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A CASE OF NON-IDENTITY DIFFERENCE ORDER PRESERVING TRANSFORMATION SEMIGROUP

OMELEBELE, JUDE A.* AND ASIBONG-IBE, U.I.

Department of Mathematics and Statistics, University of Port Harcourt, Rivers State, Nigeria

Abstract. We investigate the order preserving transformation semigroup $\mathcal{O}_n$ for non-identity difference order preserving transformation $\overline{ID\mathcal{O}_n}$ and obtain a subsemigroup $\bar{S}$. The properties of both $\bar{S}$ and $\overline{ID\mathcal{O}_n}$ are also obtained. We further obtain the work done and average work done by $\bar{S}$.

Keywords: subsemigroup $\bar{S}$; (non)-identity difference order preserving transformation; work done and average work done by $\bar{S}$.

2010 AMS Subject Classification: 20M20.

1.0 INTRODUCTION

The subject "identity difference transformation semigroup" originated from the work of Adeniji and Makanjuola, 2012 [1], where they obtained some combinatorial results on the order of the subsemigroups of the transformation semigroup $|IDPT_n|, |IDT_n|, |IDO_n|, |IDI_n|, |IDPO_n|$ and $|IDPOI_n|$. Order of Nilpotents $|N_n|$, idempotents, $|E(S)|$, and fix of the subsemigroups of the transformation semigroups were obtained.

Adeniji et al [2] amongst others extended their earlier studies to the Identity difference order-preserving transformation semigroups and obtained the cardinalities of fixed points, nilpotent and chain decompositions of the subsemigroups $OIDT_n$, $OIDI_n$, and $OIDP_n$.

*Corresponding author
E-mail address: judeomelebele@gmail.com
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In [3] it was shown that IDT_n is a subsemigroup of the full transformation semigroup T_n. Also the paper examined congruence property of the Green's relations $\mathcal{L}$ and $\mathcal{R}$ on DT_n.

In effort to study the properties of non-identity difference order-preserving transformation semigroup $\overline{ID\mathcal{O}_n}$ it is found that the set does not generally form a subsemigroup of the transformation, except for a particular case of $n = 3$, $\overline{ID\mathcal{O}_3}$ form a three elements semigroup $\bar{S}$ with some unique properties studied here. The semigroup $\bar{S}$ provides us result which can be seen in a more general form for all $n \geq 3$ as shown in section 4. Section 3 shows that $\overline{ID\mathcal{O}_n}$ cannot form a subsemigroup of the transformation semigroup for all $n \geq 4$.

In further observation on this interesting situation, the concept of work done by transformation semigroup as studied by James East in 2006 [5] was investigated on $\bar{S}$ and some results were obtained as shown in section 5.

To explain further, let us consider some definitions and preliminary studies.

2.0 PRELIMINARY

Some basic definitions

2.1 Let S be a non empty set. S is called identity difference if $(\max(\mathrm{im}) - \min(\mathrm{im})) \leq 1$.

S is called order preserving if $\alpha x \leq \alpha y \;\; \forall\, x, y \in S$.

2.2 Let $\overline{IDO_n} \subseteq S \;\forall\, n \geq 3$ is a non empty set. $\overline{IDO_n}$ is called a non-identity difference if $(\max(\mathrm{im}\alpha) - \min(\mathrm{im}\alpha)) \geq 2$. $\overline{IDO_n}$ is called order preserving transformation if $\alpha x \leq \alpha y \;\; \forall\, x, y \in \overline{IDO_n}$

2.3 Let $\bar{S} \subseteq \overline{IDO_n} \;\forall\, n \geq 3$ be a non empty set. $\bar{S}$ is a semigroup if for any three elements $\alpha, \beta, \gamma \in \bar{S}$. and $(\alpha * \beta) * \gamma = \alpha * (\beta * \gamma) \in \bar{S}$. Where $\mathrm{im}\,\alpha = \{i, i+1, i+2, \; . \; . \; ., n\}$, $\mathrm{im}\beta = \{i, \; . \; . \; ., i\, n\}$ and $\mathrm{im}\gamma = \{i, n, \; . \; . \; ., n\}$ for all $n \geq 3, i = 1$.

3.0 MAIN RESULTS

3.1 Observations on $\overline{IDO_n}$.

In this section, it is shown that the set of a non-identity difference order preserving transformation $\overline{IDO_n}$ are not generally semigroup except for $n = 3$. The elements of $\overline{IDO_3}$ are stated below; if for any three elements $\alpha, \beta, \gamma \in \overline{IDO_3}$ we have $\alpha = \begin{pmatrix} x_1 & x_2 & x_3 \\ 1 & 2 & 3 \end{pmatrix}$, $\beta = \begin{pmatrix} \{x_1 & x_2\} & x_3 \\ 1 & & 3 \end{pmatrix}$ and $\gamma = \begin{pmatrix} x_1 & \{x_2 & x_3\} \\ 1 & 3 & \end{pmatrix}$ and by observation this set of elements form a semigroup called $\overline{IDO_3}$. Meanwhile, a prove of this case where $\overline{IDO_n}$ is not a semigroup is shown in lemma 3 below for $n \geq 4$.

Some properties of non-identity difference order preserving transformation semigroup $\overline{\mathrm{IDO_n}}$.

The $\overline{\mathrm{IDO_n}}$ satisfy the property that

- $(\max(\mathrm{im}) - \min(\mathrm{im})) \geq 2$.
- The elements of $\overline{\mathrm{ID}\mathcal{O}_3}$ form a semigroup and its elements are all idempotent.
- The elements of $\overline{\mathrm{ID}\mathcal{O}_\mathrm{n}}$ $\forall$ n ≥ 4 are not subsemigroups of S.

However, these facts are shown below.

Lemma 1. Let $\mu \in \overline{\mathrm{IDO_n}}$ $\mathrm{Max}\big(\mathrm{im}(\mu) - \mathrm{Min}(\mathrm{im}\mu)\big) \geq 2$.

Proof

Suppose that $\mu \in \overline{\mathrm{IDO_n}}$, if $\mathrm{Max}\big(\mathrm{im}(\mu) - \mathrm{Min}(\mathrm{im}\mu)\big) \geq 2$. Then $(\mathrm{n} - \mathrm{i}) \geq 2$. That is,

If $\mu = \begin{pmatrix} \mathrm{X_1} & \mathrm{X_2} & . & . & . & \mathrm{X_n} \\ \mathrm{i} & \mathrm{i+1} & & & & \mathrm{n} \end{pmatrix}$ $\forall$ $\mathrm{i} = 1, \mathrm{n} \geq 3$ implies that, for $\mathrm{n} = 3$, $\max\big(\mathrm{im}(\mu) - \min(\mathrm{im}\mu)\big) = 3 - 1 = 2$, for $\mathrm{n} = 4$ $\max\big(\mathrm{im}(\mu) - \min(\mathrm{im}\mu)\big) = 4 - 1 = 3$, and so on. If there exist $\rho \in \overline{\mathrm{IDO_3}}$ where $\rho = \begin{pmatrix} \{\mathrm{X_1} & \mathrm{X_2} & \mathrm{X_3}\} & \mathrm{X_n} \\ & \mathrm{i} & & \mathrm{n} \end{pmatrix}$ $\forall$ $\mathrm{i} = 1, \mathrm{n} \geq 3$, then for $\mathrm{n} = 3$, $\max(\mathrm{im}(\rho)) - \min(\mathrm{im}(\rho)) = 3 - 1 = 2$, for $\mathrm{n} = 4$ we have $(4 - 1 = 3)$ and so on. This therefore tells us that $\mathrm{Max}\big(\mathrm{im}(\mu) - \mathrm{Min}(\mathrm{im}\mu)\big) \geq 2$. □

Lemma 2. $\overline{\mathrm{ID}\mathcal{O}_3}$ form a subsemigroup of $\mathcal{O}_\mathrm{n}$ and its elements are all idempotent.

Proof

Consider the elements $\mathrm{p}, \mathrm{q}, \mathrm{r} \in \overline{\mathrm{ID}\mathcal{O}_3}$ and supposed that $\mathrm{imp} = \{\mathrm{i},\ \mathrm{i}+1,\ \mathrm{i}+2\}$, $\mathrm{imq} = \{\mathrm{i},\ \mathrm{i},\ \mathrm{i}+2\}$ and $\mathrm{imr} = \{\mathrm{i}, \mathrm{i}+2,\ \mathrm{i}+2)$ $\forall$ $\mathrm{i} = 1$, if these elements are closed with respect to multiplication and associative then, $\overline{\mathrm{ID}\mathcal{O}_3}$ a semigroup.

$\mathrm{im\ p} = \{\mathrm{i}, \mathrm{i}+1, .\ .\ ., \mathrm{n}\}$, $\mathrm{imq} = \{\mathrm{i},\ .\ .\ ., \mathrm{i}, \mathrm{n}\}$ and $\mathrm{imr} = \{\mathrm{i}, \mathrm{n},\ .\ .\ ., \mathrm{n}\}$ for all $\mathrm{n} \geq 3, \mathrm{i} = 1$.

To show;

For all $\mathrm{p}, \mathrm{q} \in \overline{\mathrm{ID}\mathcal{O}_3}$, if $\mathrm{imp}(\mathrm{X_1}) = \mathrm{i}, \mathrm{imq}(\mathrm{X_1}) = \mathrm{i}, = \mathrm{imp}(\mathrm{X_2}) = \mathrm{i}+1, \mathrm{imq}(\mathrm{X_2}) = \mathrm{i}, \mathrm{imp}(\mathrm{X_3}) = \mathrm{i}+2$, $\mathrm{imq}(\mathrm{X_3}) = \mathrm{i}+2$ then, $\mathrm{p} * \mathrm{q} = \mathrm{p} \in \overline{\mathrm{ID}\mathcal{O}_3}$. Hence $\overline{\mathrm{ID}\mathcal{O}_3}$ is closed with respect to multiplication. Also, for $\mathrm{r} \in \overline{\mathrm{ID}\mathcal{O}_3}$, with $\mathrm{imr}(\mathrm{X_1}) = \mathrm{i}, \mathrm{imr}(\mathrm{X_2}) = \mathrm{i}+2$, $\mathrm{imr}(\mathrm{X_3}) = \mathrm{i}+2$. Then $(\mathrm{p} * \mathrm{q}) * \mathrm{r} = \mathrm{p} * \mathrm{q} * \mathrm{r} = \mathrm{p} * (\mathrm{q} * \mathrm{r}) = \mathrm{p} \in \overline{\mathrm{ID}\mathcal{O}_3}$. Hence, $\overline{\mathrm{ID}\mathcal{O}_3} \subseteq \mathcal{O}_\mathrm{n}$. Furthermore, it is easily seen that p, q and r are idempotent, since $\mathrm{p}^2 = \mathrm{p}$, $\mathrm{q}^2 = \mathrm{q}$ and $\mathrm{r}^2 = \mathrm{r}$. Hence $\overline{\mathrm{ID}\mathcal{O}_3}$ is an idempotent semigroup. □

Lemma 3. et $a, b, c \in \overline{ID\mathcal{O}_n}$. The elements of $\overline{ID\mathcal{O}_n}$ $n \geq 4$ is not a subsemigroup of $\mathcal{O}_n$.

Proof

Suppose that, $a_n, b_n \in \overline{ID\mathcal{O}_n}$ for $n \geq 4$, if $a_n = \begin{pmatrix} \{X_1 & X_2 & . & . & .X_{n-1}\} & X_n \\ & i & & & & n-1 \end{pmatrix}$, $b_n = \begin{pmatrix} \{X_1 & . & . & . & X_{n-1}\} & X_n \\ & & i & & & n \end{pmatrix}$ and $\forall$ $i = 1$. Then $a_n * b_n = y_n \notin \overline{ID\mathcal{O}_n}$ rather $y_n = \begin{pmatrix} \{X_1 & X_2 & X_3 & . & . & . & X_5\} \\ & & i & & & & \end{pmatrix} \in ID\mathcal{O}_n$.

So, the set of elements of $\overline{ID\mathcal{O}_n}$ is not closed under multiplication for $n \geq 4$.

Also, the associativity property is not true since if there exist $c_n \in \overline{IDO_n}$ Such that $c_n = \begin{pmatrix} \{X_1 & X_2 & \cdots & X_{n-1}\} & X_n \\ & & i+1 & & n \end{pmatrix} \forall\ i = 1$ then, $(a_n * b_n) * c_n \neq a_n * (b_n * c_n)$ since, $(a_n * b_n) * c_n = y_n * c_n = z_n \notin \overline{IDO_n}$ but in IDO_n. Hence, the elements of $\overline{IDO_n}$ $\forall\ n \geq 4$ does not form a semigroup.

For the purpose of illustration, consider the elements $a, b, c \in \overline{IDO_4}$ where $a = \begin{pmatrix} \{X_1 & X_2 & X_3\} & X_4 \\ & 1 & & 3 \end{pmatrix}$, $b = \begin{pmatrix} \{X_1 & X_2 & X_3\} & X_4 \\ & 1 & & 4 \end{pmatrix}$, $c = \begin{pmatrix} \{X_1 & X_2 & X_3\} & X_4 \\ & 2 & & 4 \end{pmatrix}$. Then $a * b = z = \begin{pmatrix} \{X_1 & X_2 & X_3 & X_4\} \\ & 1 & & \end{pmatrix} \notin \overline{IDO_4}$, but $z \in IDO_4$. This is not closed with respect to multiplication. Hence $\overline{IDO_4}$ Is not a semigroup.

$(a * b) * c = z * c = y = \begin{pmatrix} \{X_1 & X_2 & X_3 & X_4\} \\ & 2 & & \end{pmatrix}$, but $y \notin \overline{IDO_4}$, instead y is in IDO_4. Hence the operation $*$ is not associative with respect to multiplication. As such, $\overline{IDO_4}$ is not a semigroup.

With $n = 5$; we observe that for all $', b' \in \overline{IDO_5}$ $a' * b' = z' = \begin{pmatrix} \{X_1 & X_2 & X_3 & X_4 & X_5\} \\ & & 1 & & \end{pmatrix} \notin \overline{IDO_5}$, but $z' \in IDO_5$. So the operation $*$ is not closed with respect to multiplication. Also, $\forall\ c' \in \overline{IDO_5}$, $(a' * b') * c' = z' * c' = y' \notin \overline{IDO_5}$ but in IDO_5 that is, $y' = \begin{pmatrix} \{X_1 & X_2 & X_3 & X_4 & X_5\} \\ & & 2 & & \end{pmatrix} \in IDO_5$. □

4.0 SOME PROPERTIES OF THE SUBSEMIGROUP $\bar{S}$.

In this section, the construction of $\bar{S}$ and the respective generalizations are shown.

Since the elements of O_n (where $n = 3$) and IDO_3 are known we proceed by stating the elements of $\overline{IDO_3}$ below,

$$\bar{S} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix} = \overline{IDO_3}.$$

That the table below with $a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}$, and $c = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \end{pmatrix}$ is a semigroup is explanatory.

Table of elements of $\overline{IDO_3}$ as a semigroup.

$*$	a	b	c
a	a	b	c
b	b	b	b
c	c	c	c

$\bar{S} = \{a, b, c\}$ is a semigroup.

The set of $\mathcal{L}$-classes are $\{a\}, \{b, c\}$ and that of $\mathcal{R}$- classes are $\{a\}, \{b\}, \{c\}$of $\bar{S}$.

While for $\mathcal{H}$- Classes;$H_a = \{a\}$, $H_b = \{b\}$, $H_c = \{c\}$ and the $\mathcal{D}$ -classes are, $D_a = \{a\}, D_b = \{b\}, \{b, c\}$, $D_c = \{c\}, \{b, c\}$. Basically, we have only two $\mathcal{D}$-classes since $D_b \cap D_c = \{b, c\}$, so, $D_b = D_c$. Hence D_a and $D_b = D_c$ are the two classes.

For $n \geq 4$ the semigroup $\bar{S}$ has only three elements α, β, γ of which are idempotent and the left and right principal ideals are equal, that is $\bar{S}\alpha = \alpha\bar{S} = \bar{S}$, where $\alpha = \begin{pmatrix} X_1 & X_2 & . & . & . & X_n \\ 1 & 2 & . & . & . & n \end{pmatrix}$, $\beta = \begin{pmatrix} \{X_1 & . & . & . & X_{n-1}\} & X_n \\ & & 1 & & & n \end{pmatrix}$ and $\gamma = \begin{pmatrix} X_1 & \{ X_2 & . & . & . & X_n\} \\ 1 & & & n & & \end{pmatrix}$.

Theorem 4. Let $\{\beta, \gamma\} \in \bar{S}$ then,

a. $\beta \mathcal{L} \gamma \; iff \; \bar{S}\beta = \bar{S}\gamma \;\; and \; |im\beta| = |im\gamma|$

b. $\beta^n \mathcal{L} \gamma^n \; iff \; \bar{S}\beta = \bar{S}\gamma \;\; and \; |im\beta| = |im\gamma|$

c. $\beta \mathcal{R} \gamma \; iff \; \beta\bar{S} = \beta, \; \gamma\bar{S} = \gamma \; and \; |im\beta| = |im\gamma|$

d. $\beta \mathcal{D} \gamma \; iff \; |im\beta| = |im\gamma|$

Proof

a. Suppose that, $(\beta, \gamma) \in \mathcal{L}(\bar{S}) \; \forall \; \beta, \gamma \in \bar{S}$. Then, $\bar{S}\beta = \{\beta, \gamma\} = \bar{S}\gamma$, that is, if

$\beta = \begin{pmatrix} \{X_1 & . & . & . & X_{n-1}\} & X_n \\ & & i & & & n \end{pmatrix} \; and \; \gamma = \begin{pmatrix} X_1 & \{ X_2 & . & . & . & X_n\} \\ i & & & n & & \end{pmatrix}, \forall \; i = 1, n \geq 3.$

There exist $\alpha \in \bar{S}$ such that, $\alpha = \begin{pmatrix} X_1 & X_2 & . & . & . & X_n \\ i & i+1 & . & . & . & n \end{pmatrix}$ and as such, $\bar{S}\beta = \bar{S}\gamma = \{\beta, \gamma\}$. Also, since in β, $im\beta = \{i \;\; . \;\; . \;\; .; n\}$ and in γ, $im\gamma = \{i, n \;\; . \;\; . \;\; . n\}$. This tells us that for all $n \geq 3 \; and \; i = 1$ we have $|im\beta| = |im\gamma|$. Suppose that in $\bar{S}$, $|im\beta| \neq |im\gamma|$ it implies that, either $|im\alpha| = |im\beta|$ with $\bar{S}\alpha = \bar{S}\beta$ or $|im\alpha| = |im\gamma|$ with $\bar{S}\alpha = \bar{S}\gamma$. This is clearly a contradiction since by observation in $\bar{S}$ $|im\alpha| = 3, |\mathrm{im}\beta| = 2$, and $|im\gamma| = 2$ respectively $(\forall \; n \geq 3)$ also $\bar{S}\alpha \neq \bar{S}\beta$ and $\bar{S}\alpha \neq \bar{S}\gamma$. Hence $|im\beta| = |im\gamma|$ and $\bar{S}\beta = \bar{S}\gamma$ of which by hypothesis, $\beta \mathcal{L} \gamma \; \forall \; \beta, \gamma \in \bar{S}$.

b. Suppose that, $(\beta^n, \gamma^n) \in \mathcal{L}(\bar{S}) \; \forall \; \beta, \gamma \in \bar{S}$ with

$$\beta = \begin{pmatrix} \{X_1 & . & . & . & X_{n-1}\} & X_n \\ & & i & & & n \end{pmatrix}, \beta^2 = \beta, \beta^3 = \beta \; and \; respectively, \beta^n = \beta \; for \; all \; n.$$

Similarly, since $\gamma = \begin{pmatrix} X_1 & \{ X_2 & . & . & . & X_n\} \\ i & & & n & & \end{pmatrix}, \gamma^2 = \gamma, \gamma^3 = \gamma \; and \; respectively, \; \gamma^n = \gamma \; for \; all \; n.$

As such, it follows from (a) above that if $(\beta^n, \gamma^n) \in \mathcal{L}(\bar{S}) \; \forall \; \beta, \gamma \in \bar{S}$ then, $\bar{S}\beta^n = \bar{S}\gamma^n$ and since, $\bar{S}\beta^n = \bar{S}\beta$ and $\bar{S}\gamma^n = \bar{S}\gamma$. Hence $\bar{S}\beta = \bar{S}\gamma$ and without loss of generality, $im\beta^n = im\beta$ and $im\gamma^n = im\gamma$. Hence $|im\beta^n| = |im\gamma^n|$ implying that, $|im\beta| = |im\gamma|$. The converse is clearly seen in (a) above.

Suppose that $((\beta, \gamma) \in \mathcal{R}(\bar{S}) \; \forall \; \beta, \gamma \in \bar{S}$ then by hypothesis, $\beta\bar{S} = \gamma\bar{S}$, but by the formation of $\bar{S}$ $\beta\bar{S}$ implies that, $\beta\alpha = \beta^2 = \beta\gamma = \beta$ and $\gamma\bar{S}$ implies that $\gamma\alpha = \gamma^2 = \gamma\beta = \gamma$, as such, $\beta\bar{S} \neq \gamma\bar{S}$ hence $\beta\bar{S} = \beta$ and $\gamma\bar{S} = \gamma$. Clearly we see that $im\beta = im\gamma$, hence $|im\beta| = |im\gamma|$.

c. Suppose $\{\beta, \gamma\} \in \mathcal{D}$ then $\beta \mathcal{L} \gamma$ and $\gamma \mathcal{R} \gamma$. That is, if there exist $\alpha \in \bar{S} \; \alpha\beta = \beta, and \; \alpha\gamma = \gamma$, then

$$\beta = \beta\gamma = \alpha\beta\gamma = \alpha\beta\gamma^2 = \alpha\beta\gamma\gamma = \alpha\gamma\gamma \; (since \; \beta = \beta\gamma \rightarrow \beta\beta^{-1} = \gamma) = \gamma\gamma = \gamma^2 = \gamma,$$

$\gamma = \alpha\gamma = \alpha\beta\beta^{-1} = \alpha\beta\beta\beta^{-1} = \alpha\beta = \beta. \; Hence \; \beta \mathcal{L} \gamma.$

Similarly, $\gamma\alpha = \gamma, and\ \gamma\alpha = \gamma$, that is,

$\gamma = \gamma\alpha = \alpha\gamma = \alpha\gamma\beta = \gamma\alpha\beta = \gamma\beta\alpha = \gamma\beta\beta\alpha = \gamma\beta\gamma$. Hence, $\gamma\mathcal{R}\gamma$. Thus $\beta\mathcal{D}\gamma$ and $|im\beta| = |im\gamma|$ □

Theorem 5. Let $a, b, c \in \bar{S}$. Then the following properties hold;

(i). $\bar{S}$ is a commutative monoid, (**ii).** $\bar{S}$ forms rectangular band and (**iii).** $\bar{S}$ form a semilattice

Proof

i. Observe that $\bar{S}$ is a commutative semigroup since if $, b, c \in \bar{S}$ $im\ a(X_1) = i$, $im\ a(X_2) = i + 1, . . ., ima(X_n) = n$, $im\ b(X_1) = i, . . ., im\ b(X_{n-1}) = i$, $imb(X_n) = n$, and $im\ c(X_1) = i$, $im\ c(X_2) = n, . . ., imc(X_n) = n$, $n \geq 3, i = 1$.

$(a * b) = (b * a) \subseteq \bar{S}$, $\forall\ n \geq 3$ and dually, $(a, b) * (b, c) = (b, c) * (a, b)$. Hence $\bar{S}$ is a commutative semigroup. If a is an identity element in $\bar{S}$, then $a * b = b = b * a$ and $a * c = c = c * a$ for all $b, c \in \bar{S}$. This implies that, $\bar{S}$ contain an identity element. Hence $\bar{S}$ is a commutative monoid.

ii. Also, since in $\bar{S}$ $bab = abb = bba = bab = b$ and $cac = acc = cac = cca = cac = c$ $\forall\ a, b, c \in \bar{S}$. Hence $\bar{S}$ is a rectangular band.

iii. That $\bar{S}$ form a semilattice can easily be seen. □

5.0 WORK DONE BY $\bar{S}$

Following the approach of James East in [4] on the work titled "work done by transformation semigroup" we obtain the work done and average work done by $\bar{S}$ respectively.

Combinatorially, we obtain that

$$\mathcal{W}(\bar{S}) = 2(n-2) + \binom{n-2}{n-3}(n-3) = n^2 - 3n + 2$$

$$\bar{\mathcal{W}}(\bar{S}) = \frac{2(n-2) + \binom{n-2}{n-3}(n-3)}{|\bar{S}| = 3} = \frac{n^2 - 3n + 2}{3}$$

Representing $\mathcal{W}(\bar{S})$ and $\bar{\mathcal{W}}(\bar{S})$ on the table below for $n \geq 3$.

$n\ (n \geq 3)$	3	4	5	6	7	8	9	10	11	12
$\mathcal{W}(\bar{S}) = n^2 - 3n + 2$	2	6	12	20	30	42	56	72	90	110
$\bar{\mathcal{W}}(\bar{S}) = \frac{\mathcal{W}(\bar{S})}{\lvert\bar{S}\rvert = 3}$	0.6667	2	4	6.6667	10	14	18.6667	24	30	36.6667
$\lvert\bar{S}\rvert = 3 + \sum_{i=0}^{n}(i-1) + (1-i)$	3	3	3	3	3	3	3	3	3	3

Table 1

Theorem 6. Let $a, b, c \in \bar{S}$ a 3 elements semigroup of the order preserving transformation semigroup S.

a. $\mathcal{W}(\bar{S}) = n^2 - 3n + 2$ and b. $\bar{\mathcal{W}}(\bar{S}) = \frac{n^2-3n+2}{3}$

Proof

Let $\mathcal{W}(\bar{S})$ represent work done by $\bar{S}$. Suppose $\mathcal{W}(\bar{S}) = 2(n-2) + \binom{n-2}{n-3}(n-3)$ where,

$\binom{n-2}{n-3}$ tells us that there are (n-2) ways (n-3) can be presented in $\mathcal{W}(\bar{S})$ for $n \geq 3$.

2(n-2) implies that for every n there are 2(n-2) which must be added to the presiding value and (n-3) implies that for every n there are (n-3) multiple to (n-2) ways (n-3) can be presented, as such;

$$2(n-2) + \frac{(n-2)!}{(n-3)!(1)!}(n-3) = 2(n-2) + \frac{(n-2)!}{(n-3)!}(n-3)$$

$$= 2(n-2) + \frac{(n-2)!(n-3)}{(n-3)!}$$

$$= 2(n-2) + \frac{(n-2)(n-3)!}{(n-3)!}(n-3)$$

$$= 2(n-2) + (n-2)(n-3)$$

$$= n - 2(2 + n - 3)$$

$$= (n-2)(n-1) = n^2 - 3n + 2$$

Therefore, $2(n-2) + \frac{(n-2)!}{(n-3)!(1)!}(n-3) = n^2 - 3n + 2$.

Obviously, the average work done by $\bar{S}$ is give as $\bar{\mathcal{W}}(\bar{S}) = \frac{\mathcal{W}(\bar{S})}{|\bar{S}|=3} = \frac{n^2-3n+2}{3} \; \forall \, n \geq 3$. □

6.0 SUMMARY

The investigation of $\mathcal{O}_n$ for $\overline{ID\mathcal{O}_n}$ yield a crucial result which has some significance in semigroup theorem. The significant results we obtain in this study reveal that the elements of non-identity difference order preserving transformation $\overline{ID\mathcal{O}_n}$ form a semigroup for $n = 3$ but cease to be a semigroup for $n \geq 4$ as shown in section 3 above. Also, a close investigation on the said elements for $n = 3$ gave another view as regards $\overline{ID\mathcal{O}_3}$ characteristics which can be seen for $n \geq 4$. By these approaches we were able to obtain a semigroup $\bar{S}$ that has only three elements for all $n \geq 3$. Further investigation on the said semigroup $\bar{S}$ gave us the properties that we were able to build on as shown in section 4 and 5 respectively.

Conflict of Interests

The authors declare that there is no conflict of interests.

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