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SOME RESULTS ABOUT A CLASS OF SYMMETRIC NUMERICAL SEMIGROUPS

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Abstract. In this paper, we will give some results about the numerical semigroups such that $S_k = < 5, 5k+3 >$ where $k \geq 1$, $k \in \mathbb{Z}$. Also, we will obtain Arf closure of these symmetric numerical semigroups.

Keywords: symmetric numerical semigroup; Arf closure; genus.

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1. INTRODUCTION

Let $\mathbb{N}= \ 0,1,2,...,n,...$ and $\mathbb{Z}$ be integer set. S is called a numerical semigroup if

(i) $s_1 + s_2 \in S$, for $s_1, s_2 \in S$

(ii) $\gcd \ S \ = 1$

(iii) $0 \in S$

where $S \subseteq \mathbb{N}$ (Here, $\gcd \ S \ =$ greatest common divisor the elements of S).

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A numerical semigroup S can be written that

$S = < a_1, a_2, ..., a_n > = \left\{ \sum_{i=1}^{n} c_i a_i : c_i \in \mathbb{N} \right\}$ (for detail see [4]).

$T \subset \mathbb{N}$ is minimal system of generators of S if $< T > = S$ and there isn't any subset $M \subset T$ such that $< M > = S$ .Also, $m(S) = \min \ x \in S : x > 0$ is called as multiplicity of S (See [3]). Let S be a numerical semigroup, then $F(S) = \max \ \mathbb{Z} \backslash S$ is called as Frobenius number of S. $n(S) = Card \ \ 0,1,2,...,F(S) \ \cap S$ is called as the determine number of S (see [5]).

If S is a numerical semigroup such that $S = < a_1, a_2, ..., a_n >$, then we observe that $S = < a_1, a_2, ..., a_n > = \ s_0 = 0, s_1, s_2, ..., s_{n-1}, s_n = F(S)+1, \rightarrow ...$, where $s_i < s_{i+1}$, $n = n(S)$ and the arrow means that every integer greater than $F(S)+1$ belongs to S for $i = 1, 2, ..., n = n(S)$ (see [6]).

If $t \in \mathbb{N}$ and $t \notin S$, then t is called gap of S. We denote the set of gaps of S, by $H(S)$, i.e, $H(S) = \mathbb{N} \backslash S$. The $G(S) = \#(H(S))$ is called the genus of S. It known that $G(S) = F(S) + 1 - n(S)$ (see [4]).

S is called symmetric numerical semigroup if $F(S) - u$ belongs to S, for $u \in \mathbb{Z} \backslash S$. It is known the numerical semigroup $S = < a_1, a_2 >$ is symmetric and $F(S) = a_1 a_2 - a_1 - a_2$. In this case, we write $n(S) = \dfrac{F(S)+1}{2}$ (see [1]).

A numerical semigroup S is called Arf if $s_1 + s_2 - s_3 \in S$, for all $s_1, s_2, s_3 \in S$ such that $s_1 \geq s_2 \geq s_3$. The smallest Arf numerical semigroup containing a numerical semigroup S is called the Arf closure of S, and it is denoted by $Arf(S)$ (for detail see [2, 3]). If S is a numerical semigroup such that $S = < a_1, a_2, ..., a_n >$, then $L(S) = \langle a_1, a_2 - a_1, a_3 - v_1, ..., a_n - v_1 \rangle$ is called Lipman numerical semigroup of S, and it is known that

$L_0(S)=S\subseteq L_1(S)=L(L_0(S))\subseteq L_2=L(L_1(S))\subseteq ...\subseteq L_m=L(L_{m-1}(S))\subseteq ...\subseteq \mathbb{N}$ (see [7]).

2. MAIN RESULTS

Theorem 1. Let $S_k=<5,5k+3>$ be numerical semigroups, where $k\geq 1,\ k\in\mathbb{Z}$. Then, we have

(a) $F(S_k)=20k+7$

(b) $n(S_k)=10k+4$

(c) $G(S_k)=10k+4$.

Proof. Let $S_k=<5,5k+3>$ be numerical semigroups, where $k\geq 1,\ k\in\mathbb{Z}$. Then, S_k is symmetric and we find that

(a) $F(S_k)=5(5k+3)-5-5k-3=20k+7$.

(b) $n(S_k)=\dfrac{F(S_k)+1}{2}=\dfrac{20k+7+1}{2}=10k+4$.

(c) $G(S_k)=20k+7+1-10k-4=10k+4$ from $G(S_k)=F(S_k)+1-n(S_k)$.

Theorem 2. Let $S_k=<5,5k+3>$ be numerical semigroups, where $k\geq 1,\ k\in\mathbb{Z}$. Then,

$Arf(S_k)=\ 0,5,10,15,...,5k,5k+3,5k+5,\rightarrow ...$.

Proof. Let $S_k=<5,5k+3>$ be numerical semigroups, where $k\geq 1,\ k\in\mathbb{Z}$. Then, we have

$L_i(S_k)=<5,5k+(3-5i)>$ for $i=0,1,2,...,k-2$. In this case,

If $5<5k+(3-5i)$ then $m_i=5$.

If $5>5k+(3-5i)$ then $m_i=3$. So, we write $L_{k-1}(S_k)=<5,6>,\ m_{k-1}=5$

and $L_k(S_k)=<5,1>=<1>=\mathbb{N},\ m_k=1$.

Thus, we obtain $Arf(S_k)=\ 0,5,10,15,...,5k,5k+3,5k+5,\rightarrow ...$.

Corollary 3. Let $S_k =< 5, 5k+3 >$ be numerical semigroups, where $k \geq 1, k \in \mathbb{Z}$. Then, we have

(a) $F(Arf(S_k)) = 5k+4$

(b) $n(Arf(S_k)) = k+2$

(c) $G(Arf(S_k)) = 4k+3$.

Proof. Let $S_k =< 5, 5k+3 >$ be numerical semigroups, where $k \geq 1, k \in \mathbb{Z}$. Then, we write that $F(Arf(S_k)) = 5k+4$ from Theorem 2. On the other hand, we find that

$$n(Arf(S_k)) = \#(\ 0,1,2,...,5k+4 \ \cap Arf(S)) = \#(\ 0,5,10,...,5k,5k+3 \ ) = k+2$$

and we obtain

$$G(Arf(S_k)) = 5k+4+1-k-2 = 4k+3$$

since $G(Arf(S_k)) = F(Arf(S_k)) + 1 - n(Arf(S_k))$.

Corollary 4. Let $S_k =< 5, 5k+3 >$ be numerical semigroups, where $k \geq 1, k \in \mathbb{Z}$. Then, we have

(a) $F(S_k) = 4F(Arf(S_k)) - 9$

(b) $n(S_k) = 10n(Arf(S_k)) - 16$

(c) $G(S_k) = 2G(Arf(S_k)) + 2(k-1)$.

Proof. Let $S_k =< 5, 5k+3 >$ be numerical semigroups, where $k \geq 1, k \in \mathbb{Z}$. We write that

(a) $4F(Arf(S_k)) - 9 = 4(5k+4) - 9 = 20k+7 = F(S_k)$. However, we find that

(b) $10n(Arf(S_k)) - 16 = 10(k+2) - 16 = 10k+4 = n(S_k)$,

(c) $2G(Arf(S_k)) + 2(k-1) = 2(4k+3) + 2k - 2 = 10k+4 = G(S_k)$.

Corollary 5. Let $S_k =< 5, 5k+3 >$ be numerical semigroups, where $k \geq 1, k \in \mathbb{Z}$. Then, it satisfies following conditions:

(a) $F(S_{k+1}) = F(S_k) + 20$

(b) $n(S_{k+1}) = n(S_k) + 10$

(c) $G(S_{k+1}) = G(S_k) + 10$.

Corollary 6. Let $S_k = < 5, 5k+3 >$ be numerical semigroups, where $k \geq 1$, $k \in \mathbb{Z}$. Then, it satisfies following conditions:

(a) $F(Arf(S_{k+1})) = F(Arf(S_k)) + 5$

(b) $n(Arf(S_{k+1})) = n(Arf(S_k)) + 1$

(c) $G(Arf(S_{k+1})) = G(Arf(S_k)) + 4$.

Example 7. We put $k = 1$ **i**n $S_k = < 5, 5k+3 >$ symmetric numerical semigroups. Then we have $S_1 = < 5,8 > = 0,5,8,10,13,15,18,20,23,24,25,26,28,\rightarrow\ldots$. In this case, we obtain

$F(S_1) = 27$, $n(S_1) = 14$, $H(S_1) = 1,2,3,4,6,7,9,11,12,14,17,19,22,27$, $G(S_1) = 14$,

$Arf(S_1) = 0,5,8,10,\rightarrow\ldots$, $F(Arf(S_1)) = 9$, $n(Arf(S_1)) = 3$, $H(Arf(S_1)) = 1,2,3,4,6,7,9$

and $G(Arf(S_1)) = 7$. Thus, we find that

$4F(Arf(S_1)) - 9 = 4.9 - 9 = 27 = F(S_1)$, $10n(Arf(S_1)) - 16 = 10.3 - 16 = 14 = n(S_1)$

and $2G(Arf(S_1)) + 2(1-1) = 2G(Arf(S_1)) = 2.7 = 14 = G(S_1)$.

If $k = 2$ then we write

$$S_2 = < 5,13 > = 0,5,10,13,15,18,20,23,25,26,28,30,31,33,35,36,38,39,40,41,43,44,45,46,48,\rightarrow\ldots .$$

Thus, we have $F(S_2) = 47$, $n(S_2) = 24$, $G(S_2) = 24$, $Arf(S_2) = 0,5,10,13,15,\rightarrow\ldots$,

$F(Arf(S_2)) = 14$, $n(Arf(S_2)) = 4$ and $G(Arf(S_2)) = 11$.

So, we write that $F(S_1) + 20 = 27 + 20 = 47 = F(S_2)$,

$n(S_1) + 10 = 14 + 10 = 24 = n(S_2)$ and $G(S_1) + 10 = 14 + 10 = 24 = G(S_2)$. Also, we obtain that

$F(Arf(S_1))+5=9+5=14=F(Arf(S_2))$, $n(Arf(S_1))+1=3+1=4=n(Arf(S_2))$

and $G(Arf(S_1))+4=7+4=11=G(Arf(S_2))$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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