group G, Milan in [10] showed under an assumption on $ker(\varphi)$, that S has weak containment if and only if G is amenable and $ker(\varphi)$ has weak containment. The author particularly showed that all graph inverse semigroups have weak containment and that Nica's inverse semigroup $T_{G,P}$ of a quasi-lattice ordered group (G,P) has weak containment if and only if (G,P) is amenable.

For a Clifford ω-semigroup, we have the following result from [6].

Theorem 7.1 Let T be the Clifford ω-semigroup $\cup\{G_n : n \in Z_+\}$. Then T has weak containment if and only if each G_n is amenable.

Corollary 7.2 *Let $S = \cup_i^n G_i$ be a Clifford semigroup with identity. Then S is amenable if and only if each G_i is amenable.*

Proof. By Theorem 7.1, S has weak containment if and only if each G_i is amenable. Now suppose there exists a map $\varphi : S \to G_s$ where G_s is the maximum group homomorphic image of S. If each G_i is amenable, then G_s is amenable. The result follows from [13, Proposition 4.1].

Let S be an inverse semigroup and $p \in E(S)$. We set

$$S_p = \{s \in S : ss^{-1} = s^{-1}s = p\}$$

where s^{-1} denotes the inverse of s. Then S_p is a group with identity p and S_p contains other subgroups of S with identity p. Thus S_p is called maximal subgroup of S at p.

Theorem 7.3

Let S be an inverse semigroup with finite $E(S)$. Let S_p be a maximal subgroup of S. If $P_L(S)$ is a set of pointwise closure of the set of positive definite functions on S and $P(S)$ is a set of positive functions on S. Then the following are equivalent:

(i) $P_L(S) = P(S)$.

(ii) S_p is amenable.

(iii) S has weak containment.

(iv) $C_r^*(S) = C^*(S)$.

Proof. $(i) \Rightarrow (ii)$ If $P_L(S) = P(S)$, then by [13, Proposition 3.6], S_p is amenable.

$(ii) \Rightarrow (iii)$ Suppose S_p is amenable, then by Corollary 7.2, S has weak containment.

$(iii) \Rightarrow (iv)$ Suppose S has weak containment, then by [10], $C_r^*(S) = C^*(S)$. $(iv) \Rightarrow (i)$ Since $P_L(S) = P(S)$ implies that S is amenable [13] and $C_r^*(S) = C^*(S)$ equally implies that S is amenable [10]. Then the result follows.

For an inverse ω-semigroup, we have the following result from [6].

Theorem 7.4 Let Ω be an inverse ω-semigroup which is not a Clifford ω-semigroup, so that Ω is a finite chain of the form $G_0 \cup G_1 \cup ... \cup G_{k-1} \cup \Omega_k$ where Ω_k is simple (or bisimple).

(i) $C^*(\Omega)$ is star isomorphic to the finite sequence algebra determined by $C^*(G_i) = i = 0, 1, ..., k-1$ and $C^*(\Omega_k)$.

(ii) Ω has weak containment if and only if G_i is amenable for $i = 0, 1, ...k-1$ and Ω_k has weak containment.

8. Open questions.

Question 8.1 If S is a non-discrete semigroup. Can S have a weak containment property?

Question 8.2 It has been observed that amenability in semigroup is associated with finiteness. Is there no infinite semigroup that is amenable?

Conflict of Interests

The authors declare that there is no conflict of interests.

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