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MODIFIED SEMI-LINEAR UNIFORM SPACES AND NEW TYPES OF CONTRACTIONS

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Abstract. In 2009 Tallafha, A. and Khalil, R. defined a new type of uniform space namely, semi-linear uniform space [8]. Later Tallafha, A. in [9], [10] and [11], Alhihi, S. in [1] and Tallafha, A. and Alhihi, S. in [12], gave more properties of semi-linear uniform spaces. Also Lipschitz condition and contraction mapping on semi-linear uniform spaces are defined, which enables one to study fixed point for such functions.

In this article we shall define a modified semi-linear uniform space and a new type of contractions on semi-linear uniform spaces, finally we ask the following natural question. If f is a contraction from a complete modified semi-linear uniform space (X,Γ) to itself, is f has a unique fixed point.

Keywords: uniform spaces; semi-linear spaces; contraction; fixed point; best approximation.

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1. Introduction

Banach contraction principle is a classical and powerful tool in nonlinear analysis, more precisely a self-mapping f on a complete metric space (X,d) such that $d(f(x),f(y)) \leq cd(x,y)$

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for all $x, y \in X$, where $c \in [0,1)$, has a unique fixed point. Since then, the Banach contraction principle has been generalized and investigated in several direction.

In 1909, when Luitzen Brouwer proved the first fixed point theorem named after him, fixed point theory has played very important roles in many different fields. We can find a lot of demonstrations in optimization theory, approximation theory, differential equations, variational inequalities, complementary problems, equilibrium theory, game theory, economics theory, and so forth.

Fixed point theorems are developed for single-valued or set-valued mappings of metric spaces, topological vector spaces, posets and lattices, Banach lattices,.... Among the themes of fixed point theory, the topic of approximation of fixed points of mappings is particularly important because it is useful for proving the existence of fixed points of mappings. It can be applied to prove the solvability of optimization problems, differential equations, variational inequalities, and equilibrium problems.

In a wide range of mathematical problems the existence of a solution is equivalent to the existence of a fixed point for a suitable map. The existence of a fixed point is therefore of paramount importance in several areas of mathematics and other sciences. Fixed point results provide conditions under which maps have solutions. The theory itself is a beautiful mixture of analysis (pure and applied), topology, and geometry. Over the last 50 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena.

Contractions are usually discussed in metric and normed space and never been studied in other weaker spaces, we will study fixed point theory in a space which is a beautiful mixture of analysis and topology, a space which is weaker than metric space and stronger than topological space, called semi-linear uniform space. Semi-linear uniform space is a new type of uniform space given by Tallafha, A. and Khalil,.R. in 2009 [10], by assuming certain conditions. A uniform space $(X, \mathscr{F})$, is a non-empty set X, with a uniform structure $\mathscr{F}$. Uniform spaces are topological spaces with additional structure that is used to define uniform properties such as completeness, uniform continuity and uniform convergence. The notion of uniformity has been investigated by several mathematician as Weil [16], [17], and [18]. L.W.Cohen [4], and

[5]. Graves [7].The theory of uniform spaces was given by Burbaki in [3], Also Wiel´s in his booklet [16] define uniformly continuous mapping. we refer the reader to [6] for more properties of uniform spaces.

In [11], [12] and [13]. Tallafha, A. defined another set valued map called δ on $X \times X$, which is used with ρ to give more properties of semi-linear uniform spaces. Finally he studied the relation between ρ and δ. Tallafha, also defined Lipschitz condition**,** and contraction mapping on semi-linear uniform spaces, which enables one to study fixed point for such functions, since best approximation, Lipschitz condition**,** and contractions are usually discussed in metric and normed spaces, and never been studied in other weaker spaces, we believe that the structure of semi-linear uniform spaces is very rich and all the known results on approximation and fixed point theory can be generalized.

In [1], Alhihi, S. and in [14] Tallafha, A. and Alhihi, S. gave mor properties of semi-linear uniform space. Also Alhihi, S. and Al-Fayyad M. [2] gave the important topological properties of semi-linear uniform spaces.

Tallafha, A. and Alhihi, S. [14], [15] established another properties of semi-linear uniform and they asked the following question. If f is a contraction from a complete semi-linear uniform space (X,Γ) to it self, is f has a unique fixed point.

In [9] Rawashdeh A. and Tallafha, A. answered the above question negatively, they gave an example of a complete semi-linear uniform space (X,Γ) and a contraction $f:(X,\Gamma)\to(X,\Gamma)$ which has infinitely many fixed points.

In this articles we shall define a modified semi-linear uniform space and a new types of contractions on semi-linear uniform spaces, finally we asked the following natural question. If f is a contraction from a complete modified semi-linear uniform space (X,Γ) to it self, is f has a unique fixed point.

2. Semi-Linear Uniform Spaces

In this section we shall give the definitions and notation of uniform and semi-linear uniform spaces.

Let X be a non empty set and A,B be two subsets of $X\times X$, i.e., A,B are relations on X. The inverse relation of A will be denoted by $A^{-1}=\{(y,x):(x,y)\in A\}$ and the composition of A,B will be denoted by $A\circ B$.Clearly the composition is associated but not commutative. For a relation A on X and a natural number n the relation nA is defined inductively by $1A=A$ and $nA=(n-1)A\circ A$.The identity relation will be denoted by $\Delta=\{(x,x):x\in X\}$. Let D_X be the set of all relations on X such that each element V of D_X contains the diagonal Δ and $V=V^{-1}$, D_X is called the family of all entourages of the diagonal.

Let $x_0\in X$ and $A\in D_X$, the ball with center x_0 and radius A is defined by $B(x_0,A)=\{x\in X:(x_0,x)\in A\}$. For $E\subseteq X$, $A\in D_X$, by the A ball about E we mean $B(E,A)=\bigcup_{x\in E}B(x,A)$.

Let $\mathscr{F}$ be a sub-collection of D_X. Then we have the following definition.

Definition 2.1. $[4]$.*The pair $(X,\mathscr{F})$ is called a uniform space if:*

(i) $V_1\cap V_2\in\mathscr{F}$ for all V_1,V_2 in $\mathscr{F}$.

(ii) for every $V\in\mathscr{F}$, there exists $U\in\mathscr{F}$ such that $U\circ U\subseteq V$.

(iii) $\bigcap\{V:V\in\mathscr{F}\}=\Delta$.

(iv) if $V\in\mathscr{F}$ and $V\subseteq W\in D_X$, then $W\in\mathscr{F}$.

If the condition $V=V^{-1}$ is omitted then the space is quasiuniform space.

In 2009 Tallafha, A. and Khalil, R. [10] defined anew type of Uniform spaces called, semi-linear uniform spaces.

Now we shall give the required definitions and notations of semi-linear uniform spaces.

Definition 2.2. $[10]$. *Let Γ be a sub collection of D_X, the pair (X,Γ) is called a semi-linear uniform space if,*

(i) Γ is a chain.

(ii) for every $V\in\Gamma$, there exists $U\in\Gamma$ such that $U\circ U\subseteq V$.

(iii) $\bigcap_{V\in\Gamma}V=\Delta$.

(iv) $\bigcup_{V\in\Gamma}V=X\times X$.

Definition 2.3. $[10]$. *Let (X,Γ) be a semi-linear uniform space, for $(x,y)\in X\times X$, let $\Gamma_{(x,y)}=\{V\in\Gamma:(x,y)\in V\}$. Then, the set valued map ρ on $X\times X$ is defined by $\rho(x,y)=\bigcap\{V:V\in\Gamma_{(x,y)}\}$.*

Clearly from the above definition for all $(x,y) \in X \times X$, we have $\rho(x,y) = \rho(y,x)$ and $\Delta \subseteq \rho(x,y)$.

In [12] Tallafha A. defined anew set valued map δ which is useful in the study of the theory of semi-linear uniform spaces. For more properties of semi-linear uniform spaces, we refer the readers to [2], [10], [11], [12], [13] and [14].

Definition 2.4. [12]. *Let (X,Γ) be a semi-linear uniform space. Then, the set valued map δ on $X \times X$ is defined by,*

$$\delta(x,y) = \left\{ \begin{array}{ll} \bigcup\{V : V \in \Gamma^c_{(x,y)}\} & if\ x \neq y \\ \Delta & if\ x = y \end{array} \right\},$$

where $\Gamma^c_{(x,y)} = \Gamma \backslash \Gamma_{(x,y)} = \{V \in \Gamma : (x,y) \notin V\}$

In 1928, K. Menger defined convex metric space, then Khalil R. define M- space. In [9] Rawashdeh, A. and Tallafha, A. showed that convex metric space and M- space are equivalent except uniqueness, and used these results to give mor properties of semi-linear uniform spaces.

Definition 2.5. [10]. *Let (X,d) be a metric space. For $x \in X, r > 0$, let $B[x,r] = \{t : d(x,t) \leq r\}$. A metric space (X,d) is convex, if for all $x,y \in X$, $B[x,r_1] \cap B[y,r_2] \neq \phi$ whenever $r_1 + r_2 \geq d(x,y)$.*

Definition 2.6. [9]. *A metric space (X,d) is M-space, if for all $(x,y) \in X \times X$, and $\lambda = d(x,y)$, if $\alpha \in [0,\lambda]$, there exists a unique $z_\alpha \in X$ such $B[x,\alpha] \cap B[y,\lambda - \alpha] = \{z_\alpha\}$.*

In Definition 2.5, if r_1 or $r_2 = 0$, then $B[x,r_1] \cap B[y,r_2] \neq \phi$. Therefore we have the following.

Proposition 2.7. [9].*1- If a metric space (X,d) is convex and $r_1, r_2 \geq 0$, such that $d(x,y) \leq r_1 + r_2$ then $B[x,r_1] \cap B[y,r_2] \neq \phi$.*

2- If a metric space (X,d) is M-space, then for all $(x,y) \in X \times X$ and $\alpha \in [0, d(x,y)]$ there exists a unique $z_\alpha \in X$ such that $d(x,z_\alpha) = \alpha$ and $d(y,z_\alpha) = d(x,y) - \alpha$.

Proposition 2.8. [9].*Convex and M-spaces are equivalent except uniqueness.*

3. More properties of semi-linear uniform spaces

Alhihi, S. in [1] gave more properties of semi-linear uniform spaces using the set valued maps ρ and δ.

Definition 3.1. [1]. *For $n\in\mathbb{N}$ and $A\in\Lambda\subseteq D_X$. Define $\frac{1}{n}A$ by, $\frac{1}{n}A=\bigcup\limits_{U\in\Gamma}\{U:nU\subseteq A\}$.*

Clearly $\frac{1}{n}\Delta=\Delta$ and $\frac{1}{n}A\in D_X$ for all $A\in\Lambda$. But $\frac{1}{n}A$ need not be an element of Γ, even if $A\in\Gamma$. But we have.

Theorem 3.2. [1]. *Let $A\in\Lambda$, and σ a sub collection of Λ. For $n\in\mathbb{N}$, we have.*

(i) $n\left(\frac{1}{n}A\right)\subseteq A$

(ii) If $B\in\Lambda$ satisfies $nB\subseteq A$, then $B\subseteq\frac{1}{n}A$.

(iii) $\frac{1}{n+1}A\subseteq\frac{1}{n}A$

(iv) $\frac{1}{n}A\subseteq A$

(v) $\frac{1}{n}\bigcap\limits_{A\in\sigma}A=\bigcap\limits_{A\in\sigma}\frac{1}{n}A$

(vi) $\bigcup\limits_{A\in\sigma}\frac{1}{n}A\subseteq\frac{1}{n}\bigcup\limits_{A\in\sigma}A$

Let $n\in\mathbb{N}$ and $A_\alpha\in D_X$, $\alpha\in\Lambda$, replacing A by $\bigcup\limits_{\alpha\in\Lambda}A_\alpha$,or by $\bigcap\limits_{\alpha\in\Lambda}A_\alpha$, we have.

Corollary 3.3. [1]. *For $x,y\in X$ where (X,Γ) is a semi-linear uniform spaces, we have,*

(i) $\frac{1}{n}\ \delta(x,y)=\begin{cases}\bigcup\limits_{V\in\Gamma^c_{(x,y)}}\frac{1}{n}V & if\ x\neq y\\ \Delta & if\ x=y\end{cases}$

(ii) $\frac{1}{n}\rho(x,y)=\frac{1}{n}\bigcap\limits_{V\in\Gamma_{(x,y)}}V\subseteq\bigcap\limits_{V\in\Gamma_{(x,y)}}\frac{1}{n}V$

Also in [1] Alhihi gave the following propositions and definitions.

Proposition 3.4. [1]. *Let x,y be any two points in semi-linear uniform spaces (X,Γ), then.*

1- $n\left(\frac{1}{n}\rho(x,y)\right)\subseteq\rho(x,y)$.

2- $n\left(\frac{1}{n}\delta(x,y)\right)\subseteq\delta(x,y)\subseteq\frac{1}{n}\left(n\delta(x,y)\right)$.

Definition 3.5. [1]*.Let $x.y$ be any points in semi-linear uniform spaces (X,Γ). For $r\in\mathbb{Q}^+$, $r\delta(x,y)=n\left(\frac{1}{m}\delta(x,y)\right)$, where $r=\frac{n}{m}$, $n,m\in\mathbb{N}$ and the greatest common devisor of n,m is1.*

Rawashdeh, A. and Tallafha, A. in [9], proved the following important property of semi-linear uniform space induced by a convex metric space (X,d).

Lemma 3.6. [9]*.Let (X,Γ_d) be a semi-linear uniform space induced by a convex metric space (X,d). Then $V_{\varepsilon_1}\circ V_{\varepsilon_2}=V_{\varepsilon_1+\varepsilon_2}$, where $V_\varepsilon=\{(s,t)\in X\times X:d(s,t)<\varepsilon\}$.*

Lemma 3.7. [9] .*Let* (X,Γ_d) *be a semi-linear uniform space induced by a convex metric space* (X,d)*. Then*

1- $\rho(x,y)=\{(s,t)\in X\times X : d(s,t)\leq d(x,y)\}$.

2- $\delta(x,y)=\{(s,t)\in X\times X : d(s,t)< d(x,y)\}$.

3- $n\delta(x,y)=\{(s,t)\in X\times X : d(s,t)< nd(x,y)\}$.

4- $n\rho(x,y)=\{(s,t)\in X\times X : d(s,t)\leq nd(x,y)\}$.

5- $\frac{1}{m}\delta(x,y)=\{(s,t)\in X\times X : md(s,t)< d(x,y)\}$.

Theorem 3.8. [9] . *Let* $(X,\Gamma_X),(Y,\Gamma_Y)$ *be two semi-linear uniform spaces induced by the metric spaces* (X,d_X) *and* (Y,d_Y) *respectively and* (x_n) *be a sequence in* X*. Then.*

1- $f:(X,d_X)\to(Y,d_Y)$ *is continuous if and only if* $f:(X,\Gamma_X)\longrightarrow(Y,\Gamma_Y)$ *is continuous.*

2- $f:(X,d_X)\to(Y,d_Y)$ *is uniformly continuous if and only if* $f:(X,\Gamma_X)\longrightarrow(Y,\Gamma_Y)$ *is uniformly continuous.*

3- (x_n) *is converge in* (X,d) *if and only* (x_n) *is converge in* (X,Γ).

4- (x_n) *is Caushy in* (X,d) *if and only* (x_n) *is Caushy in* (X,Γ).

5- $f:(X,d_X)\longrightarrow(X,d_X)$ *is a contraction if and only if if:* $f:(X,\Gamma_X)\longrightarrow(X,\Gamma_X)$ *is a contraction.*

In [14] Tallafha A. and Alhihi, S. asked the following question. If (X,Γ) is a complete semi-linear space and $f:(X,\Gamma)\longrightarrow(X,\Gamma)$ is a contraction. Does f has a unique fixed point.

In [9], Rawashdeh, A. and Tallafha, A. answered this question negatively, they gave the following example.

Example 3.9. . *Let* $\Gamma=\{V_\varepsilon,\varepsilon>0\}, V_\varepsilon=\{(x,y):x^2+y^2<\varepsilon\}\cup\{\Delta\}$*.Then* (X,Γ) *is a semi-linear uniform space which is not metrizable. Now* (X,Γ) *is complete since if* x_n *is Cushy, then* $x_n\to 0$ *or* x_n *has a constant tail, so* x_n *converge. Let* $f(x)=x\ sin(x)$*, so* $f(x)$ *is a contraction and has* $\left\{\frac{\pi}{2}+2n\pi : n=0,1,2\right\}\cup\{0\}$ *as fixed points.*

4. Main results

In [13], Tallafha gave an example of a space which is semi-linear uniform spaces, but not metrizable. Till now, to define a function f that satisfies Lipschitz condition, or to be a contraction, it should be defined on a metric space to another metric space. The main idea of this paper is to define a new type of semi-linear uniform space, on which we cane define contraction functions.

In [12], Tallafha, A., in [1], Alhihi, S. and in [9], Rawashdeh, A. and Tallafha, A., gave the definitions of contraction, r-contraction and strong contraction from a semi-linear uniform space to it self, some of the definitions given are using the set valued function $\rho(x,y)$ and some are using the set valued function $\delta(x,y)$.

In this article we shall define anew type of contractions called $m-$contraction and gave an example of $m-$contraction not a contraction.

we start this section by the following definitions.

Definition 4.1. [13]. *Let $f:(X,\Gamma)\longrightarrow(X,\Gamma)$, then f satisfied Lipschitz condition if there exist $m,n\in\mathbb{N}$ such that $m\delta(f(x),f(y))\subseteq n\delta(x,y)$. Moreover if $m>n$, then we call f a contraction.*

In [1] Alhihi, S. gave a new definition of Lipschitz condition and contraction**.** Called $r-$Lipschitz condition and $r-$contraction.

Definition 4.2. [1]. *Let $f:(X,\Gamma)\longrightarrow(X,\Gamma)$, then f satisfied $r-$Lipschitz condition if there exist $r\in\mathbb{Q}^{+}$ such that $\delta(f(x),f(y))\subseteq r\delta(x,y)$. Moreover if $r<1$ then we call f a $r-$contraction.*

Also Alhihi, S. asked the following**.**

Question. Let (X,Γ) be semi-linear uniform spaces and $f:(X,\Gamma)\longrightarrow(X,\Gamma)$. what is the relation between Lipschitz condition and $r-$Lipschitz condition. contraction. and $r-$ contraction.

In [9] Rawashdeh, A. and Tallafha, A. defined the following.

Definition 4.3. [9]. *A semi-linear uniform space (X,Γ) called a strong semi-linear uniform space if Γ satisfies the following additional condition for all $V\in\Gamma$,we have $\bigcup_{n=1}^{\infty}nV=X\times X$.*

Definition 4.4. [9]. *Let (X,Γ) be a semi-linear uniform space. $f:(X,\Gamma)\longrightarrow(X,\Gamma)$ is strong contraction if there exists $m,n\in N$ such that $m>n$ and $m\,\rho(f(x),f(y))\subseteq n\rho(x,y)$ and $(m+1)$ $\rho(f(x),f(y))\nsubseteq n\rho(x,y)$.*

We know that every metric spaces induces a semi-linear uniform space, wile to induces a strong semi-linear uniform space, we need a metric space (X,d) with mid point property.

Lemma 4.5. [9]. *Every metric space (X,d) with mid point property, induces a strong semi-linear uniform space (X,Γ), where $\Gamma = \{V_\varepsilon : \varepsilon > 0\}$, $V_\varepsilon = \{(x,y) \in X \times X : d(x,y) < \varepsilon\}$.*

Now we shall define a new type of contraction called, $m-$contraction.

Definition 4.6. *Let (X,Γ) be a semi-linear uniform space. $f : (X,\Gamma) \longrightarrow (X,\Gamma)$ is called $m-$contraction if there exists $m \in \{2,3,4,...\}$ such that $m\delta(f(x), f(y)) \subseteq \delta(x,y)$, for all $x,y \in X$.*

Clearly $m-$contraction is a contraction but the covers need not be true.

Example 4.7. *Let $X = \mathbb{R}$, for $t \in (0,\infty)$, let $V_t = \{(x,y) : y-t < x < y+t, y \in \mathbb{R}\}$, and*

$\Gamma = \{Vt : 0 < t < \infty\}$. Then $(\mathbb{R},\Gamma)$ is a semi-linear uniform space. Let $f : (X,\Gamma) \longrightarrow (X,\Gamma)$ be defined by $f(x) = \frac{3}{4}x$,by Lemma 3.7 and similar argument,we have the following

$$.4\delta(f(x), f(y)) = \left\{(s,t) : d(s,t) < 4\left|\tfrac{3}{4}x - \tfrac{3}{4}y\right|\right\}$$

$$= \{(s,t) : d(s,t) < 3|x-y|\} = 3\delta(x,y), \text{ so } f \text{ is a contraction.}$$

Now using Lemma 3.7, we have, $\delta(f(x), f(y)) = \delta(\frac{3}{4}x, \frac{3}{4}y) = \left\{(s,t) : d(s,t) < \left|\frac{3}{4}x - \frac{3}{4}y\right|\right\}$ and $\delta(x,y) = \{(s,t) : d(s,t) < |x-y|\}$. So $2\delta(f(x), f(y))$ not a subset of $\delta(x,y)$, hence $m\delta(f(x), f(y))$ not a subset of $\delta(x,y)$ for all $m \in \mathbb{N}$.

The following lemmas are used in the proof the main theorem.

Lemma 4.8. *Let (X,Γ) be a semi-linear uniform space induced by a convex metric space (X,d), then $\delta(x,y) = \bigcup_{\varepsilon < d(x,y)} V_\varepsilon$.*

Proof. Let (X,Γ) be a semi-linear uniform space induced by a convex metric space (X,d), and $(s,t) \in \delta(x,y)$, then there exist $V \in \Gamma^c_{(x,y)}$, such that $(s,t) \in V$, since $V = V_\varepsilon$ for some $\varepsilon > 0$, and $V \in \Gamma^c_{(x,y)}$, implies $\varepsilon < d(x,y)$, therefore $(s,t) \in V_\varepsilon$ and $\varepsilon < d(x,y)$.

Conversely, if $(s,t) \in \bigcup_{\varepsilon < d(x,y)} V_\varepsilon$,then $(s,t) \in V_\varepsilon$ for some $\varepsilon < d(x,y)$, hence $d(s,t) < e < d(x,y)$. By Lemma 3.7, the result follows. $\square$

Lemma 4.9. *Let (X,Γ) be a semi-linear uniform space induced by a convex metric space or M-space, (X,d), then $\delta(x,z)\circ\delta(z,y)=\{(s,t):d(s,t)<d(x,z)+d(x,y)\}$*

Proof. Let (X,Γ) be a semi-linear uniform space induced by a convex metric space (X,d). By Lemma 4.9, 2.8 and since (X,d) is convex we have,

$$\begin{aligned}\delta(x,z)\circ\delta(z,y) &= \left(\bigcup_{V\in\Gamma^c_{(x,z)}} V\right)\circ\left(\bigcup_{U\in\Gamma^c_{(x,z)}} U\right)=\bigcup_{V\in\Gamma^c_{(x,z)}}\bigcup_{U\in\Gamma^c_{(x,z)}} V\circ U\\ &=\bigcup_{\varepsilon<d(x,z)}\bigcup_{t<d(z,y)} V_\varepsilon\circ U_t=\bigcup_{t+\varepsilon<d(x,z)+d(z,y)} U_{\varepsilon+t}\\ &=\bigcup_{r<d(x,z)+d(z,y)} U_r=\{(s,t):d(s,t)<d(x,z)+d(z,y)\}.\end{aligned}$$

□

Now we shall define a modified semi-linear uniform space (X,Γ).

Definition 4.10. *We shall call a semi-linear uniform space (X,Γ) a modified semi-linear uniform space, if Γ satisfies the following additional conditions.*

1- For all $V\in\Gamma$,we have $\bigcup_{n=1}^{\infty} nV=X\times X$.

2- $\delta(x,y)\subseteq\delta(x,z)\circ\delta(z,y)$ for all x,y and $z\in X$.

Clearly Example 3.9 is a semi-linear uniform space which is not a modified semi-linear uniform space.

In $[1]$, Alhihi define $\frac{m}{n}A$ by $\frac{m}{n}A=m\left(\frac{1}{n}A\right)$, where m,n are relatively prime, that is, the greatest common devisor of m,n is 1. If m,n are not relatively prime then $\frac{m}{n}=\frac{m_1}{n_1}$, where m_1,n_1 are relatively prime, so we have the following definition.

Definition 4.11. *Let $m,n\in\mathbb{N}$, be such that the greater common deviser $gcd(m,n)=k$. Let $m_1=\frac{m}{k}$ and $n_1=\frac{n}{k}$ then we define $\frac{m}{n}A$ by $\frac{m_1}{n_1}A$.*

Now we shall give the main theorem, which shoes that every convex metric space induced a modified semi-linear uniform space.

Theorem 4.12. *Let (X,Γ) be a semi-linear uniform space induced by a convex metric space (X,d), then (X,Γ) a modified semi-linear uniform space.*

Proof. Let (X,Γ) be a semi-linear uniform space induced by a convex metric space (X,d), then $\Gamma=\{U_\varepsilon:\varepsilon>0\}$.

1- For all $V \in \Gamma, V = V_{\varepsilon}$ for some $\varepsilon > 0$,therefor $\bigcup_{n=1}^{\infty} nV = \bigcup_{n=1}^{\infty} nV_{\varepsilon} = \bigcup_{n=1}^{\infty} V_{n\varepsilon} = X \times X$.

2- Let x, y and $z \in X$. By Lemma 3.7 and Lemma 4.9, we have

$$\begin{aligned}\delta(x,y) &= \{(s,t) : d(s,t) < d(x,y)\} \\ &\subseteq \{(s,t) : d(s,t) < d(x,z) + d(z,y)\} \\ &= \delta(x,z) \circ \delta(z,y) \text{ for all } x, y \text{ and } z \in X.\end{aligned}$$

□

Now we shall show that 2−contraction is $\frac{1}{2}$−contraction.

Proposition 4.13. *Let (X,Γ) be a modified semi-linear uniform space. If $f : (X,\Gamma) \longrightarrow (X,\Gamma)$ is m−contraction $m \in \{2,3,...\}$, then f is $\frac{1}{m}$−contraction $\left(\frac{1}{m} < 1\right)$.*

Proof. Let (X,Γ) be a modified semi-linear uniform space, and $f : (X,\Gamma) \longrightarrow (X,\Gamma)$ is m−contraction, $m \in \{2,3,...\}$. Then for all $x, y \in X$, we have $m\delta(f(x), f(y)) \subseteq \delta(x,y)$. Now we want to show $\delta(f(x), f(y)) \subseteq \frac{1}{m}\ \delta(x,y)$. Let $(s,t) \in \delta(f(x), f(y))$, then, there exist $U \in \Gamma^{c}_{(f(x),f(y))}$ such that $(s,t) \in U$. Since $mU \subseteq m\delta(f(x), f(y)) \subseteq \delta(x,y)$, then $U \subseteq \frac{1}{m}\ \delta(x,y)$, hence $(s,t) \in \frac{1}{m}\ \delta(x,y)$. □

The following Proposition is a beautiful consequences of Definition 4.11 .

Proposition 4.14. *Let (X,Γ) be semi-linear uniform space, then $\frac{1}{m}\delta(x,y) \circ \frac{1}{n}\ \delta(x,y) = \left(\frac{1}{m} + \frac{1}{n}\right)\delta(x,y)$, for all $x, y \in X$.*

Proof. Let (X,Γ) be a semi-linear uniform space and $x, y \in X$. By Definition 4.11, $\frac{1}{m}\delta(x,y) \circ \frac{1}{n}\ \delta(x,y) = \frac{n}{mn}\delta(x,y) \circ \frac{m}{nm}\ \delta(x,y) = n\left(\frac{1}{mn}\delta(x,y)\right) \circ m\left(\frac{1}{mn}\delta(x,y)\right) = (n+m)\left(\frac{1}{mn}\delta(x,y)\right) = \frac{n+m}{mn}\delta(x,y) = \left(\frac{1}{m} + \frac{1}{n}\right)\delta(x,y)$. □

Now we shall end this article by the following natural questions.

Question 1. Is there a modified semi-linear uniform space which is not metrizable?

Question 2. Let (X,Γ) be a complete modified semi-linear uniform space, and $f : (X,\Gamma) \longrightarrow (X,\Gamma)$ be a contraction. Is f has unique fixed?

Conflict of Interests

The authors declare that there is no conflict of interests.

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