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LAGUERRE EQUATION AND FRACTIONAL LAGUERRE POLYNOMIALS

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Abstract. In this paper we introduce the concept of regular singular fractional point and use the technique of fractional power series to solve the fractional Laguerre equation. Then we get the factional Laguerre polynomials.

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1. Introduction

Fractional differential equations proved to be very important in applied sciences. That is why there is so much work on fractional calculus and fractional differential equations. The subject of fractional derivative is as old as calculus. In 1695, L'Hopital asked if the expression $\frac{d^{0.5}}{dx^{0.5}}f$ has any meaning. Since then, many researchers have been trying to generalize the concept of the usual derivative to fractional derivatives. These days, many definitions for the fractional derivative are available. Most of these definitions use an integral form. The most popular definitions are:

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(i) Riemann - Liouville Definition: If n is a positive integer and $\alpha \in [n-1, n)$, the α^{th} derivative of f is given by

$$T_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}} dx.$$

(ii) Caputo Definition. For $\alpha \in [n-1, n)$, the α derivative of f is

$$T_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx.$$

Now, all definitions are attempted to satisfy the usual properties of the standard derivative. The only property inherited by all definitions of fractional derivative is the linearity property. However, the following are the setbacks of one definition or another:

(i) The Riemann-Liouville derivative does not satisfy $T_a^\alpha(1) = 0$ ($T_a^\alpha(1) = 0$ for the Caputo derivative), if α is not a natural number.

(ii) All fractional derivatives do not satisfy the known product rule:

$$T_a^\alpha(fg) = fT_a^\alpha(g) + gT_a^\alpha(f).$$

(iii) All fractional derivatives do not satisfy the known quotient rule:

$$T_a^\alpha(f/g) = \frac{gT_a^\alpha(f) - fT_a^\alpha(g)}{g^2}.$$

(iv) All fractional derivatives do not satisfy the chain rule:

$$T_a^\alpha(f \circ g)(t) = f^{(\alpha)}\left(g(t)\right) g^{(\alpha)}(t).$$

(v) All fractional derivatives do not satisfy: $T^\alpha T^\beta f = T^{\alpha+\beta} f$ in general

(vi) Caputo definition assumes that the function f is differentiable.
(v) $T_1(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

In [5], a new definition called $\alpha-$conformable fractional derivative was introduced:

Let $\alpha \in (0,1)$, and $f : E \subseteq (0,\infty) \to R$. For $x \in E$ let: $D^{\alpha} f(x) = \lim\limits_{\varepsilon \to 0} \frac{f(x+\varepsilon x^{1-\alpha})-f(x)}{\varepsilon}$. If the limit exists then it is called the $\alpha-$conformable fractional derivative of f at x.

For $x = 0$, $D^{\alpha} f(0) = \lim\limits_{x \to 0} D^{\alpha} f(0)$ if such limit exists.

The new definition satisfies:

$1. D^{\alpha}(af+bg) = aD^{\alpha}(f)+bD^{\alpha}(g)$, for all $a,b \in \mathbb{R}$.

$2. D^{\alpha}D_{\alpha}(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

Further, for $\alpha \in (0,1]$ and f,g be $\alpha-$differentiable at a point t, with $g(t) \neq 0$. Then

3. $T_{\alpha}D^{\alpha}(fg) = fD^{\alpha}(g)+gD^{\alpha}(f)$.

4. $D^{\alpha}(\frac{f}{g}) = \frac{gD^{\alpha}(f)-fD^{\alpha}(g)}{g^2}$

We list here the fractional derivatives of certain functions,

(1) $1. D^{\alpha}(t^p) = p\, t^{p-\alpha}$.
(2) $D^{\alpha}(\sin \frac{1}{\alpha}t^{\alpha}) = \cos \frac{1}{\alpha}t^{\alpha}$.
(3) $D^{\alpha}(\cos \frac{1}{\alpha}t^{\alpha}) = -\sin \frac{1}{\alpha}t^{\alpha}$.
(4) $D^{\alpha}(e^{\frac{1}{\alpha}t^{\alpha}}) = e^{\frac{1}{\alpha}t^{\alpha}}$.

On letting $\alpha = 1$ in these derivatives, we get the corresponding ordinary derivatives.

One should notice that a function could be $\alpha-$conformable differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$. Then $D^{\frac{1}{2}}\,(f)(t) = 1$. Hence $D^{\frac{1}{2}}(f)(0) = 1$. But $D^{1}(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives.

For more on fractional calculus and its applications we refer to [1] to [11]

2. Fractional Laguerre Equation

The equation

$$xy'' + (1-x)y' + ny = 0................(1)$$

is called Laguerre differential equation. It is a well known and important equation that appears in the quantum mechanical description of the hydrogen atom. The point $x = 0$ is a regular singular point for the equation. Power series technique is a method to solve such equation. In this paper we are interested in some fractional form of the Laguerre equation. More precisely we will study the equation:

$$x^{\alpha}D^{\alpha}D^{\alpha}y+(\alpha-x^{\alpha})D^{\alpha}y+k\alpha y....................(2)$$

where $\alpha\in(0,1]$, and k is a natural number.

Often we will use $D^{n\alpha}$ to denote $D^{\alpha}.....D^{\alpha}$ n-times

Definition 2.1. The point $x=0$ is called an $\alpha-$regular singular point for the equation $D^{\alpha}D^{\alpha}y+P(x)D^{\alpha}y+Q(x)y=0$ if

$\lim\limits_{x\longrightarrow 0^{+}}\frac{P(x)}{x^{\alpha}}$ and $\lim\limits_{x\longrightarrow 0^{+}}\frac{Q(x)}{x^{2\alpha}}$ both exist.

Clearly, $x=0$ is an $\alpha-$regular singular point for equation (2).

Definition 2.2.[6]. A series $\sum\limits_{n=0}^{\infty}a_{n}x^{n\alpha}$ is called a fractional power series.

If $D^{n\alpha}f$ exits for all n in some interval $[0,\lambda]$, then one can write f in the form of a fractional power series.

Now, let us start solving equation (2)

Procedure:

$$x^{\alpha}D^{\alpha}D^{\alpha}y+(\alpha-x^{\alpha})D^{\alpha}y+\alpha py=0.............(*)$$

Put

$$y=\sum a_{n}x^{\alpha n}\ ,\qquad a_{0}\neq 0$$

Then

$$D^{\alpha}y=\sum_{1}n\alpha a_{n}x^{\alpha n-\alpha}\ ,,D^{\alpha}D^{\alpha}y=\sum_{2}n\alpha(n\alpha-\alpha)a_{n}x^{\alpha n-2\alpha}$$

Substitute to get

$$\sum_{2}n\alpha(n\alpha-\alpha)x^{\alpha n-\alpha}a_{n}+\sum_{1}n\alpha^{2}a_{n}x^{\alpha n-\alpha}-\sum_{1}n\alpha\, x^{\alpha n}a_{n}+\alpha p\sum_{0}a_{n}x^{\alpha n}=0$$

$$\sum_{1}(n+1)n\alpha^{2}a_{n+1}x^{\alpha n}+\sum_{0}(n+1)\alpha^{2}a_{n+1}x^{\alpha n}-\sum_{1}n\alpha a_{n}x^{\alpha n}+\sum_{0}\alpha p\, a_{n}x^{\alpha n}=0$$

$$\alpha^2 a_1 + \alpha p a_0 + \sum_1 \left\{ \left[(n+1)n\alpha^2 + (n+1)\alpha^2 \right] a_{n+1} + (\alpha p - n\alpha) a_n \right\} x^{n+\alpha} = 0$$

Hence

$$a_1 = \frac{-p}{\alpha} a_0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$((n+1)n\alpha^2 + (n+1)\alpha^2) a_{n+1} + (\alpha p - n\alpha) a_n = 0 \ldots\ldots\ldots (4)$$

From equation (4) we get

$$a_{n+1} = \frac{-\alpha(n-p)}{\alpha^2(n+1)(n+1)} a_n \quad \text{So} \quad a_{n+1} = \frac{n-p}{\alpha(n+1)^2} \ldots\ldots\ldots\ldots (5)$$

Hence if $n = 1$ we get from 3 and 5 we get

$$a_2 = \frac{(1-p)}{\alpha(2)(2)} a_1 = \frac{p\,(p-1)}{\alpha^2(2!)^2} a_0$$

Similarly,

$$a_3 = \frac{-p\,(p-1)(p-2)}{\alpha^3(3!)^2} a_0$$

and for general $n = 3$ we have

$$a_r = (-1)^r \; \frac{p\,(p-1)(p-2)\ldots\ldots\ldots\ldots(p-r+1)}{\alpha^r (r\,!)^2} a_0 \ldots\ldots\ldots\; r \leq p, \; and \quad a_{p+1} = 0$$

Consequently

$$y = a_0 \sum (-1)^r \; \frac{p\,(p-1)(p-2)\ldots\ldots\ldots\ldots(p-r+1)}{\alpha^r (r\,!)^2} \; x^{\alpha r}$$

But

$$\binom{p}{r} = \frac{p!}{(p-r)!r!} \qquad\qquad r \leq p$$

so

$$y = a_0 \sum_{r=0} (-1)^r \; \frac{p\,(p-1)(p-2).........(p-r\;+1)(p-r)(p-r-1).......(3)(2)(1)}{\alpha^r (p-r)(p-r-1).......(3)(2)(1)(r\,!)^2} x^{\alpha r}$$

$$y = a_0 \sum (-1)^r \; \frac{p!}{\alpha^r (p-r)!r!r!} x^{\alpha r}$$

$$y = a_0 \sum_{r=0}^{p} (-1)^r \; \binom{p}{r} \frac{x^{\alpha r}}{\alpha^r r!}$$

By taking $a_0 = 1$

$$y(x) = \sum_{r=0}^{p} (-1)^r \; \frac{p!}{\alpha^r (p-r)!(r!)^2} \, x^{\alpha r}..............(6)$$

3. The Laguerre Polynomials And Generating Function

From equation (6) one can see that for each value of p we get a form of a solution which is a fractional polynomial to be called Laguerre fractional polynomial. The following are the Laguerre polynomials:

$$L_0(x) = 1$$

$$L_1(x) = \sum_{r=0}^{1} (-1)^r \; \frac{1!}{\alpha^r (1-r)!(r!)^2} \, x^{\alpha r} = (1 - \frac{x^\alpha}{\alpha})$$

$$L_2(x) = \sum_{r=0}^{2} (-1)^r \; \frac{2!}{\alpha^r (2-r)!(r!)^2} \, x^{\alpha r} = \left[1 - \frac{2}{\alpha} x^\alpha + \frac{x^{2\alpha}}{2!\alpha^2}\right] = \frac{1}{2!} \left[2 - 4\frac{x^\alpha}{\alpha} + (\frac{x^\alpha}{\alpha})^2\right]$$

And similarly one can find the Leguerre polynomials of all orders.

The function $F(x,t)$ is called generating function for the Laguerre polynomial if $F(x,t) = \sum_{p=0}^{\infty} L_p(\frac{x^\alpha}{\alpha}) t^p$.

Theorem3.1. The generating function of the Laguerre Polynomials is: $\frac{1}{1-t} Exp\left(\left(-\frac{x^\alpha}{\alpha} t\right)(1-t)\right)$

Proof.

$$\begin{aligned}\frac{1}{1-t}\sum_{r=0}^{\infty}\frac{\left(-\frac{x^{\alpha}}{\alpha}t\right)^{r}}{r\,!(1-t)^{r}} &= \frac{1}{1-t}\sum_{r=0}^{\infty}(-1)^{r}\,\frac{\left(\frac{x^{\alpha}}{\alpha}\right)^{r}t^{\,r}}{r\,!(1-t)^{r}}\\ &= \sum_{r=0}^{\infty}(-1)^{r}\,\frac{\left(\frac{x^{\alpha}}{\alpha}\right)^{r}t^{\,r}}{r\,!(1-t)^{r+1}}\end{aligned}$$

But

$$\frac{1}{1-t}=\sum_{0}^{\infty}t^{\,s}$$

$$\begin{aligned}\frac{1}{(1-t)^{2}} &= \sum_{1}s\,t^{\,s-1}\\ &= \sum_{0}(s+1)\,t^{\,s}\end{aligned}$$

$$\begin{aligned}\frac{1}{(1-t)^{3}} &= \sum_{2}s\,(s-1)t^{\,s-2}\\ &= \sum_{0}(s+2)(s+1)\,t^{\,s}\end{aligned}$$

$$\frac{1}{(1-t)^{r+1}}=\sum_{0}\frac{(s+r)!}{s\,!r\,!}t^{\,s}$$

Hence

$$\begin{aligned}\frac{1}{1-t}\sum_{r=0}^{\infty}\frac{\left(-\frac{x^{\alpha}}{\alpha}t\right)^{r}}{r\,!(1-t)^{r}} &= \sum_{r=0}^{\infty}(-1)^{r}\,\frac{\left(\frac{x^{\alpha}}{\alpha}\right)^{r}t^{\,r}}{r\,!}\,\sum_{s=0}^{\infty}\frac{(s+r)!}{s\,!r\,!}t^{\,s}\\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^{r}\,\frac{\left(\frac{x^{\alpha}}{\alpha}\right)^{r}(s+r)!}{(r\,!)^{2}s\,!}\,t^{\,r+s}\end{aligned}$$

Put $r+s=p$. We get

$$\begin{aligned} &= \sum_{p=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{r}\,\frac{p!}{(r\,!)^{2}(p-r)\,!}\left(\frac{x^{\alpha}}{\alpha}\right)^{r}\,t^{\,p}\\ &= \sum_{p=0}^{\infty}L_{P}\,\left(\frac{x^{\alpha}}{\alpha}\right)\,t^{\,p}\end{aligned}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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