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INVERSE SEMIGROUP AMALGAMS WITH A LOWER BOUNDED CORE

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Abstract. We consider inverse semigroup amalgams $[S_1, S_2; U]$ such that for any $u \in U$ and $e \in E(S_i)$ with $u \geq e$ in S_i, where $i \in \{1, 2\}$, there exists $f \in E(U)$ with $u \geq f \geq e$ in S_i; we say that U is lower bounded in S_1 and S_2. We construct and describe the Schützenberger automata of $S_1 *_U S_2$ and give conditions for decidable word problem. The homomorphisms of the Schützenberger graphs of $S_1 *_U S_2$ are studied and conditions are given for $S_1 *_U S_2$ to be completely semisimple. In the case when S_1 and S_2 have decidable word problems and U is finite, we show that $S_1 *_U S_2$ has decidable word problem.

Keywords: inverse semigroups; amalgams; Schützenberger automata; word problem; completely semisimple; subgroups.

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1. Introduction

It was proved by Hall [11] (1975) that any amalgam of inverse semigroups is strongly embedded into an inverse semigroup. It follows that any amalgam $[S_1, S_2; U]$ is strongly embedded into the amalgamated free product $S_1 *_U S_2$, in the variety of inverse semigroups. Since Hall's result, the structure of $S_1 *_U S_2$ and conditions for decidability of the word problem, in the general case, have been open problems, although some special cases have been studied.

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It was shown by Birget, Margolis and Meakin [3] (1991) that $S_1 *_U S_2$ can have undecidable word problem. Haataja, Margolis and Meakin in [10] (1996) considered full amalgams. Lower bounded amalgams were studied by the author in [1] and [2] (1997). Stephen [18] (1998) reproved Hall's result.

Cherubini, Meakin and Piochi [6] (1997) showed that the amalgamated free product of free inverse semigroups can have decidable word problem. In [7] (2005), they showed that amalgams of finite inverse semigroups have decidable word problem. Further work was done by Cherubini, Jajcayová, Mazzuchelli, Nuccio and Rodaro in [5], [15], [8], [9] and [4] (2008–2015).

In Algorithm 4.20, a method is given for constructing the Schützenberger automata of $S_1 *_U S_2$. Theorem 4.26 describes the Schützenberger automata of $S_1 *_U S_2$. Results are given concerning homomorphisms of Schützenberger graphs, which lead to conditions for $S_1 *_U S_2$ to be completely semisimple. In Theorem 4.40, a list of conditions is given for $S_1 *_U S_2$ to have decidable word problem. As an example, Corollary 4.41 shows that $S_1 *_U S_2$ has decidable word problem when S_1 and S_2 have decidable word problems and U is finite.

2. Preliminaries

A semigroup S is called an *inverse semigroup* if, for every element $s \in S$, there is a unique element s^{-1}, called the *inverse of* s, such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s$. The set $E(S) = \{e \in S : e^2 = e\}$ is called the *semilattice of idempotents* of S. The *natural partial order* of S is defined by $a \leq b$ if and only if $a = eb$, for some $e \in E(S)$, for $a, b \in S$. A subsemigroup U of S is called an *inverse subsemigroup* of S if the inverse of each element of U is also contained in U. For results on inverse semigroups, see Howie [12] and Petrich [14].

A *presentation* for an inverse semigroup S is a pair $\langle X \mid R \rangle$, where X is a non-empty set and R is a binary relation on $(X \cup X^{-1})^+$ such that $S \cong (X \cup X^{-1})^+/\tau$, letting τ denote the congruence generated by R and the Vagner congruence ρ. We say that the inverse semigroup S is *presented* by the *generators* X and *relations* R and write $S = Inv\langle X \mid R \rangle$.

The presentation $\langle X \mid R \rangle$ is then studied by considering the *Schützenberger automaton* $\mathscr{A}(X, R, w)$ of w, for $w \in (X \cup X^{-1})^+$. The automaton $\mathscr{A}(X, R, w)$ has underlying graph $S\Gamma(X, R, w)$, consisting of vertices $R_{w\tau}$, the $\mathscr{R}$-class of S containing $w\tau$, and an edge from s to t labeled by y, if $s, t \in R_{w\tau}$ and $y \in X \cup X^{-1}$ such that $s \cdot y\tau = t$. The initial and terminal states are the vertices $ww^{-1}\tau$ and $w\tau$, respectively. If a presentation has been specified then we also denote $\langle X \mid R\rangle$, $S\Gamma(X,R,w)$, $\mathscr{A}(X,R,w)$ by $\langle S\rangle$, $S\Gamma(S,w)$, $\mathscr{A}(S,w)$, respectively. For results on presentations, see Stephen [16], [17] and [18].

For a non-empty set X, an *inverse word graph* Γ *over* X is a connected graph with edges labeled over $X\cup X^{-1}$, such that for every edge from v_1 to v_2 labeled by y, there is an *inverse edge* from v_2 to v_1 labeled by y^{-1}. The inverse word graph Γ is *deterministic* if no two distinct edges have the same initial vertex and label. The vertex and edge sets are denoted $V(\Gamma)$ and $E(\Gamma)$, respectively.

A V-equivalence η is an equivalence relation on $V(\Gamma)$. The *quotient* of Γ under η is defined to be the graph Γ/η with vertices $V(\Gamma/\eta) = V(\Gamma)/\eta$ and an edge from $v_1\eta$ to $v_2\eta$, labeled by y, if there is an edge from v_1 to v_2 in Γ labeled by y. The quotient Γ/η is also an inverse word graph over X and η induces a homomorphism from Γ onto Γ/η.

If $v_1, v_2 \in V(\Gamma)$ and w labels a path, or edge, from v_1 to v_2 then we write this as $v_1 \to^{w} v_2$. We have a path $v_1\eta \to^{w} v_2\eta$ in Γ/η if and only if there are paths $x_1 \to^{w_1} y_1$, $x_2 \to^{w_2} y_2$, …, $x_n \to^{w_n} y_n$ in Γ, where $n \geq 1$, $v_1\eta x_1$, $y_1\eta x_2$, …, $y_{n-1}\eta x_n$, $y_n\eta v_2$ and $w_1w_2\cdots w_n = w$.

The V-equivalence η is called *determinising* if Γ/η is deterministic. The *least determinising V-equivalence containing* η is defined as the intersection η^* of all determinising V-equivalences that contain η. The *determinised form* of Γ is the quotient Γ/id^*, where id is the identity relation.

Result 2.1. [17, Lemma 4.3, Theorem 4.4] *The determinised form Γ/id^* of an inverse word graph Γ over X is a well-defined deterministic inverse word graph over X. For $v_1, v_2 \in V(\Gamma)$, we have $v_1 id^* = v_2 id^*$ if and only if there is a path $v_1 \to^{w} v_2$, for some word w freely reducible to* 1.

The composition of V-equivalences $\eta \circ id^*$ is equal to η^*. Thus, for vertices $v_1, v_2 \in V(\Gamma)$, we have $v_1\eta^* v_2$ if and only if there is a path $v_1\eta \to^{w} v_2\eta$ in Γ/η, where w is freely reducible to 1. Then $v_1\eta^* v_2$ if and only if there are paths $x_1 \to^{w_1} y_1$, $x_2 \to^{w_2} y_2$, …, $x_n \to^{w_n} y_n$ in Γ, where $n \geq 1$, $v_1\eta x_1$, $y_1\eta x_2$, …, $y_{n-1}\eta x_n$, $y_n\eta v_2$ and $w_1w_2\cdots w_n$ is freely reducible to 1.

Further, we have a path $v_1\eta^* \to^{w} v_2\eta^*$ in Γ/η^* if and only if there exist paths $x_1\eta \to^{w_1} y_1\eta$, $x_2\eta \to^{w_2} y_2\eta$, …, $x_n\eta \to^{w_n} y_n\eta$ in Γ/η, where $n \geq 1$, $v_1\eta^* x_1$, $y_1\eta^* x_2$, …, $y_{n-1}\eta^* x_n$, $y_n\eta^* v_2$ and $w_1w_2\cdots w_n = w$. Thus, using the above, we a path $v_1\eta^* \to^{w} v_2\eta^*$ in Γ/η^* if and only if there exist paths $x_1 \to^{w_1} y_1$, $x_2 \to^{w_2} y_2$, …, $x_n \to^{w_n} y_n$ in Γ, where $n \geq 1$, $v_1\eta x_1$, $y_1\eta x_2$, $y_2\eta x_3$, …, $y_{n-1}\eta x_n$, $y_n\eta v_2$ and $w_1w_2\cdots w_n$ is freely reducible to w.

A *(birooted) inverse automaton over* X is a triple $\mathscr{A} = (\alpha, \Gamma, \beta)$, where Γ is an inverse word graph over X and $\alpha, \beta \in V(\Gamma)$. The vertices α and β are called the *initial* and *terminal roots* of $\mathscr{A}$, respectively. We also denote the vertices of $\mathscr{A}$ by $V(\mathscr{A})$. The *language* $L[\mathscr{A}]$ of $\mathscr{A}$ is the set of all words that label paths from α to β. If η is a V-equivalence on the vertices of Γ then the quotient automaton $\mathscr{A}/\eta$ is given by $(\alpha\eta, \Gamma/\eta, \beta\eta)$. The *determinised form* of $\mathscr{A}$ is the quotient $\mathscr{A}/id^*$.

Result 2.2. [17, Theorem 2.5] *If $\mathscr{A}$ and $\mathscr{A}'$ are deterministic inverse automata over X with $L[\mathscr{A}] \subseteq L[\mathscr{A}']$ then there is a unique homomorphism from $\mathscr{A}$ into $\mathscr{A}'$. Thus if $L[\mathscr{A}] = L[\mathscr{A}']$ then we have $\mathscr{A} \cong \mathscr{A}'$.*

An inverse automaton $\mathscr{A}$ over X has *decidable language* if there exists an algorithm that decides, on input $w \in (X \cup X^{-1})^+$, whether or not $w \in L[\mathscr{A}]$.

Result 2.3. [17, Theorems 3.1 and 3.9] *For $S = Inv\langle X \mid R\rangle$, the language $L[\mathscr{A}(X,R,w)]$ consists of all words $w_1 \in (X \cup X^{-1})^+$ such that $w_1 \geq w$ in S. We have $w = w_1$ in S if and only if $\mathscr{A}(X,R,w) \cong \mathscr{A}(X,R,w_1)$.*

Result 2.4. *The word problem for $S = Inv\langle X \mid R\rangle$ is decidable if and only if the Schützenberger automata of $\langle X \mid R\rangle$ have decidable languages.*

Proof. The proof follow from Result 2.3. □

An inverse automaton $\mathscr{A}$ over X is called an *approximate automaton* of $\mathscr{A}(X,R,w)$ if we have $L[\mathscr{A}] \subseteq L[\mathscr{A}(X,R,w)]$ and there exists $w_1 \in L[\mathscr{A}]$ with $w_1 = w$ in $S = Inv\langle X \mid R\rangle$, written $\mathscr{A} \rightsquigarrow \mathscr{A}(X,R,w)$. An inverse word graph Γ over X is called an *approximate graph* if we have $(\alpha, \Gamma, \beta) \rightsquigarrow \mathscr{A}(X,R,w)$, for some $\alpha, \beta \in V(\Gamma)$ and some w. Multiplication of disjoint automata $\mathscr{A}_1 = (\alpha_1, \Gamma_1, \beta_1)$ and $\mathscr{A}_2 = (\alpha_2, \Gamma_2, \beta_2)$ is defined by $\mathscr{A}_1 \times \mathscr{A}_2 = (\alpha_1\eta, (\Gamma_1 \cup \Gamma_2)/\eta, \beta_2\eta)$, where η is the V-equivalence generated by $\{(\beta_1, \alpha_2)\}$.

Result 2.5. [17, Theorem 5.2] *If we have $\mathscr{A}_1 \rightsquigarrow \mathscr{A}(X,R,w_1)$ and $\mathscr{A}_2 \rightsquigarrow \mathscr{A}(X,R,w_2)$ then $\mathscr{A}_1 \times \mathscr{A}_1 \rightsquigarrow \mathscr{A}(X,R,w_1w_2)$.*

The *linear automaton* of $w = y_1y_2\cdots y_n \in (X \cup X^{-1})^+$, for $y_k \in X \cup X^{-1}$, is the inverse automaton $(\alpha_w, \Gamma_w, \beta_w)$, with vertices $v_0 = \alpha_w$, v_1, …, v_{n-1}, $v_n = \beta_w$ and edges $v_{k-1} \to^{y_k} v_k$, $v_k \to^{y_k^{-1}} v_{k-1}$, for $k = 1, 2, \ldots, n$.

If (α, Γ, β) is an inverse automaton over X and (r,s) is a relation in R such that Γ contains a path $v_1 \to^r v_2$, and no path $v_1 \to^s v_2$, then we perform an *elementary expansion*, relative to $\langle X \mid R \rangle$, by taking the disjoint union $\Gamma \cup \Gamma_s$ and then forming the automaton $(\alpha\eta, (\Gamma \cup \Gamma_s)/\eta, \beta\eta)$, where η is the V-equivalence generated by $\{(v_1, \alpha_s), (v_2, \beta_s)\}$. We refer to this construction as *sewing on* the linear automaton of s from v_1 to v_2.

If we take the disjoint union of Γ with any automaton $(\alpha_1, \Gamma_1, \beta_1)$ and then take the quotient by the V-equivalence generated by $\{(v_1, \alpha_1), (v_2, \beta_1)\}$, then we also refer to this operation as *sewing on* $(\alpha_1, \Gamma_1, \beta_1)$ from v_1 to v_2. If $v_1 = v_2$ then we refer to this operation as *sewing on* $(\alpha_1, \Gamma_1, \beta_1)$ at v_1.

If Γ has two edges $v_1 \to^y v_2$ and $v_1 \to^y v_3$, for some $y \in X \cup X^{-1}$, then we can perform an *elementary determination*, relative to $\langle X \mid R \rangle$, by taking the quotient of Γ by the V-equivalence generated by $\{(v_2, v_3)\}$. An elementary expansion or determination of an inverse automaton is just an elementary expansion or determination of the underlying graph.

Result 2.6. [17, Lemmas 4.1, 5.5, 5.6] *If $\mathscr{B}$ is obtained from the automaton $\mathscr{A}$ by an elementary expansion or elementary determination, relative to $\langle X \mid R \rangle$, then $L[\mathscr{A}] \subseteq L[\mathscr{B}]$. Further, if $\mathscr{A} \rightsquigarrow \mathscr{A}(X,R,w)$ then $\mathscr{B} \rightsquigarrow \mathscr{A}(X,R,w)$.*

Result 2.7. *If $\mathscr{B}$ is the determinised form of the automaton $\mathscr{A}$, relative to $\langle X \mid R \rangle$, then $L[\mathscr{B}] \subseteq \{z \in (X \cup X^{-1})^+ : z\tau \geq y\tau$, for some $y \in L[\mathscr{A}]\}$.*

Proof. We have $\mathscr{B} = \mathscr{A}/id^*$. Thus if $z \in L[\mathscr{B}]$ then there exists $y \in L[\mathscr{A}]$ that is freely reducible to z. Hence $z\tau \geq y\tau$ for some $y \in L[\mathscr{A}]$. □

Result 2.8. *If $\mathscr{A} \rightsquigarrow \mathscr{A}(X,R,w)$ and $\mathscr{B}$ is the determinised form of $\mathscr{A}$ then $\mathscr{B} \rightsquigarrow \mathscr{A}(X,R,w)$.*

Proof. The proof follows from Results 2.3 and 2.7. □

A deterministic inverse automaton over X is *closed*, relative to $\langle X \mid R \rangle$, if no elementary expansions can be performed.

Result 2.9. [17, Theorem 5.10] *If $\mathscr{A} \rightsquigarrow \mathscr{A}(X,R,w)$ and $\mathscr{A}$ is deterministic and closed, relative to $\langle X \mid R \rangle$, then $\mathscr{A} \cong \mathscr{A}(X,R,w)$.*

If Γ is an inverse graph over X then we say *there is a path from vertex v_1 to vertex v_2 labeled by $s \in S$*, and write $v_1 \to^s v_2$, if there is a path $v_1 \to^w v_2$, for some $w \in (X \cup X^{-1})^+$ with $w\tau = s \in S$.

If Γ is closed, relative to $\langle X \mid R\rangle$, and we have a path $v_1 \to^w v_2$, for some $w \in (X \cup X^{-1})^+$ with $w\tau = s$, then we have a path $v_1 \to^y v_2$, for every $y \in (X \cup X^{-1})^+$ with $y\tau \geq s$.

As defined in Stephen [17], an *expansion*, relative to $\langle X \mid R\rangle$, consists of performing an elementary expansion and then taking the determinised form.

Result 2.10. [16] *The category of all inverse automata over X is cocomplete.*

That is, every directed system of inverse automata over X has a direct limit. For an inverse automaton $\mathscr{A}$ over X, the *closed form of $\mathscr{A}$, relative to $\langle X \mid R\rangle$*, is the direct limit of the directed system of all automata obtained from $\mathscr{A}$ by finite elementary expansions and determinations, relative to $\langle X \mid R\rangle$.

Result 2.11. [18, Theorem 3.3] *The automaton $\mathscr{A}(X,R,w)$ is the closed form, relative to $\langle X \mid R\rangle$, of the linear automaton of w.*

Result 2.12. [18, Lemma 3.4] *If $\mathscr{B}$ is the closed form of the automaton $\mathscr{A}$, relative to $\langle X \mid R\rangle$, then we have $L[\mathscr{B}] = \{z \in (X \cup X^{-1})^+ : z\tau \geq y\tau$, for some $y \in L[\mathscr{A}]\}$.*

Let $S_1 = Inv\langle X_1 \mid R_1\rangle$ and $S_2 = Inv\langle X_2 \mid R_2\rangle$, where $X_1 \cap X_2 = \emptyset$. Then the free product $S_1 * S_2$, in the variety of inverse semigroups, has presentation $\langle X \mid R\rangle$, where $X = X_1 \cup X_2$ and $R = R_1 \cup R_2$. If Γ is an inverse word graph over X then each edge of Γ is labeled from $X_i \cup X_i^{-1}$, for some $i \in \{1,2\}$, and is said to be *colored* by i. A subgraph of Γ is *monochromatic* if all its edges have the same color. A *lobe* of Γ is a maximal monochromatic connected subgraph of Γ. The coloring of edges extends to coloring of lobes. Two lobes are said to be *adjacent* if they share common vertices, called *intersections*. A path in Γ is called *simple* if it contains no repeated vertex, other than perhaps its first and last, in which case it is a *simple cycle*. The graph Γ is called *cactoid* if it has finitely many lobes and every simple cycle is monochromatic. An inverse automaton is called *cactoid* if is underlying graph is cactoid.

Result 2.13. [13, Theorem 4.1] *The Schützenberger automata of the free product $S_1 * S_2 = Inv\langle X \mid R\rangle$ are precisely, up to isomorphism, the cactoid inverse automata over X, where the lobes are isomorphic to Schützenberger graphs of either $\langle X_1 \mid R_1\rangle$ or $\langle X_2 \mid R_2\rangle$.*

Construction 2.14. [13, Section 3] Let $\mathscr{A} = (\alpha, \Gamma, \beta)$ be a cactoid inverse automaton over X with lobes that approximate graphs relative to either $\langle X_1 \mid R_1\rangle$ or $\langle X_2 \mid R_2\rangle$. Let Δ be a lobe with

$(v,\Delta,v) \rightsquigarrow \mathscr{A}(X_i,R_i,y)$, for some $v \in V(\Delta)$ and $y \in (X_i \cup X_i^{-1})^+$, and let $(v_1,\Delta_1,v_1) \cong \mathscr{A}(X_i,R_i,y)$ be disjoint from Δ. Construct the quotient $\mathscr{A}' = (\alpha\eta, (\Gamma \cup \Delta_1)/\eta, \beta\eta)$, where η is the least V-equivalence identifying v with v_1 and determinising Δ and Δ_1.

Result 2.15. [13, Propositions 3.1, 3.2, 3.3] *If $\mathscr{A}'$ is obtained from $\mathscr{A}$ by an application of Construction 2.14 then $\mathscr{A}'$ is a cactoid inverse automaton over X with lobes that approximate graphs relative to either $\langle X_1 \mid R_1\rangle$ or $\langle X_2 \mid R_2\rangle$. Further, if $\mathscr{A} \rightsquigarrow \mathscr{A}(X,R,w)$ then $\mathscr{A}' \rightsquigarrow \mathscr{A}(X,R,w)$.*

Result 2.16. [13, Theorem 3.4] *Starting with the linear automaton of w, any sequence obtained by repeated applications of Construction 2.14 terminates finitely in $\mathscr{A}(X,R,w)$.*

Result 2.17. [13, Corollary 3.5] *The free product $S_1 * S_2 = Inv\langle X \mid R\rangle$ has decidable word problem if $S_1 = Inv\langle X_1 \mid R_1\rangle$ and $S_2 = Inv\langle X_2 \mid R_2\rangle$ both have decidable word problems.*

3. Directed systems of inverse automata

Notation 3.1. Suppose we have an operation, *Construction Q*, say, that we can apply to inverse automata over X. Let I denote the set of all automata obtained from a deterministic automaton $\mathscr{A}$ by finitely many applications of Construction Q. Define a relation by $\mathscr{B} \leq \mathscr{C}$ if and only if $\mathscr{B} = \mathscr{C}$ or $\mathscr{C}$ is obtained from $\mathscr{B}$ by finite applications of Construction Q, for $\mathscr{B},\mathscr{C} \in I$.

Lemma 3.2. *Suppose Construction Q satisfies the following, for $\mathscr{B},\mathscr{C},\mathscr{D} \in I$:*

(A) *$\mathscr{B} \leq \mathscr{C}$ implies $\mathscr{C}$ is a deterministic and $L[\mathscr{B}] \subseteq L[\mathscr{C}]$.*

(B) *$\mathscr{B} \leq \mathscr{C}$ and $\mathscr{B} \leq \mathscr{D}$ imply $\mathscr{C} \leq \mathscr{E}$ and $\mathscr{D} \leq \mathscr{E}$, for some $\mathscr{E} \in I$.*

Then the automata of I form a directed system. The direct limit is the quotient of the disjoint union $\sqcup_{\mathscr{B} \in I}\mathscr{B}$, under the V-equivalence η defined by $v_1 \eta v_2$ if and only if $\mathscr{B} \leq \mathscr{D}$ and $\mathscr{C} \leq \mathscr{D}$, for some $\mathscr{D} \in I$, where the images of v_1 and v_2 are identified in $\mathscr{D}$, for $v_1 \in V(\mathscr{B})$, $v_2 \in V(\mathscr{C})$ and $\mathscr{B},\mathscr{C} \in I$. For any w in the language of the direct limit, there exists $\mathscr{B}$ in the directed system with $w \in L[\mathscr{B}]$.

Proof. It is immediate that the relation $\leq$ is reflexive and transitive. If $\mathscr{B} \leq \mathscr{C}$ and $\mathscr{C} \leq \mathscr{B}$ then condition (A) implies $\mathscr{B} \cong \mathscr{C}$, by Result 2.2. Hence $\leq$ defines a partial order on I. If $\mathscr{C},\mathscr{D} \in I$ then $\mathscr{A} \leq \mathscr{C}$ and $\mathscr{A} \leq \mathscr{D}$ imply $\mathscr{C} \leq \mathscr{E}$ and $\mathscr{D} \leq \mathscr{E}$, for some $\mathscr{E} \in I$, by condition (B). Thus $(I,\leq)$ determines a directed set.

If $\mathscr{B},\mathscr{C} \in I$ and $\mathscr{B} \leq \mathscr{C}$ then we have a unique homomorphism $\alpha_{\mathscr{B},\mathscr{C}} : \mathscr{B} \to \mathscr{C}$, where $\alpha_{\mathscr{B},\mathscr{B}}$ is the identity map on $\mathscr{B}$, by Result 2.2. If $\mathscr{B},\mathscr{C},\mathscr{D} \in I$ and $\mathscr{B} \leq \mathscr{C} \leq \mathscr{D}$ then $\alpha_{\mathscr{B},\mathscr{C}} \circ \alpha_{\mathscr{C},\mathscr{D}} = \alpha_{\mathscr{B},\mathscr{D}}$, by uniqueness. Thus the automata in I, with the homomorphisms $\alpha_{\mathscr{B},\mathscr{C}}$, determine a directed system.

It is immediate that η is reflexive and symmetric. Suppose $v_1\eta v_2$ and $v_2\eta v_3$, for some $v_1 \in V(\mathscr{B})$, $v_2 \in V(\mathscr{C})$ and $v_3 \in V(\mathscr{D})$, where $\mathscr{B},\mathscr{C},\mathscr{D} \in I$. There exist $\mathscr{E},\mathscr{F} \in I$ with $\mathscr{B},\mathscr{C} \leq \mathscr{E}$ and $\mathscr{C},\mathscr{D} \leq \mathscr{F}$, where v_1 and v_2 are identified in $\mathscr{E}$, and v_2 and v_3 are identified in $\mathscr{F}$. There exists $\mathscr{G} \in I$ with $\mathscr{E},\mathscr{F} \leq \mathscr{G}$. Thus v_1 and v_3 are identified in $\mathscr{G}$ and so $v_1\eta v_3$. Hence η is transitive. Therefore η defines a V-equivalence. Put $\mathscr{H} = (\sqcup_{\mathscr{B}\in I}\mathscr{B})/\eta$.

Suppose $v_1 \to^x v_3$ is an edge in $\mathscr{B}$ and $v_2 \to^x v_4$ is an edge in $\mathscr{C}$, where $v_1\eta v_2$, for some $\mathscr{B},\mathscr{C} \in I$. There exists $\mathscr{D} \in I$ such that $\mathscr{B},\mathscr{C} \leq \mathscr{D}$, where the images of v_1 and v_2 are identified in $\mathscr{D}$. Then v_3 and v_4 are identified in $\mathscr{D}$, since $\mathscr{D}$ is deterministic. Thus v_3 and v_4 are identified in $\mathscr{H}$. Hence $\mathscr{H}$ is deterministic. For each $\mathscr{B} \in I$, we have a unique homomorphism $\beta_{\mathscr{B}} : \mathscr{B} \to \mathscr{H}$, induced by η. For all $\mathscr{B},\mathscr{C} \in I$ with $\mathscr{B} \leq \mathscr{C}$, we have $\alpha_{\mathscr{B},\mathscr{C}} \circ \beta_{\mathscr{C}} = \beta_{\mathscr{B}}$, by uniqueness.

Suppose we have an inverse automaton $\mathscr{H}'$ over X and a homomorphism $\gamma_{\mathscr{B}} : \mathscr{B} \to \mathscr{H}'$, for each $\mathscr{B} \in I$, such that $\alpha_{\mathscr{B},\mathscr{C}} \circ \gamma_{\mathscr{C}} = \gamma_{\mathscr{B}}$, for all $\mathscr{B},\mathscr{C} \in I$ with $\mathscr{B} \leq \mathscr{C}$. If $v_1\eta v_2$, where $v_1 \in V(\mathscr{B})$ and $v_2 \in V(\mathscr{C})$, for some $\mathscr{B},\mathscr{C} \in I$, then there exists $\mathscr{D} \in I$ with $\mathscr{B},\mathscr{C} \leq \mathscr{D}$, such that $(v_1)\alpha_{\mathscr{B},\mathscr{D}} = (v_2)\alpha_{\mathscr{C},\mathscr{D}}$. Thus we have $(v_1)\gamma_{\mathscr{B}} = (v_1)\alpha_{\mathscr{B},\mathscr{D}} \circ \gamma_{\mathscr{D}} = (v_2)\alpha_{\mathscr{C},\mathscr{D}} \circ \gamma_{\mathscr{D}} = (v_2)\gamma_{\mathscr{C}}$. Hence we have a map $\delta : V(\mathscr{H}) \to V(\mathscr{H}')$, defined by $(v\eta)\delta = (v)\gamma_{\mathscr{B}}$, for $v \in V(\mathscr{B})$ and $\mathscr{B} \in I$. For $\mathscr{B} \in I$ and any edge $v_1 \to^y v_2$ in $\mathscr{B}$, we define $(v_1\eta \to^y v_2\eta)\delta = (v_1 \to^y v_2)\gamma_{\mathscr{B}}$. Then we have a homomorphism $\delta : \mathscr{H} \to \mathscr{H}'$ with $\beta_{\mathscr{B}} \circ \delta = \gamma_{\mathscr{B}}$, for $\mathscr{B} \in I$. Now suppose $\delta' : \mathscr{H} \to \mathscr{H}'$ is a homomorphism such that $\beta_{\mathscr{B}} \circ \delta' = \gamma_{\mathscr{B}}$, for $\mathscr{B} \in I$. Then $(v)\beta_{\mathscr{B}} \circ \delta' = (v\eta)\delta' = (v)\gamma_{\mathscr{B}}$, for all $v \in V(\mathscr{B})$, and it follows that $\delta' = \delta$. Hence $\mathscr{H}$ is the direct limit of the directed system.

Suppose $w = x_1x_2\cdots x_n \in L[\mathscr{H}]$. Put $\mathscr{A} = (\alpha,\Gamma,\beta)$. Then there exists $\mathscr{B}_k \in I$ containing an edge $v_k \to^{x_k} y_k$, for each k, with $\alpha\eta v_1$, $y_k\eta v_{k+1}$, for $k \leq n-1$, and $y_n\eta\beta$. Put $y_0 = \alpha$, $v_{n+1} = \beta$ and $\mathscr{B}_0 = \mathscr{B}_{n+1} = \mathscr{A}$. Then, for $0 \leq k \leq n$, there exists $\mathscr{C}_k \in I$ with $\mathscr{B}_k,\mathscr{B}_{k+1} \leq \mathscr{C}_k$ and the images of the vertices y_k and v_{k+1} are identified in $\mathscr{C}_k$. By induction on n, there exists $\mathscr{B} \in I$ with $\mathscr{C}_0,\mathscr{C}_1,\ldots,\mathscr{C}_n \leq \mathscr{B}$. The images of y_k and v_{k+1} are identified in $\mathscr{B}$, for $0 \leq k \leq n$. Thus we have $w \in L[\mathscr{B}]$. □

In Stephen [16], it was shown that the direct limit of the directed system of all automata obtained from the linear automaton of w by finitely many expansions, relative to $\langle X \mid R\rangle$, is the automaton $\mathscr{A}(X,R,w)$. This result is included in the following.

Corollary 3.3. *Let $\mathscr{A}$ be a deterministic inverse automaton over X and let R_1, R_2 denote sets of relations. Then we have a directed system of all automata obtained from $\mathscr{A}$ by finite applications of performing an elementary expansion, relative to $\langle X \mid R_1\rangle$, and taking the closed form, relative to $\langle X \mid R_2\rangle$. The direct limit $\mathscr{E}$ is the closed form of $\mathscr{A}$, relative to $\langle X \mid R_1 \cup R_2\rangle$. Thus if we have $\mathscr{A} \rightsquigarrow \mathscr{A}(X,R_1\cup R_2,w)$ then $\mathscr{E} \cong \mathscr{A}(X,R_1\cup R_2,w)$.*

Proof. If $\mathscr{C}$ is obtained from $\mathscr{B}$ by performing an elementary expansion, relative to $\langle X \mid R_1\rangle$, and then taking the closed form, relative to $\langle X \mid R_2\rangle$, then $\mathscr{C}$ is deterministic and $L[\mathscr{B}] \subseteq L[\mathscr{C}]$. Thus condition (A) of Lemma 3.2 is satisfied.

Let v_1,v_2,v_3,v_4 denote vertices of $\mathscr{B}$. Let $\mathscr{C}_1$ be obtained from $\mathscr{B}$ by sewing on the linear automaton of s from v_1 to v_2. Then let $\mathscr{C}_2$ be the closed form of $\mathscr{C}_1$, relative to $\langle X \mid R_2\rangle$. Next let $\mathscr{C}_3$ be obtained from $\mathscr{C}_2$ by sewing on the linear automaton of t from the image of v_3 to the image of v_4. Then let $\mathscr{C}_4$ be the closed form of $\mathscr{C}_3$, relative to $\langle X \mid R_2\rangle$. Similarly, let $\mathscr{D}_1$ be obtained from $\mathscr{B}$ by sewing on the linear automaton of t from v_3 to v_4. Then let $\mathscr{D}_2$ be the closed form of $\mathscr{D}_1$, relative to $\langle X \mid R_2\rangle$. Next let $\mathscr{D}_3$ be obtained from $\mathscr{D}_2$ by sewing on the linear automaton of s from the image of v_1 to the image of v_2. Then let $\mathscr{D}_4$ be the closed form of $\mathscr{D}_3$, relative to $\langle X \mid R_2\rangle$.

It is immediate that $L[\mathscr{B}] \subseteq L[\mathscr{D}_1]$. Now $L[\mathscr{D}_1] \subseteq L[\mathscr{D}_2]$, by Result 2.12. Thus $L[\mathscr{B}] \subseteq L[\mathscr{D}_2]$. It follows that $L[\mathscr{C}_1] \subseteq L[\mathscr{D}_3]$. Then, from Result 2.12, we have $L[\mathscr{C}_2] \subseteq L[\mathscr{D}_4]$. It follows that $L[\mathscr{C}_3] \subseteq L[\mathscr{D}_4]$, since there is a path labeled by t from the image of v_3 to the image of v_4 in $\mathscr{D}_4$. Then, again from Result 2.12, we have $L[\mathscr{C}_4] \subseteq L[\mathscr{D}_4]$. A similar proof shows that $L[\mathscr{D}_4] \subseteq L[\mathscr{C}_4]$. Hence $\mathscr{C}_4 \cong \mathscr{D}_4$, by Result 2.2. It now follows that condition (B) of Lemma 3.2 is satisfied.

Hence, from Lemma 3.2, we have a directed system of all automata obtained from $\mathscr{A}$ by finite applications of performing an elementary expansion, relative to $\langle X \mid R_1\rangle$, and then taking the closed form, relative to $\langle X \mid R_2\rangle$. Let $\mathscr{E}$ denote the direct limit of this directed system and let $\mathscr{F}$ denote the closed form of $\mathscr{A}$, relative to $\langle X \mid R_1 \cup R_2\rangle$.

Let $w \in L[\mathscr{E}]$. Then, by Lemma 3.2, there exists an automaton $\mathscr{B}$ in the directed system for $\mathscr{E}$ such that $w \in L[\mathscr{B}]$. The automaton $\mathscr{B}$ is obtained from $\mathscr{A}$ by performing an elementary expansion, relative to $\langle X \mid R_1\rangle$, and taking the closed form, relative to $\langle X \mid R_2\rangle$, a finite number of times. If $\mathscr{A}'$ is obtained from $\mathscr{A}$ by an elementary expansion, relative to $\langle X \mid R_1\rangle$, then $z \in L[\mathscr{A}']$ implies $z \geq y$ in $Inv\langle X \mid R_1\rangle$, for some $y \in L[\mathscr{A}]$, by Result 2.12. If $\mathscr{A}''$ is the closed form of $\mathscr{A}'$, relative to $\langle X \mid R_2\rangle$, then $z \in L[\mathscr{A}'']$ implies $z \geq y$ in $Inv\langle X \mid R_2\rangle$, for some $y \in L[\mathscr{A}']$, by Result 2.12. It now follows that $z \in L[\mathscr{B}]$ implies $z \geq y$ in $Inv\langle X \mid R_1 \cup R_2\rangle$, for some $y \in L[\mathscr{A}]$. Thus $w \geq y$ in $Inv\langle X \mid R_1 \cup R_2\rangle$, for some $y \in L[\mathscr{A}]$. Hence $L[\mathscr{E}] \subseteq L[\mathscr{F}]$, by Result 2.12.

Conversely, let $w \in L[\mathscr{F}]$. There exists $\mathscr{C}$ in the directed system for $\mathscr{F}$ with $w \in L[\mathscr{C}]$. Then $\mathscr{C}$ is obtained from $\mathscr{A}$ by a finite sequence of elementary expansions and elementary determinations, relative to $\langle X \mid R_1 \cup R_2\rangle$. Now if $\mathscr{A}'$ is obtained from $\mathscr{A}$ by either an elementary expansion or an elementary determination, relative to $\langle X \mid R_1 \cup R_2\rangle$, then there exists an automaton $\mathscr{B}$ obtained from $\mathscr{A}$ by an elementary expansion, relative to $\langle X \mid R_1\rangle$, and taking the closed form, relative to $\langle X \mid R_2\rangle$, with $L[\mathscr{A}'] \subseteq L[\mathscr{B}]$. If $\mathscr{A}''$ is obtained from $\mathscr{A}'$ by an elementary expansion or an elementary determination, relative to $\langle X \mid R_1 \cup R_2\rangle$, then there exists an automaton $\mathscr{B}'$ where either $\mathscr{B}' = \mathscr{B}$ or $\mathscr{B}'$ is obtained from $\mathscr{B}$ by an elementary expansion, relative to $\langle X \mid R_1\rangle$, and taking the closed form, relative to $\langle X \mid R_2\rangle$, with $L[\mathscr{A}''] \subseteq L[\mathscr{B}']$. It follows that there exists $\mathscr{B}''$ in the directed system for $\mathscr{E}$ with $L[\mathscr{C}] \subseteq L[\mathscr{B}'']$ and so $w \in L[\mathscr{E}]$. Thus $L[\mathscr{F}] \subseteq L[\mathscr{E}]$. Hence $\mathscr{E} \cong \mathscr{F}$, by Result 2.2.

Suppose $\mathscr{A} \rightsquigarrow \mathscr{A}(X, R_1 \cup R_2, w)$. By Results 2.3 and 2.12, we have $\mathscr{E} \rightsquigarrow \mathscr{A}(X, R_1 \cup R_2, w)$. Since $\mathscr{E}$ is closed, relative to $\langle X \mid R_1 \cup R_2\rangle$, we have $\mathscr{E} \cong \mathscr{A}(X, R_1 \cup R_2, w)$, by Result 2.9. □

Corollary 3.4. *Let $\mathscr{A}$ be a deterministic inverse automaton over X. Let V_1 be a subset of the vertices of $\mathscr{A}$ and let R_1, R_2 be sets of relations. Then we have a directed system of all automata obtained from $\mathscr{A}$ by finite applications of an elementary expansion relative to $\langle X \mid R_1\rangle$, between vertices of V_1 or their images, and taking the closed form relative to $\langle X \mid R_2\rangle$.*

Proof. The proof is similar to that in Corollary 3.3. □

4. A GENERALISATION OF LOWER BOUNDED AMALGAMS

Definition 4.1. We extend the terminology of [1] and [2] by calling an inverse subsemigroup U *lower bounded in S* if for any $u \in U$ and $e \in E(S)$ with $u \geq e$ there exists $f \in E(U)$ with $u \geq f \geq e$. The lower bounded subsemigroup condition is illustrated in Figure 4.1.

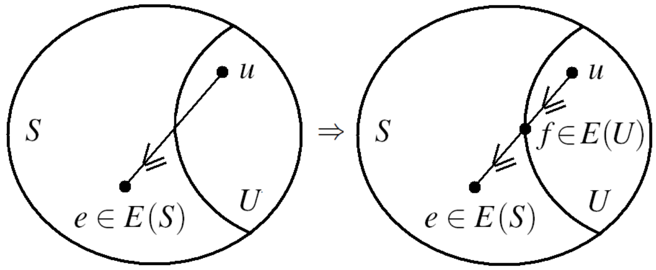

FIGURE 1. The lower bounded subsemigroup condition.

In this section, let $[S_1, S_2; U; \phi_1, \phi_2]$ be an amalgam of inverse semigroups where U is lower bounded in S_1 and S_2.

Notation 4.2. For $i \in \{1,2\}$, we assume that S_i has a given inverse semigroup presentation, which we refer to by $\langle S_i \rangle$. We then denote the Schützenberger automaton of $s \in S_i$, relative to $\langle S_i \rangle$, by $\mathscr{A}(S_i, s)$. If Δ is isomorphic to a Schützenberger graph of $\langle S_i \rangle$ and $v \in V(\Delta)$ then we let $e_i(v)$ denote the unique idempotent of $E(S_i)$ such that $(v, \Delta, v) \cong \mathscr{A}(S_i, e_i(v))$.

We have an inverse semigroup presentation $\langle S_1 * S_2 \rangle = \langle S_i \cup S_2 \rangle$ for the free product $S_1 * S_2$, in the variety of inverse semigroups. We denote the Schützenberger automaton of z, relative to $\langle S_1 * S_2 \rangle$, by $\mathscr{A}(S_1 * S_2, z)$, for any word z in the generators of S_1 and S_2.

For $u \in E(U)$, let $w_i(u)$ be a word in the generators of S_i that equals u in S_i, for $i \in \{1,2\}$. We have a presentation $\langle S_1 *_U S_2 \rangle = \langle S_1 \cup S_2 \mid W \rangle$ for the amalgamated free product $S_1 *_U S_2$, where $W = \{(w_1(u), w_2(u)) : u \in U\}$. We assume that $w_1(u)$ and $w_2(u)$ are calculable, for $u \in U$. We let $\mathscr{A}(S_1 *_U S_2, z)$ denote the Schützenberger automaton of z, relative to $\langle S_1 *_U S_2 \rangle$.

The following generalises the definitions of [1, Sections 2 and 3].

Definition 4.3. Let Γ be an inverse word graph over the generators of S_1 and S_2. If v is a vertex of Γ that belongs to a lobe colored by i then we denote this lobe by $\Delta_i(v)$, for $i \in \{1,2\}$. A *lobe path* is a finite sequence of lobes $\Delta_1, \Delta_2, \ldots, \Delta_n$, where Δ_k is adjacent to Δ_{k+1}, for $1 \le k \le n-1$. The path is *reduced* if it is not of the form $\Delta_1, \Delta_2, \Delta_1$ and all the lobes in the sequence are distinct, except possibly the first and last. There is a unique reduced lobe path between any two lobes if and only if there are no non-trivial reduced lobe loops. The *lobe graph* of Γ is the graph with vertices consisting of the lobes of Γ and edges consisting of all (Δ_1, Δ_2), from a lobe Δ_1 colored by 1 to a lobe Δ_2 colored by 2, whenever Δ_1 and Δ_2 are adjacent in Γ. The lobe graph of an inverse automaton, over the generators of S_1 and S_2, is the lobe graph of the underlying graph. The lobe graph of Γ is a tree if and only if there are no non-trivial reduced lobe loops. The graph Γ is cactoid if and only if the lobe graph of Γ is a finite tree and adjacent lobes share precisely one intersection. A lobe is *extremal* if it is adjacent to precisely one other lobe.

The graph Γ has the *idempotent property relative to* $\langle S_i \rangle$, for $i \in \{1,2\}$, if for every loop $v \rightarrow^s v$ in Γ, where $s \in S_i$, there is a loop $v \rightarrow^e v$, for some $e \in E(S_i)$ with $s \geq e$ in S_i. An inverse automaton over the generators of S_1 and S_2 has the idempotent property relative to $\langle S_i \rangle$ if its underlying graph does. The Schützenberger graphs of $\langle S_i \rangle$ have the idempotent property relative to $\langle S_i \rangle$. A product of automata with the idempotent property relative to $\langle S_i \rangle$ also has the idempotent property relative to $\langle S_i \rangle$. We say that Γ has the *idempotent property* if it has the idempotent property relative to $\langle S_1 \rangle$ and $\langle S_2 \rangle$. An inverse automaton over the generators of S_1 and S_2 has the idempotent property if its underlying graph does. The Schützenberger graphs of $\langle S_1 * S_2 \rangle$ have the idempotent property, by Result 2.13.

Suppose Γ is a cactoid graph with the idempotent property. Let Γ' be obtained from Γ by closing a lobe Δ of Γ colored by i, relative to $\langle S_i \rangle$. Let $v \in V(\Delta)$ and let v' denote the image of v in Γ'. Then $w \in L[(v', \Delta_i(v'), v')]$ implies $w \geq y$ in S_i, for some $y \in L[(v, \Delta, v)]$, by Result 2.12. In which case, $w \geq y \geq e$ in S_i, for some $e \in E(S_i)$ with $e \in L[(v, \Delta, v)]$, by the idempotent property. Thus $\Delta_i(v')$ has the idempotent property relative to $\langle S_i \rangle$. If v is an intersection of Δ then $(v', \Delta_j(v'), v')$ is isomorphic to a product $\Pi_x(x, \Delta_j(x), x)$, where the product is taken over all intersections x of Δ that are identified with v' in Γ', where $\Delta_j(x)$ denotes the lobe of Γ colored by j that contains x. Thus $\Delta_j(v')$ has the idempotent property relative to $\langle S_j \rangle$. It follows that Γ' also has the idempotent property. More generally, if Γ is a cactoid graph with the idempotent property then the closed form of Γ, relative to $\langle S_1 * S_2 \rangle$, also has the idempotent property.

The graph Γ has the *equality property* if, for every intersection v, there is loop $v \rightarrow^u v$ in $\Delta_1(v)$ if and only if there is loop $v \rightarrow^u v$ in $\Delta_2(v)$, for all $u \in U$. An inverse automaton over the generators of S_1 and S_2 has the equality property if its underlying graph does.

For any intersection v of Γ, the set of *related pairs* of v consists of (v,v) and all pairs (v_1, v_2) of vertices for which we have a path $v \rightarrow^u v_1$ in $\Delta_1(v)$ and a path $v \rightarrow^u v_2$ in $\Delta_2(v)$, for some $u \in U$. If (v_1, v_2) is a related pair then v_1 and v_2 are called its *coordinates*. The graph Γ has the *separation* *property* if the related pairs of any two intersections, not belonging to the same pair of lobes, share no common coordinates. An inverse automaton over the generators of S_1 and S_2 has the separation property if its underlying graph does.

Lemma 4.4. *Let Γ be an inverse word graph over the generators of S_1 and S_2 that is closed, relative to $\langle S_1 * S_2 \rangle$, and has the equality property. Then, for any intersection v of Γ, the set of related pairs of v defines a partial one-one map between $V(\Delta_1(v))$ and $V(\Delta_2(v))$.*

Proof. Suppose (v_1, v_2) and (v_1', v_2) are related pairs of v. Then we have paths $v \to^{u} v_1$, $v \to^{u'} v_1'$ in $\Delta_1(v)$ and paths $v \to^{u} v_2$, $v \to^{u'} v_2$ in $\Delta_2(v)$, for some $u, u' \in U^1$. Since Γ has the equality property, we then have a loop $v \to^{u'u^{-1}} v$ in $\Delta_1(v)$. Since Γ is closed, relative to $\langle S_1 * S_2 \rangle$, we must have $v_1 = v_1'$. Similarly, if (v_1, v_2) and (v_1, v_2') are related pairs of v then $v_2 = v_2'$. We have a partial one-one map $V(\Delta_1(v)) \to V(\Delta_2(v))$, defined by $v_1 \to v_2$, for each related pair (v_1, v_2) of v. □

The following generalises [1, Construction 2.1].

Construction 4.5. Let $\mathscr{A}$ be a cactoid inverse automaton over the generators of S_1 and S_2 that is closed, relative to $\langle S_1 * S_2 \rangle$. Suppose v is an intersection of $\mathscr{A}$ and we have a loop $v \to^{f} v$ in $\Delta_i(v)$, for some $f \in E(U)$, and no loop $v \to^{f} v$ in $\Delta_j(v)$, for some $i \in \{1, 2\}$ and $j = 3 - i$. Let $\mathscr{A}'$ be the closed form relative to $\langle S_1 * S_2 \rangle$ of the automaton obtained from $\mathscr{A}$ by sewing on the linear automaton of $w_j(f)$ at the intersection v. In Figure 2, the circles and dots represent lobes and vertices of $\mathscr{A}$, arrows represent paths and the dashed arrow represents the linear automaton of $w_j(f)$.

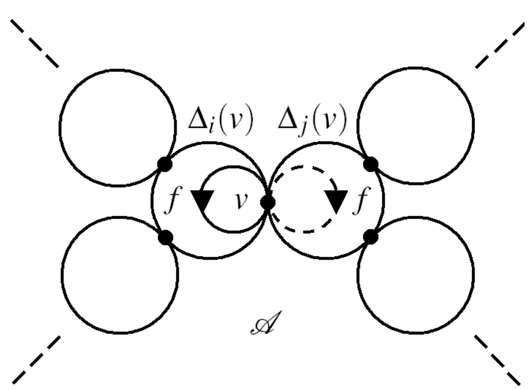

FIGURE 2. Construction 4.5 illustrated.

Lemma 4.6. *Let $\mathscr{A}$ be a cactoid inverse automaton over the generators of S_1 and S_2 that is closed, relative to $\langle S_1 * S_2\rangle$.*

(i) *We have a directed system of all automata obtained from $\mathscr{A}$ by finite applications of Construction 4.5.*

(ii) *The direct limit $\mathscr{B}$ is cactoid and closed, relative to $\langle S_1 * S_2\rangle$.*

(iii) *We have a graph homomorphism from the lobe tree of $\mathscr{A}$ onto the lobe tree of $\mathscr{B}$.*

(iv) *If $\mathscr{A} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$ then $\mathscr{B} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$.*

(v) *If $\mathscr{A}$ has the idempotent property then $\mathscr{B}$ has the idempotent property.*

(vi) *If $\mathscr{B}$ has the idempotent property then $\mathscr{B}$ has the equality property.*

Proof. Only (vi) assumes that U is lower bounded in S_1 and S_2:

(i) We have a directed system from Corollary 3.4.

(ii) Put $\mathscr{A} = (\alpha_1, \Gamma_1, \beta_1)$ and $\mathscr{B} = (\alpha_2, \Gamma_2, \beta_2)$. Since $\mathscr{A}$ is cactoid, it follows that any automaton in the directed system is also cactoid. Thus $\mathscr{B}$ must be cactoid. Let $v_1 \to^r v_2$ be a path in $\mathscr{B}$, where (r,s) is some relation of S_1 or S_2. Let $\alpha_2 \to^{w_1} v_1$ and $v_2 \to^{w_2} \beta_2$ be paths in $\mathscr{B}$. Then, from Lemma 3.2, there is some automaton $\mathscr{A}'$ in the directed system with $w_1 r w_2 \in L[\mathscr{A}']$. Since $\mathscr{A}'$ is closed, relative to $\langle S_1 * S_2\rangle$, we have $w_1 s w_2 \in L[\mathscr{A}']$. Thus $w_1 s w_2 \in L[\mathscr{B}]$. Hence $\mathscr{B}$ is closed, relative to $\langle S_1 * S_2\rangle$.

(iii) For $\mathscr{A}'$ in the directed system for $\mathscr{B}$, the homomorphism from $\mathscr{A}$ into $\mathscr{A}'$ induces a homomorphism from the lobe tree of $\mathscr{A}$ onto the lobe tree of $\mathscr{A}'$. It follows that the homomorphism from $\mathscr{A}$ into $\mathscr{B}$ induces a homomorphism from the lobe tree of $\mathscr{A}$ onto the lobe tree of $\mathscr{B}$.

(iv) Suppose $\mathscr{A} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$. Let $w' \in L[\mathscr{B}]$. From Lemma 3.2, there is some $\mathscr{A}'$ in the directed system for $\mathscr{B}$ with $w' \in L[\mathscr{A}']$. It follows, from Results 2.3, 2.6 and 2.12, that $\mathscr{A}' \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$. Thus $w' \in L[\mathscr{A}(S_1 *_U S_2, w)]$. Hence $\mathscr{B} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$.

(v) Suppose $\mathscr{A}$ has the idempotent property. If z defines an idempotent of $E(S_i)$, where $i \in \{1,2\}$, then, for $y \in L[(\alpha_z, \Gamma_z, \beta_z)/\eta]$, we have $y \geq z$ in S_i, where $(\alpha_z, \Gamma_z, \beta_z)$ is the linear automaton of z and η is the V-equivalence generated by (α_z, β_z). Thus any automaton obtained from $\mathscr{A}$ by sewing on the linear automaton of z, at some vertex, has the idempotent property. From Definition 4.3, the idempotent property is preserved under taking the closed form, relative to $\langle S_1 * S_2\rangle$, of a cactoid graph. It follows that any automaton in the directed system for $\mathscr{B}$ has the idempotent property. If we have a path $\alpha_2 \to^{w_1} v$ and a loop $v \to^{s} v$ in $\mathscr{B}$, where $s \in S_i$ and $i \in \{1,2\}$, then, letting $\alpha_1 \to^{w} \beta_1$ be a path in $\mathscr{A}$, there is some automaton $\mathscr{A}'$ in the directed system with $w_1 s w_1^{-1} w \in L[\mathscr{A}']$, by Lemma 3.2. Since the automaton $\mathscr{A}'$ has the idempotent property, we have $w_1 e w_1^{-1} w \in L[\mathscr{A}']$, for some $e \in E(S_i)$ with $s \geq e$ in S_i. We then have a loop $v \to^{e} v$ in $\mathscr{B}$. Hence $\mathscr{B}$ has the idempotent property.

(vi) Suppose $\mathscr{B}$ has the idempotent property. Let v be an intersection of $\mathscr{B}$ and suppose we have a loop $v \to^{w_i(u)} v$ in $\Delta_i(v)$, for some $u \in U$ and some $i \in \{1,2\}$. Put $j = 3 - i$. From the idempotent property, we have a loop $v \to^{e} v$ in $\Delta_i(v)$, for some $e \in E(S_i)$ with $u \geq e$. Since U is lower bounded in S_i, there exists $f \in E(U)$ with $u \geq f \geq e$ in S_i. Since $\mathscr{B}$ is closed, relative to $\langle S_1 * S_2\rangle$, we have a loop $v \to^{w_i(f)} v$ in $\Delta_i(v)$. Let v_1 denote an intersection of $\mathscr{A}$ that is a preimage of v and let $\alpha_1 \to^{w} \beta_1$ and $\alpha_1 \to^{z} v_1$ be paths in $\mathscr{A}$. We have $z \cdot w_i(f) \cdot z^{-1} w \in L[\mathscr{B}]$. There is some automaton $\mathscr{A}' = (\alpha_1', \Gamma_1', \beta_1')$ in the directed system for $\mathscr{B}$ such that $z \cdot w_i(f) \cdot z^{-1} w \in L[\mathscr{A}']$, by Lemma 3.2. Thus $\mathscr{A}'$ contains a path $\alpha_1' \to^{z} v'$ and a loop $v' \to^{w_i(f)} v'$, where v' is an intersection. We can obtain $\mathscr{A}''$ from $\mathscr{A}'$ by an application of Construction 4.5 such that we have a loop $v'' \to^{w_j(f)} v''$ in $\Delta_j(v'')$, letting v'' denote the image of v' in $\mathscr{A}''$. Thus we have a loop $v \to^{w_j(f)} v$ in $\Delta_j(v)$. Since $u \geq f$ in U, we have a loop $v \to^{w_j(u)} v$ in $\Delta_j(v)$. Hence $\mathscr{B}$ has the equality property.

□

Lemma 4.7. *Let $(\alpha_2, \Gamma_2, \beta_2)$ be the direct limit of the directed system of all automata obtained from $(\alpha_1, \Gamma_1, \beta_1)$, by finite applications of Construction 4.5. Let $\gamma_1, \delta_1 \in V(\Gamma_1)$ and let γ_2, δ_2 denote their respective images in Γ_2. Then $(\gamma_2, \Gamma_2, \delta_2)$ is the direct limit $\mathscr{C}$ of the directed system of all automata obtained from $(\gamma_1, \Gamma_1, \delta_1)$, by finite applications of Construction 4.5.*

Proof. Let $\gamma_1 \to^{w_1} \alpha_1$ and $\beta_1 \to^{w_2} \delta_1$ be paths in Γ_1. Let $y \in L[(\gamma_2, \Gamma_2, \delta_2)]$. Then we have $w_1^{-1} y w_2^{-1} \in L[(\alpha_2, \Gamma_2, \beta_2)]$ and so there is some automaton $(\alpha_1', \Gamma_1', \beta_1')$ in the directed system for $(\alpha_2, \Gamma_2, \beta_2)$ with $w_1^{-1} y w_2^{-1} \in L[(\alpha_1', \Gamma_1', \beta_1')]$, from Result 3.2. Now $(\alpha_1', \Gamma_1', \beta_1')$ is obtained from $(\alpha_1, \Gamma_1, \beta_1)$ by finite applications of Construction 4.5. Let γ_1' and δ_1' be the respective images of γ_1 and δ_1 in Γ_1'. It is immediate that $(\gamma_1', \Gamma_1', \delta_1')$ is obtained from $(\gamma_1, \Gamma_1, \delta_1)$ by the same applications of Construction 4.5. Thus $(\gamma_1', \Gamma_1', \delta_1')$ is in the directed system for $\mathscr{C}$. Since Γ_1' is deterministic, we have $y \in L[(\gamma_1', \Gamma_1', \delta_1')]$. Thus $y \in L[\mathscr{C}]$ and so $L[(\gamma_2, \Gamma_2, \delta_2)] \subseteq L[\mathscr{C}]$. Similarly, $L[\mathscr{C}] \subseteq L[(\gamma_2, \Gamma_2, \delta_2)]$. Hence $(\gamma_2, \Gamma_2, \delta_2) \cong \mathscr{C}$, by Result 2.2. □

Lemma 4.8. *Let $\mathscr{B}$ denote the direct limit of the directed system of all automata obtained from $\mathscr{A}$, by finite applications of Construction 4.5. Then there exists $\mathscr{A}'$ in the directed system for $\mathscr{B}$ such that the homomorphism from $\mathscr{A}'$ into $\mathscr{B}$ induces an isomorphism of the lobe trees.*

Proof. Put $\mathscr{A} = (\alpha_1, \Gamma_1, \beta_1)$ and let $w \in L[\mathscr{A}]$. Suppose the homomorphism from $\mathscr{A}$ into $\mathscr{B}$ maps distinct lobes Δ_1 and Δ_2 of $\mathscr{A}$ into a lobe of $\mathscr{B}$. Let the images of $v_1 \in V(\Delta_1)$ and $v_2 \in V(\Delta_2)$ be identified in $\mathscr{B}$. Let $\alpha_1 \to^{w_1} v_1$, $\alpha_1 \to^{w_2} v_2$ be paths in Γ_1. Then we have $w_1 w_2^{-1} w \in L[\mathscr{B}]$. There exists $\mathscr{A}'$ in the directed system for $\mathscr{B}$ with $w_1 w_2^{-1} w \in L[\mathscr{A}']$, from Result 3.2. Put $\mathscr{A}' = (\alpha_1', \Gamma_1', \beta_1')$. There are paths $\alpha_1' \to^{w_1} v_1'$ and $\alpha_1' \to^{w_2} v_1'$ in Γ_1'. Thus the images of v_1 and v_2 in $\mathscr{A}'$ are identified. Hence the homomorphism from $\mathscr{A}$ into $\mathscr{A}'$ maps Δ_1 and Δ_2 into a lobe of $\mathscr{A}'$. Since $\mathscr{A}$ has finitely many lobes, it follows that there exists $\mathscr{A}'$ in the directed system for $\mathscr{B}$ such that any lobes of $\mathscr{A}$ that are identified in $\mathscr{B}$ are also identified in $\mathscr{A}'$. Thus the lobe trees of $\mathscr{A}'$ and $\mathscr{B}$ must be isomorphic. □

Construction 4.9. Let $\mathscr{B} = (\alpha_2, \Gamma_2, \beta_2)$ be a cactoid inverse automaton over the generators of S_1 and S_2 that is closed, relative to $\langle S_1 * S_2 \rangle$. Suppose v_2 is an intersection, $\Delta_i(v_2)$ is extremal, where $i \in \{1,2\}$, $\alpha_2, \beta_2 \notin V(\Delta_i(v_2)) \backslash \{v_2\}$, and for any loop $v_2 \to^{y} v_2$ in $\Delta_i(v_2)$ there exists a loop $v_2 \to^{f} v_2$ in $\Delta_j(v_2)$, where $j = 3 - i$ and $f \in E(U)$ with $y \geq f$ in S_i. Then put $\mathscr{B}' \cong (\alpha_2, \Sigma_2, \beta_2)$, where Σ_2 is the subgraph of Γ_2 consisting of all the lobes except $\Delta_i(v_2)$.

Lemma 4.10. *Suppose $\mathscr{B}'$ is obtained from $\mathscr{B}$ by Construction 4.9. If we have $\mathscr{B} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$ then $\mathscr{B}' \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$.*

Proof. We adopt the notation of Construction 4.9. We have $L[\mathscr{B}'] \subseteq L[\mathscr{B}] \subseteq L[\mathscr{A}(S_1 *_U S_2, w)]$. Let $w' \in L[\mathscr{B}]$ such that $w' = w$ in $S_1 *_U S_2$. If $w' \in L[\mathscr{B}']$ then it is immediate that $\mathscr{B}' \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$. Otherwise we have $w' = z_0 y_1 z_1 \cdots y_n z_n$, for some $n \geq 1$, some paths $\alpha_2 \to^{z_0} v_2$, $v_2 \to^{z_k} v_2$, $v_2 \to^{z_n} \beta_2$ in Σ_2, where $k \leq n-1$, and some loops $v_2 \to^{y_k} v_2$ in $\Delta_i(v_2)$, where $k \leq n$, allowing z_0 and z_n to be 1. Then, by assumption, there is a loop $v_2 \to^{f_k} v_2$ in $\Delta_j(v_2)$, where $f_k \in E(U)$, $j = 3 - i$ and $y_k \geq f_k$ in S_i, for $1 \leq k \leq n$. Put $z = z_0 \cdot w_j(f_1) \cdot z_1 \cdot \cdots \cdot w_j(f_n) \cdot z_n$. Then $w' \geq z$ in $S_1 *_U S_2$, where $z \in L[\mathscr{B}']$. Hence $z = w$ in $S_1 *_U S_2$ and so we have $\mathscr{B}' \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$. □

Corollary 4.11. *For any word w, there exists a word w' such that we have $w = w'$ in $S_1 *_U S_2$, the automaton $\mathscr{A}(S_1 * S_2, w')$ has at most as many lobes as $\mathscr{A}(S_1 * S_2, w)$ and the direct limit of the directed system of all automata obtained from $\mathscr{A}(S_1 * S_2, w')$, by finite applications of Construction 4.5, has the property that Construction 4.9 cannot be applied.*

Proof. Let $\mathscr{B}$ denote the direct limit of the directed system of all automata obtained from $\mathscr{A} = \mathscr{A}(S_1 * S_2, w)$, by finite applications of Construction 4.5. By Lemma 4.6, the automaton $\mathscr{B}$ has at most as many lobes as $\mathscr{A}$.

Suppose $\mathscr{B}'$ is obtained from $\mathscr{B}$ by an application of Construction 4.9. Adopt the notation of Construction 4.9. By Lemma 4.6, we have $\mathscr{B} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$. Hence, from Lemma 4.10, we have $\mathscr{B}' \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$. Thus there exists a word $z \in L[\mathscr{B}']$ such that $z = w$ in $S_1 *_U S_2$. By Lemma 4.8, there exists $\mathscr{A}' = (\alpha_1', \Gamma_1', \beta_1')$ in the directed system for $\mathscr{B}$ such that the homomorphism from $\mathscr{A}'$ into $\mathscr{B}$ induces an isomorphism of the lobe trees. From Lemma 3.2, there exists $\mathscr{A}''$ in the directed system for $\mathscr{B}$ such that $z \in L[\mathscr{A}'']$. Since we have a directed system, we can assume $\mathscr{A}' = \mathscr{A}''$.

Let v_1' denote the unique intersection of $\mathscr{A}'$ that is a preimage of v_2. Then put $\mathscr{C} = (\alpha_1', \Sigma_1', \beta_1')$, where Σ_1' is the subgraph of Γ_1' consisting of all lobes, except for $\Delta_i(v_1')$. The automata $\mathscr{C} = (\alpha_1', \Sigma_1', \beta_1')$, $\mathscr{A}' = (\alpha_1', \Gamma_1', \beta_1')$, $\mathscr{B}' = (\alpha_2, \Sigma_2, \beta_2)$ and $\mathscr{B} = (\alpha_2, \Gamma_2, \beta_2)$ are illustrated in Figure 3. For some word w', we have $\mathscr{C} \cong \mathscr{A}(S_1 * S_2, w')$. Now $L[\mathscr{C}] \subseteq L[\mathscr{B}']$. Also, $z \in L[\mathscr{B}']$

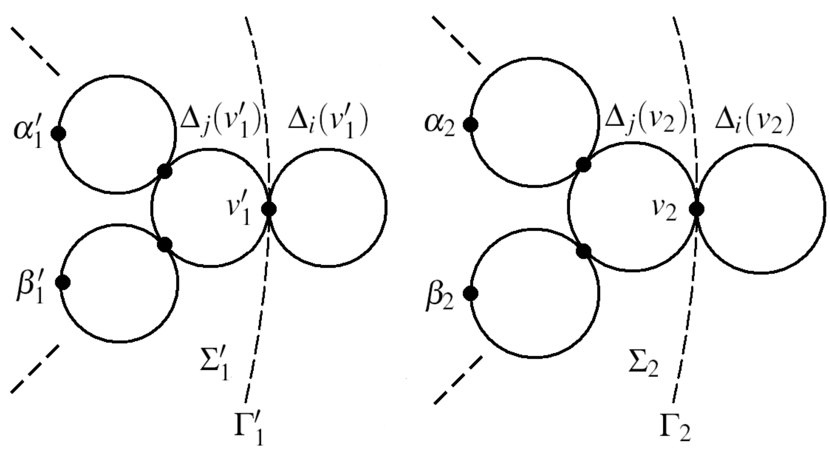

FIGURE 3. The automata $\mathscr{C}$, $\mathscr{A}'$, $\mathscr{B}'$ and $\mathscr{B}$.

implies $z \in L[\mathscr{C}]$, by the properties of $\mathscr{A}'$. Thus $\mathscr{C} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$. Hence we have $w' = w$ in

$S_1 *_U S_2$ and $\mathscr{C}$ has fewer lobes than $\mathscr{B}$. Let $\mathscr{D}$ denote the direct limit of the directed system of all automata obtained from $\mathscr{C}$, by finite applications of Construction 4.5. By Lemma 4.6, the automaton $\mathscr{D}$ has at most as many lobes as $\mathscr{C}$. Thus $\mathscr{D}$ has fewer lobes than $\mathscr{B}$. Since $\mathscr{B}$ has finite lobes, we can continue in this manner until no application of Construction 4.9 can be applied. □

The following generalises [1, Construction 3.3].

Construction 4.12. Let $\mathscr{B} = (\alpha_2, \Gamma_2, \beta_2)$ be a cactoid inverse automaton over the generators of S_1 and S_2 that is closed, relative to $\langle S_1 * S_2 \rangle$. Suppose there are paths $v_1 \to^{w_i(u)} v_0$ and $v_1 \to^{w_j(u)} v_2$ in Γ_2, where v_0 and v_1 are two intersections, for some $u \in U$, $i \in \{1,2\}$ and $j = 3 - i$. Figure 4 illustrates the situation. Since the lobe graph of Γ_2 is a tree, the unique reduced lobe path from a lobe of

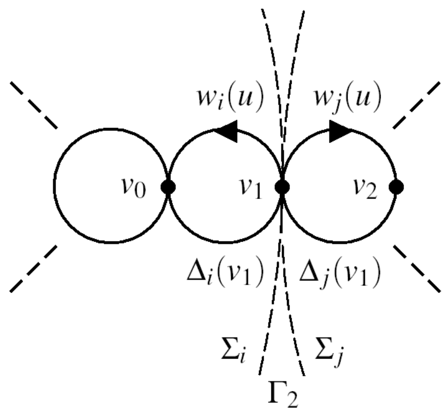

FIGURE 4. Construction 4.12 illustrated.

Γ_2 to $\Delta_i(v_1)$ either contains $\Delta_j(v_1)$ or does not. Let Σ_i denote the subgraph of Γ_2 containing $\Delta_i(v_1)$ and any lobe where the unique reduced lobe path to $\Delta_i(v_1)$ does not contain $\Delta_j(v_1)$. Similarly, let Σ_j denote the subgraph of Γ_2 containing $\Delta_j(v_1)$ and any lobe where the unique reduced lobe path to $\Delta_j(v_1)$ does not contain $\Delta_i(v_1)$. Thus $\Sigma_i \cup \Sigma_j = \Gamma_2$ and $\Sigma_i \cap \Sigma_j$ consists of v_1.

Let Σ_i^* and Σ_j^* denote disjoint copies of Σ_i and Σ_j, respectively. If $\alpha_2 \neq v_1$ then let α^* denote the unique image of α_2 in $\Sigma_i^* \cup \Sigma_j^*$. It $\alpha_2 = v_1$ then let α^* denote the image of v_1 in Σ_i^*. Define β^* similarly. Then let η denote the V-equivalence on $\Sigma_i^* \cup \Sigma_j^*$ generated by $\{(v_0, v_2)\}$, letting v_0 and v_2 denote their unique images in Σ_i^* and Σ_j^*, respectively. Put $\mathscr{C} = (\alpha^*\eta, (\Sigma_i^* \cup \Sigma_j^*)/\eta, \beta^*\eta)$. Let $\mathscr{B}'$ denote the closed form of $\mathscr{C}$ relative to $\langle S_1 * S_2 \rangle$.

Lemma 4.13. *Suppose $\mathscr{B}'$ is obtained from $\mathscr{B}$ by Construction 4.12.*

(i) *The automaton* $\mathscr{B}'$ *is cactoid and has fewer lobes than* $\mathscr{B}$.

(ii) *If* $\mathscr{B} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$ *then* $\mathscr{B}' \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$.

(iii) *If* $\mathscr{B}$ *has the idempotent property then so does* $\mathscr{B}'$.

Proof. We adopt the notation of Construction 4.12.

(i) Since v_2 is identified with v_0, the automaton $\mathscr{C}$ is cactoid and has fewer lobes than $\mathscr{B}$. Thus $\mathscr{B}'$ is cactoid and has fewer lobes than $\mathscr{B}$.

(ii) Suppose $\mathscr{B} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$. Let $\alpha_2 \to^z v_1$ denote a path in either Σ_i or Σ_j. Since $L[\mathscr{B}] \subseteq L[\mathscr{A}(S_1 *_U S_2, w)]$, the languages $L[(v_0, \Sigma_i, v_0)]$ and $L[(v_2, \Sigma_i, v_2)]$ are contained in $L[\mathscr{A}(S_1 *_U S_2, u^{-1}z^{-1}ww^{-1}zu)]$ and we have $ww^{-1}zu\mathscr{R}ww^{-1}$ in $S_1 *_U S_2$. Then the language $L[(v_0\eta, (\Sigma_1^* \cup \Sigma_j^*)/\eta, v_0\eta)]$ is contained in $L[\mathscr{A}(S_1 *_U S_2, u^{-1}z^{-1}ww^{-1}zu)]$. Since we have a path $\alpha^*\eta \to^{zu} v_0\eta$ in $(\Sigma_1^* \cup \Sigma_j^*)/\eta$, the language $L[(\alpha^*\eta, (\Sigma_1^* \cup \Sigma_j^*)/\eta, \alpha^*\eta)]$ is contained in $L[\mathscr{A}(S_1 *_U S_2, ww^{-1})]$. We show there exists $y \in L[\mathscr{C}]$ with $y = w$ in $S_1 *_U S_2$. It will follow that $\mathscr{C} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$ and so $\mathscr{B}' \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$, by Result 2.12.

Let $z \in L[\mathscr{B}]$ such that $z = w$ in $S_1 *_U S_2$. If $\alpha_2 \to^z \beta_2$ is contained in Σ_i then put $y = z$. If $\alpha_2 \to^z \beta_2$ is contained in Σ_j then put $y = z_0 z z_1$, where $z_0 = w_i(u) \cdot (w_j(u))^{-1}$ if $\alpha_2 = v_1$, else $z_0 = 1$, and $z_1 = w_j(u) \cdot (w_i(u))^{-1}$ if $\beta_2 = v_1$, else $z_1 = 1$. Then we have $y \in \mathscr{C}$ and $y = w$ in $S_1 *_U S_2$.

Now suppose $\alpha_2 \to^z \beta_2$ is not contained in Σ_i and not contained in Σ_j. Then $\alpha_2 \to^z \beta_2$ is the concatenation of subpaths $\alpha_2 \to^{w_0} v_1$, $v_1 \to^{w_1} v_1$, …, $v_1 \to^{w_n} \beta_2$, for some $n \geq 2$, where each subpath is either contained in Σ_i or contained in Σ_j. If the subpath labeled by w_k is contained in Σ_i then put $y_k = w_k$. If the subpath labeled by w_k is contained in Σ_j then put $y_k = z_0 w_k z_1$, where $z_0 = w_i(u) \cdot (w_j(u))^{-1}$ if the subpath starts at v_1, else $z_0 = 1$, and $z_1 = w_j(u) \cdot (w_i(u))^{-1}$ if the subpath ends at v_1, else $z_1 = 1$. Put $y = y_1 y_2 \cdots y_n$. Then we have $y \in \mathscr{C}$ and $y = w$ in $S_1 *_U S_2$.

(iii) Suppose $\mathscr{B}$ has the idempotent property. Then the automata (v_0, Σ_i, v_0) and (v_2, Σ_i, v_2) have the idempotent property. It follows that $\mathscr{C}$ has the idempotent property. Then $\mathscr{B}'$ has the idempotent property, since $\mathscr{C}$ is cactoid and so the idempotent property is preserved under the operation of taking the closed form, relative to $\langle S_1 * S_2 \rangle$, from the workings in Definition 4.3.

□

Corollary 4.14. *For any word w, there is a word w' with $w = w'$ in $S_1 *_U S_2$, $\mathscr{A}(S_1 * S_2, w')$ has at most as many lobes as $\mathscr{A}(S_1 * S_2, w)$ and the direct limit of the directed system of all automata obtained from $\mathscr{A}(S_1 * S_2, w')$, by finite applications of Construction 4.5, has the separation property.*

Proof. Let $\mathscr{A} = \mathscr{A}(S_1 * S_2, w)$. Let $\mathscr{B} = (\alpha_2, \Gamma_2, \beta_2)$ denote the direct limit of the directed system of all automata obtained from $\mathscr{A}$, by finite applications of Construction 4.5. By Lemma 4.6, the automaton $\mathscr{B}$ is cactoid and has the equality property, with at most as many lobes as $\mathscr{A}$.

Suppose $\mathscr{B}$ does not have the separation property. Then there are distinct intersections v_0 and v_1 of $\mathscr{B}$ and a path $v_1 \rightarrow^{w_i(u)} v_0$ in $\Delta_i(v_1)$, for some $u \in U$ and $i \in \{1,2\}$. Put $j = 3 - i$. Since $\mathscr{B}$ has the equality property, there is a path $v_1 \rightarrow^{w_j(u)} v_2$ in $\Delta_j(v_1)$. Let $\alpha_2 \rightarrow^{z} v_1$ be a path in $\mathscr{B}$.

By Lemma 4.8, there exists $\mathscr{A}' = (\alpha_1', \Gamma_1', \beta_1')$ in the directed system for $\mathscr{B}$ such that the homomorphism from $\mathscr{A}'$ into $\mathscr{B}$ induces an isomorphism of the lobe trees. By Lemma 3.2, there exists $\mathscr{A}''$ in the directed system for $\mathscr{B}$ with $z \cdot w_i(u) \cdot w_j(uu^{-1}) \cdot w_i(u^{-1}) \cdot w_j(uu^{-1}) \cdot z^{-1} w \in L[\mathscr{A}'']$. Since we have a directed system, we can assume that $\mathscr{A}' = \mathscr{A}''$. Thus we have paths $\alpha_1' \rightarrow^{z} v_1'$, $v_1' \rightarrow^{w_i(u)} v_0'$ and $v_1' \rightarrow^{w_j(u)} v_2'$ in $\mathscr{A}'$, where v_0', v_1' and v_2' are preimages of v_0, v_1 and v_2, respectively, and v_0' and v_1' are two intersections of $\mathscr{A}'$.

Let $\mathscr{A}'''$ be obtained from $\mathscr{A}'$ by an application of Construction 4.12. Then $\mathscr{A}'''$ has fewer lobes than $\mathscr{A}'$ and so $\mathscr{A}'''$ has fewer lobes than $\mathscr{B}$. Thus $\mathscr{A}'''$ has fewer lobes than $\mathscr{A}$, by Lemma 4.6. From Construction 2.14 and Result 2.15, we have $\mathscr{A}''' \cong \mathscr{A}(S_1 * S_2, w')$, for some word w'. Also, from Lemma 4.13, we have $\mathscr{A}''' \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$.

Let $\mathscr{B}'$ denote the direct limit of the directed system of all automata obtained from $\mathscr{A}'''$, by finitely many applications of Construction 4.5. Then $\mathscr{B}'$ has at most as many lobes as $\mathscr{A}'''$, by Lemma 4.6. Hence $\mathscr{B}'$ has fewer lobes than $\mathscr{B}$. Continuing in this manner, we reach such an automaton $\mathscr{A}'''$ where the direct limit $\mathscr{B}'$ has the separation property. □

The following generalises the definitions of [1, Sections 4 and 5].

Definition 4.15. Let Γ be an inverse word graph over the generators of S_1 and S_2 that is closed, relative to $\langle S_1 * S_2 \rangle$. We say that an intersection v of Γ has *identified related pairs* if every related pair of v is of the form (v', v'), for some intersection v' of $\Delta_1(v)$ and $\Delta_2(v)$. If, in addition, (v', v') is a related pair of v, for every intersection v' of $\Delta_1(v)$ and $\Delta_2(v)$, then we say that $\Delta_1(v)$ and $\Delta_2(v)$ are *assimilated* by v. Figure 5 illustrates a graph where the related pairs are not identified and a graph where the lobes are assimilated. If $\Delta_1(v)$ and $\Delta_2(v)$ are assimilated by v then they are assimilated by every

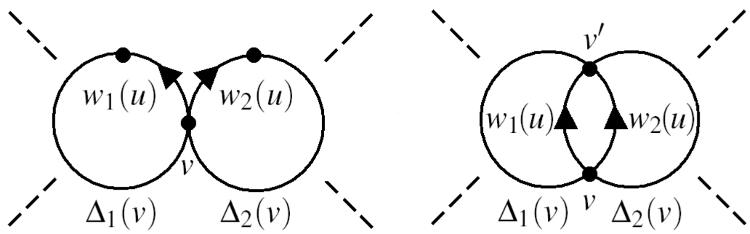

FIGURE 5. Unidentified related pairs and assimilated lobes.

intersection of $\Delta_1(v)$ and $\Delta_2(v)$. The graph Γ has the *assimilation property* if any two adjacent lobes are assimilated by a common intersection. The assimilation property implies the separation property, An inverse automaton over the generators of S_1 and S_2 that is closed, relative to $\langle S_1 * S_2 \rangle$, has the assimilation property if its underlying graph does.

If Γ has the equality property then the related pairs of any intersection v define a partial one-one map between $V(\Delta_1(v))$ and $V(\Delta_2(v))$, by Lemma 4.4. If Γ has the equality and separation properties and v is the only intersection of $\Delta_1(v)$ and $\Delta_2(v)$, then we can *assimilate* $\Delta_1(v)$ and $\Delta_2(v)$ by taking the quotient of Γ by the V-equivalence η generated by the related pairs of v. Since Γ has the separation property, each lobe of Γ is mapped isomorphically onto a lobe of Γ/η, under η. It follows that Γ/η is closed, relative to $\langle S_1 * S_2 \rangle$, and has the equality and separation properties.

More generally, suppose Γ has the equality and separation properties and adjacent lobes of Γ have precisely one intersection. Then the *assimilated form* of Γ is the quotient of Γ by the V-equivalence η generated by the related pairs of every intersection. Similarly, each lobe of Γ is mapped isomorphically onto a lobe of Γ/η, under η, and the quotient Γ/η is closed, relative to $\langle S_1 * S_2 \rangle$, with the equality, separation and assimilation properties. The assimilated form of an inverse automaton, over the generators of S_1 and S_2, is defined by taking the assimilated form of the underlying graph.

The graph Γ is *opuntoid* if it has the idempotent, equality and assimilation properties and has no non-trivial reduced lobe loops (equivalently, the lobe graph is a tree). A *subopuntoid subgraph* of an opuntoid graph Γ is a connected subgraph that is also opuntoid and formed by a collection of the lobes of Γ. An inverse automaton over the generators of S_1 and S_2 that is closed, relative to $\langle S_1 * S_2 \rangle$, is opuntoid if its underlying graph is opuntoid. A subautomaton is subopuntoid if its underlying graph is a subopuntoid subgraph.

Let Γ be an opuntoid graph. A vertex $v \in V(\Gamma)$ is a *bud* if it is not an intersection and there is a loop $v \to^f v$ in Γ, for some $f \in E(U)$. We say that the opuntoid graph Γ is *complete* if it has no buds. An opuntoid automaton over the generators of S_1 and S_2 is complete if its underlying graph is complete.

Lemma 4.16. *Let $\mathscr{B}$ denote a cactoid inverse automaton over the generators of S_1 and S_2 that is closed, relative to $\langle S_1 * S_2\rangle$, and has the idempotent, equality and separation properties. Let $\mathscr{C}$ denote the assimilated form of $\mathscr{B}$. Then $\mathscr{C}$ is opuntoid and $\mathscr{B} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$ implies $\mathscr{C} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$.*

Proof. Put $\mathscr{B} = (\alpha, \Gamma, \beta)$. Let η denote the V-equivalence generated by the related pairs of all the intersections of Γ. It follows from the discussion in Definition 4.15 that $\mathscr{C} = \mathscr{B}/\eta$ is opuntoid. Suppose $w' \in L[\mathscr{C}]$. Then there exist paths $x_1 \to^{w_1} y_1$, $x_2 \to^{w_2} y_2$, …, $x_n \to^{w_n} y_n$ in $\mathscr{B}$, where $\alpha \eta x_1$, $y_k \eta x_{k+1}$, for $1 \leq k \leq n-1$, $y_n \eta \beta$ and the word $w_1 w_2 \cdots w_n$ is equal to w'. Put $y_0 = \alpha$ and $x_{n+1} = \beta$. For $0 \leq k \leq n$, assuming $y_k \neq x_{k+1}$, there is a path $y_k \to^{a_k} x_{k+1}$ in $\mathscr{B}$ where $a_k = w_1(u^{-1}) \cdot w_2(u)$ or $a_k = w_2(u^{-1}) \cdot w_1(u)$, for some $u \in U$. Then we have $w' \geq w''$ in $S_1 *_U S_2$, where $w'' = a_0 w_1 a_1 w_2 a_2 \cdots w_n a_n \in L[\mathscr{B}]$. Thus, assuming $\mathscr{B} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$, we have $L[\mathscr{C}] \subseteq L[\mathscr{A}(S_1 *_U S_2, w)]$ and it follows that $\mathscr{C} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$. ☐

The following generalises [1, Construction 5.1].

Construction 4.17. Let $\mathscr{D}$ be an opuntoid automaton that is closed, relative to $\langle S_1 * S_2\rangle$. If v is a bud of a lobe $\Delta_i(v)$ of $\mathscr{D}$, where $i \in \{1, 2\}$, then, putting $j = 3 - i$, we form the automaton $\mathscr{E}$ from $\mathscr{D}$ by sewing on at v the linear automaton of $w_j(f)$, for every $f \in E(U)$ that labels a loop at v in $\Delta_i(v)$. In Figure 6, the dashed arrows represent the linear automata that are sewed on at v and the dashed circle represents the new lobe created. Let $\mathscr{E}'$ denote the closed form of $\mathscr{E}$, relative to $\langle S_1 * S_2\rangle$. Let v' be the image of v in $\mathscr{E}$. Then $\mathscr{E}'$ is obtained from $\mathscr{E}$ by closing $\Delta_j(v')$, relative to $\langle S_j\rangle$. Let v'' be the image of v' in $\mathscr{E}'$. Let $\mathscr{D}'$ be the quotient of $\mathscr{E}'$ by the V-equivalence generated by the related pairs of v''.

Lemma 4.18. *Let $\mathscr{D}$ be an opuntoid automaton and let $\mathscr{D}'$ be obtained from $\mathscr{D}$ by an application of Construction 4.17. Then $\mathscr{D}'$ is an opuntoid automaton and $\mathscr{D}$ is a subopuntoid subautomaton of $\mathscr{D}'$. Further, if $\mathscr{D} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$ then $\mathscr{D}' \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$.*

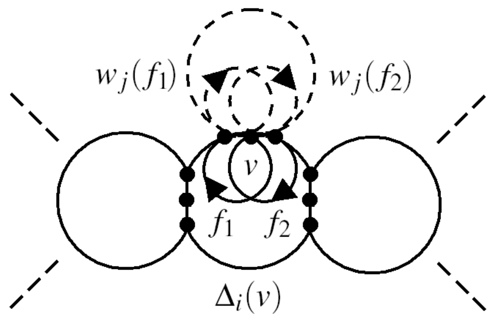

FIGURE 6. Construction 4.17 illustrated.

Proof. We adopt the notation of Construction 4.17. Put $\mathscr{D} = (\alpha, \Gamma, \beta)$. Let $\mathscr{A}(w_j(f))$ denote the linear automaton of $w_j(f)$, for $f \in E(U)$. Let α' and β' denote the respective images of α and β in $\mathscr{E}'$. Let η denote the V-equivalence generated by the related pairs of v''.

If $w'' \in L[(v'', \Delta_j(v''), v'')]$ then $w'' \geq w'$ in S_j, for some $w' \in L[(v', \Delta_j(v'), v')]$, by Result 2.12. Now $w' \in L[\Pi_{l=1}^{n} \mathscr{A}(w_j(f_l))]$, for some $f_l \in E(U)$, where f_l labels a loop at v in $\Delta_i(v)$. Thus $w'' \geq f_1 f_2 \cdots f_n$ in S_j, where $f_1 f_2 \cdots f_n$ labels a loop at v in $\Delta_i(v)$ and so $f_1 f_2 \cdots f_n$ labels a loop at v'' in $\Delta_j(v'')$. It follows that $\Delta_j(v'')$ has the idempotent property. If we also have $w'' \in U$ then $w'' \geq f_1 f_2 \cdots f_n$ implies that w'' labels a loop at v in $\Delta_i(v)$. Since $\mathscr{D}$ has the idempotent and equality properties, it now follows that $\mathscr{D}'$ has the idempotent and equality properties.

Since $\mathscr{D}$ has the assimilation property, if we have a path $v \to^{u} v_1$ in $\Delta_i(v)$, for some $u \in U$, then v_1 cannot be an intersection. Thus $\mathscr{E}'$ must have the separation property. Then $\mathscr{D}'$ is obtained from $\mathscr{E}'$ by assimilating the lobes containing v''. Since $\mathscr{D}$ is opuntoid, it now follows that $\mathscr{D}'$ is opuntoid and $\mathscr{D}$ is a subopuntoid subautomaton of $\mathscr{D}'$.

Suppose $w' \in L[\mathscr{D}']$. Then there exist paths $x_1 \to^{w_1} y_1$, $x_2 \to^{w_2} y_2$, …, $x_n \to^{w_n} y_n$ in $\mathscr{E}'$, where $\alpha' \eta x_1$, $y_k \eta x_{k+1}$, for $1 \leq k \leq n-1$, $y_n \eta \beta$ and the word $w_1 w_2 \cdots w_n$ is equal to w'. Put $y_0 = \alpha$ and $x_{n+1} = \beta$. Then, for $0 \leq k \leq n$, we have either $y_k = x_{k+1}$, in which case put $b_k = 1$, or there is a path $y_k \to^{b_k} x_{k+1}$ in $\mathscr{E}'$ where $b_k = w_1(u^{-1}) \cdot w_2(u)$ or $b_k = w_2(u^{-1}) \cdot w_1(u)$, for some $u \in U$. Thus $w' \geq z$ in $S_1 *_U S_2$, where $z = b_0 w_1 b_1 w_2 b_2 \cdots w_n b_n \in L[\mathscr{E}']$.

By Result 2.12, if $z \in L[\mathscr{E}']$ then $z \geq z'$ in $S_1 * S_2$, for some $z' \in L[\mathscr{E}]$. If $z' \notin L[\mathscr{D}]$ then we have $z' = y_1 a_1 w_1 a_2 \cdots w_{m-1} a_m y_2$, for some $y_1 \in L[(\alpha, \Gamma, v)]$, $a_k \in L[(v', \Delta_j(v'), v')]$, for all k, $w_k \in L[(v, \Gamma, v)]$, for all k, and $y_2 \in L[(v, \Gamma, \beta)]$. Now $a_k \in L[\Pi_{l=1}^{n} \mathscr{A}(w_j(f_l))]$, for some $f_l \in E(U)$, where the f_l label loops at v in $\Delta_i(v)$ and are different for each k. Hence we have $a_k \geq g_k = f_1 f_2 \cdots f_n$

in S_j, where g_k labels a loop at v in $\Delta_i(v)$, for each k. It follows that $z' \geq z''$ in $S_1 *_U S_2$, where $z'' = y_1 g_1 w_1 g_2 \cdots w_{m-1} g_m y_2 \in L[\mathscr{D}]$. Hence we have $z \geq z''$ in $S_1 *_U S_2$, for some $z'' \in L[\mathscr{D}]$.

We have shown that if $w' \in L[\mathscr{D}']$ then we have $w' \geq z''$ in $S_1 *_U S_2$, for some $z'' \in L[\mathscr{D}]$. Hence if $\mathscr{D} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$ then $L[\mathscr{D}'] \subseteq L[\mathscr{A}(S_1 *_U S_2, w)]$ and so $\mathscr{D}' \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$. □

Lemma 4.19. *Let $\mathscr{D}$ be an opuntoid automaton. Then we have a directed system of all automata obtained from the automaton $\mathscr{D}$ by finite applications of Construction 4.17. The direct limit $\mathscr{E}$ is a complete opuntoid automaton that is closed, relative to $\langle S_1 *_U S_2 \rangle$. Thus if $\mathscr{D} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$ then we have $\mathscr{E} \cong \mathscr{A}(S_1 *_U S_2, w)$.*

Proof. Put $\mathscr{D} = (\alpha_1, \Gamma_1, \beta_1)$. Let v_1 and v_2 denote buds of Γ_1. Suppose $\mathscr{D}_k$ is obtained from $\mathscr{D}$ applying Construction 4.17 at v_k, for $k = 1, 2$. We show that there is an automaton $\mathscr{D}_3$ that is obtained from both $\mathscr{D}_1$ and $\mathscr{D}_2$ by an application of Construction 4.17. By Lemma 4.18, the automaton $\mathscr{D}$ is a subopuntoid subautomaton of $\mathscr{D}_k$. Thus $L[\mathscr{D}] \subseteq L[\mathscr{D}_k]$, for $k = 1, 2$. Since $\mathscr{D}$ is embedded into $\mathscr{D}_k$, we let v_1 and v_2 denote their images in $\mathscr{D}_k$.

Suppose we do not have a path $v_1 \rightarrow^{w_i(u)} v_2$ in $\mathscr{D}$, for any $u \in U$ and $i \in \{1, 2\}$. Then v_2 is a bud of $\mathscr{D}_1$ and v_1 is a bud of $\mathscr{D}_2$. It follows that the automaton $\mathscr{D}_3$ obtained from $\mathscr{D}_1$ by applying Construction 4.17 at the bud v_2 is isomorphic to the automaton obtained from $\mathscr{D}_2$ by applying Construction 4.17 at the bud v_1.

Suppose we have a path $v_1 \rightarrow^{w_i(u)} v_2$ in $\mathscr{D}$, for some $u \in U$ and $i \in \{1, 2\}$. Then v_1 and v_2 belong to a lobe of $\mathscr{D}$ colored by i. Let Δ denote the lobe of $\mathscr{D}_1$ containing v_1 and v_2 and colored by $j = 3 - i$. We have $z \in L[(v_1, \Delta, v_1)]$ if and only if $z \geq f_1 f_2 \cdots f_n$ in S_j, for some $f_k \in E(U)$ that label loops at v_1 in $\mathscr{D}$, using workings similar to those in the proof of Lemma 4.18. We show that $z \in L[(v_2, \Delta, v_2)]$ if and only if we have $z \geq g_1 g_2 \cdots g_n$ in S_j, for some $g_k \in E(U)$ that label loops at v_2 in $\mathscr{D}$. It will then follow that $\mathscr{D}_1 \cong \mathscr{D}_2$. Suppose $z \in L[(v_2, \Delta, v_2)]$. Then uzu^{-1} labels a loop at v_1 in Δ and so $uzu^{-1} \geq f_1 f_2 \cdots f_n$, for some $f_k \in E(U)$ that label loops at v_1 in $\mathscr{D}$. Thus $z \geq u^{-1} f_1 u u^{-1} f_2 u \cdots u^{-1} f_n u$, where $g_k = u^{-1} f_k u \in E(U)$ labels a loop at v_2 in $\mathscr{D}$. Conversely, suppose $z \geq g_1 g_2 \cdots g_n$ in S_j, for some $g_k \in E(U)$ that label loops at v_2 in $\mathscr{D}$. Then $u g_1 g_2 \cdots g_n u^{-1}$ labels a loop at v_1 in Δ and so $g_1 g_2 \cdots g_n$ labels a loop at v_2 in Δ. Thus we have $z \in L[(v_2, \Delta, v_2)]$.

It now follows that conditions (A) and (B) of Result 3.2 are satisfied. We have a directed system of all automata obtained from $\mathscr{D}$ by finite applications of Construction 4.17. Let $\mathscr{E} = (\alpha_2, \Gamma_2, \beta_2)$ be the direct limit.

Let $v_1 \to^r v_2$ be a path in $\mathscr{E}$, where (r,s) is a relation of S_1 or S_2. Let $\alpha_2 \to^{z_1} v_1$ and $v_2 \to^{z_2} \beta_2$ be paths in $\mathscr{E}$. There exists $\mathscr{D}'$ in the directed system for $\mathscr{E}$ such that $z_1 r z_2 \in L[\mathscr{D}']$, by Result 3.2. Since $\mathscr{D}'$ is closed, relative to $\langle S_1 * S_2 \rangle$, we have $z_1 s z_2 \in L[\mathscr{D}']$. Thus $z_1 s z_2 \in L[\mathscr{E}]$. Hence $\mathscr{E}$ is closed, relative to $\langle S_1 * S_2 \rangle$.

Suppose we have a loop $v \to^s v$ in $\mathscr{E}$, where $s \in S_i$ and $i \in \{1,2\}$. Let $\alpha_1 \to^w \beta_1$ be a path in $\mathscr{D}$ and let $\alpha_2 \to^z v$ be a path in $\mathscr{E}$. Then we have $zsz^{-1}w \in L[\mathscr{D}']$, for some $\mathscr{D}'$ in the directed system for $\mathscr{E}$, by Result 3.2. Put $\mathscr{D}' = (\alpha_1', \Gamma_1', \beta_1')$. Thus we have a path $\alpha_1' \to^z v'$ and a loop $v' \to^s v'$ in $\mathscr{D}'$. Since $\mathscr{D}'$ has the idempotent property, by Lemma 4.18, we have a loop $v' \to^e v'$ in $\mathscr{D}'$, for some $e \in E(S_i)$ with $s \geq e$. Thus we have a loop $v \to^e v$ in $\mathscr{E}$. Hence $\mathscr{E}$ has the idempotent property.

Suppose v is an intersection of $\mathscr{E}$ such that we have a loop $v \to^{w_i(u)} v$, for some $u \in U$ and $i \in \{1,2\}$. Put $j = 3 - i$. Let $\alpha_1 \to^w \beta_1$ be a path in $\mathscr{D}$ and let $\alpha_2 \to^z v$ be a path in $\mathscr{E}$. Then we have $z \cdot w_i(u) \cdot z^{-1} w \in L[\mathscr{D}']$, for some $\mathscr{D}'$ in the directed system for $\mathscr{E}$, by Result 3.2. Put $\mathscr{D}' = (\alpha_1', \Gamma_1', \beta_1')$. Since $w \in L[\mathscr{D}']$, we must have a path $\alpha_1' \to^z v'$ and a loop $v' \to^{w_i(u)} v'$ in $\mathscr{D}'$. If v' is an intersection then we have a loop $v' \to^{w_j(u)} v'$, since $\mathscr{D}'$ has the equality property, by Lemma 4.18. Otherwise, the vertex v' is a bud and we can obtain an automaton $\mathscr{D}''$ from $\mathscr{D}'$ with the equality property, by Construction 4.17, such that we have a loop $v'' \to^{w_j(u)} v''$, letting v'' denote the image of v' in $\mathscr{D}''$. Thus we have a loop $v \to^{w_j(u)} v$ in $\mathscr{E}$. Hence $\mathscr{E}$ has the equality property.

Suppose v is an intersection of $\mathscr{E}$ and let (v_1, v_2) denote a related pair of v, other than (v,v). Let $\alpha_1 \to^w \beta_1$ be a path in $\mathscr{D}$ and let $\alpha_2 \to^z v$ be a path in $\mathscr{E}$. We have paths $v \to^{w_1(u)} v_1$ and $v \to^{w_2(u)} v_2$ in $\mathscr{E}$, for some $u \in U$. Hence we have $z \cdot w_1(uu^{-1}) \cdot w_2(uu^{-1}) \cdot z^{-1} w \in L[\mathscr{D}']$, for some automaton $\mathscr{D}'$ in the directed system for $\mathscr{E}$, by Result 3.2. Put $\mathscr{D}' = (\alpha_1', \Gamma_1', \beta_1')$. Then we have paths $\alpha_1' \to^z v'$, $v' \to^{w_1(u)} v_1'$ and $v' \to^{w_2(u)} v_2'$ in $\mathscr{D}_1'$. Since $\mathscr{D}'$ is opuntoid, with identified related pairs, by Lemma 4.18, we must have $v_1' = v_2'$. Thus $v_1 = v_2$ and so $\mathscr{E}$ has identified related pairs.

Suppose v_3 is also an intersection of $\Delta_1(v)$ and $\Delta_2(v)$. Let $v \to^{s_1} v_3$ be a path in $\Delta_1(v)$ and let $v \to^{s_2} v_3$ be a path in $\Delta_2(v)$, for some $s_1 \in S_1$ and $s_2 \in S_2$. We have $zs_1s_2^{-1}z^{-1}w \in L[\mathscr{D}'']$, for some automaton $\mathscr{D}''$ in the directed system for $\mathscr{E}$, by Result 3.2. Put $\mathscr{D}'' = (\alpha_1'', \Gamma_1'', \beta_1'')$. Then we have paths $\alpha_1'' \to^z v''$, $v'' \to^{s_1} v_3''$ and $v'' \to^{s_2} v_3''$ in $\mathscr{D}_1''$. Thus (v_3'', v_3'') is a related pair of v'', since $\mathscr{D}''$ has the assimilation property, by Lemma 4.18. Hence (v_3, v_3) is a related pair of v and so $\mathscr{E}$ has the assimilation property.

Suppose the lobe graph of $\mathscr{E}$ has a reduced lobe loop $\Delta_1, \Delta_2, \ldots, \Delta_n = \Delta_1$, for some $n \geq 4$. Then the lobes $\Delta_1, \Delta_2, \ldots, \Delta_{n-1}$ are distinct. Let v_k denote an intersection common to Δ_k and Δ_{k+1}, for $1 \leq k \leq n-1$. Put $v_n = v_1$. Let $v_k \to^{w_k} v_{k+1}$ be a path in Δ_{k+1}, for $1 \leq k \leq n-1$. Let $\alpha_1 \to^w \beta_1$ be a path in $\mathscr{D}$ and let $\alpha_2 \to^y v_1$ be a path in $\mathscr{E}$. We have $yw_1w_2\cdots w_{n-1}y^{-1}w \in L[\mathscr{D}']$, for some automaton $\mathscr{D}'$ in the directed system for $\mathscr{E}$, by Result 3.2. Then the lobes of $\mathscr{D}'$ containing the loop labeled by $w_1w_2\cdots w_{n-1}$ is a non-trivial reduced lobe loop and we have a contradiction, since $\mathscr{D}'$ is opuntoid. Hence the lobe graph of $\mathscr{E}$ contains no non-trivial reduced lobe loops. We have now proved that $\mathscr{E}$ is opuntoid.

Suppose v is a bud of $\mathscr{E}$ of a lobe $\Delta_i(v)$, for some $i \in \{1,2\}$. Let $v \to^{w_i(f)} v$ be a loop in $\Delta_i(v)$, for some $f \in E(U)$. Let $\alpha_1 \to^w \beta_1$ be a path in Γ_1 and let $\alpha_2 \to^z v$ be a path in $\mathscr{E}$. There is some $\mathscr{D}'$ in the directed system such that $z \cdot w_i(f) \cdot z^{-1}w \in L[\mathscr{D}']$, by Result 3.2. Put $\mathscr{D}' = (\alpha_1', \Gamma_1', \beta_1')$. Then we have a path $\alpha_1' \to^z v'$ and a loop $v' \to^{w_i(f)} v'$ in $\mathscr{D}'$, where v' is a bud of $\mathscr{D}'$. We can obtain an automaton $\mathscr{D}''$ from $\mathscr{D}'$, by an application of Construction 4.17, such that v'' is an intersection of $\mathscr{D}''$, letting v'' denote the image of v' in $\mathscr{D}''$. We have a contradiction, since this implies v is an intersection. Thus $\mathscr{E}$ is a complete opuntoid automaton.

Let $v_1 \to^{w_1(u)} v_2$ be a path in $\mathscr{E}$, for some $u \in U$. Then we have a loop $v_1 \to^{w_1(uu^{-1})} v_1$ in $\mathscr{E}$, since $\mathscr{E}$ is closed, relative to $\langle S_1 \rangle$. Since $\mathscr{E}$ is complete, the vertex v_1 must be an intersection. Since $\mathscr{E}$ has the equality property, we have a loop $v_1 \to^{w_2(uu^{-1})} v_1$ in $\mathscr{E}$. We have a path $v_1 \to^{w_2(u)} v_2'$ in $\mathscr{E}$, since $\mathscr{E}$ is closed, relative to $\langle S_2 \rangle$. Then the assimilation property implies that $v_2 = v_2'$. Hence we have a path $v_1 \to^{w_2(u)} v_2$ in $\mathscr{E}$. Similarly, if we have path $v_1 \to^{w_2(u)} v_2$ in $\mathscr{E}$, for some $u \in U$, then we have a path $v_1 \to^{w_2(u)} v_2$. We have shown that $\mathscr{E}$ is closed, relative to $\langle S_1 *_U S_2 \rangle$.

Finally, suppose that $\mathscr{D} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$. Let $w' \in L[\mathscr{E}]$. Then there is some automaton $\mathscr{D}'$ in the directed system with $w' \in L[\mathscr{D}']$, by Result 3.2. By Lemma 4.18, we have $\mathscr{D}' \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$. Thus $w' \in L[\mathscr{A}(S_1 *_U S_2, w)]$. Then $L[\mathscr{E}] \subseteq L[\mathscr{A}(S_1 *_U S_2, w)]$ and so $\mathscr{E} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$. Hence we have $\mathscr{E} \cong \mathscr{A}(S_1 *_U S_2, w)$, by Result 2.9. □

Algorithm 4.20. Given w in the generators of S_1 and S_2, the Schützenberger automaton of w, relative to $\langle S_1 *_U S_2\rangle$, is constructed as follows:

(i) Construct $\mathscr{A} = \mathscr{A}(S_1 * S_2, w)$, by Construction 2.14. Then $\mathscr{A}$ is cactoid, has the idempotent property and we have $\mathscr{A} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$.

(ii) The direct limit $\mathscr{B}$ of the directed system of all automata obtained from $\mathscr{A}$, by finite applications of Construction 4.5, is cactoid, has the idempotent and equality properties, has at most as many lobes as $\mathscr{A}$ and we have $\mathscr{B} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$, by Lemma 4.6.

(iii) An application of Construction 4.12 to $\mathscr{B}$ results in an automaton $\mathscr{B}'$ that is cactoid, has the idempotent and equality properties, has fewer lobes than $\mathscr{B}$ and we have $\mathscr{B}' \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$, by Lemma 4.13.

(iv) Steps (ii) and (iii) can be applied at most a finite number of times, since the initial automaton has finite lobes. The resulting automaton $\mathscr{C}$ is cactoid, has the idempotent, equality and separation properties and $\mathscr{C} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$.

(v) The assimilated form $\mathscr{D}$ of $\mathscr{C}$ is a finite-lobe opuntoid automaton and $\mathscr{D} \rightsquigarrow \mathscr{A}(S_1 *_U S_2, w)$, by Lemma 4.16.

(vi) The direct limit $\mathscr{E}$ of the directed system of all automata obtained from $\mathscr{D}$, by finite applications of Construction 4.17, is a complete opuntoid automaton and we have $\mathscr{E} \cong \mathscr{A}(S_1 *_U S_2, w)$, by Lemma 4.19.

The following generalises [1, Section 6].

Definition 4.21. Let Γ be an opuntoid graph. Let Δ_1 and Δ_2 be adjacent lobes, colored by $i \in \{1,2\}$ and $j = 3-i$, respectively. Then Δ_2 *feeds off* Δ_1 if there is a common intersection v such that, for any loop $v \to^y v$ in Δ_2, there is a loop $v \to^f v$ in Δ_2, for some $f \in E(U)$ with $y \geq f$ in S_j.

Suppose Δ_2 feeds of Δ_1 and let v' denote any intersection of Δ_1 and Δ_2. By the assimilation property, we have a path $v \to^u v'$, for some $u \in U$. If we have a loop $v' \to^y v'$ in Δ_2 then we have a loop $v \to^{uyu^{-1}} v$ in Δ_2. Thus we have a loop $v \to^f v$ in Δ_2, for some $f \in E(U)$ with $uyu^{-1} \geq f$ in S_j. Hence we have a loop $v' \to^{u^{-1}fu} v'$ in Δ_2, where $y \geq u^{-1}fu$ in S_j. Therefore, if Δ_2 feeds of Δ_1 then for any loop $v' \to^y v'$ in Δ_2 there is a loop $v' \to^g v'$ in Δ_2, for some $g \in E(U)$ with $y \geq g$ in S_j, for any intersection v' of Δ_1 and Δ_2.

For non-adjacent lobes Δ_1 and Δ_n of Γ, we say that Δ_n *feeds off* Δ_1 if there is a sequence of lobes $\Delta_1, \Delta_2, \ldots, \Delta_n$, where Δ_{k+1} is adjacent to Δ_k and Δ_{k+1} feeds off Δ_k, for $1 \leq k \leq n-1$.

Let Γ' be a subopuntoid subgraph of Γ. A lobe of Γ that does not belong to Γ' is called *external* to Γ'. An extremal lobe of Γ' is a called a *parasite* if it feeds off the unique lobe of Γ' to which it is adjacent. The subgraph Γ' is *parasite-free* if it has no parasites. The subgraph Γ' is a *host* of Γ if it has finitely many lobes, is parasite-free, and every lobe of Γ that is external to Γ' feeds off some lobe of Γ'. A host of an opuntoid automaton is a host of its underlying graph.

Lemma 4.22. *Every finite-lobe opuntoid graph has a host.*

Proof. The proof follows from the properties of finite trees, since the lobe graph of an opuntoid graph is a tree, and is similar to that of [1, Lemma 6.1]. □

Lemma 4.23. *Let Γ be an opuntoid graph. Then a host of Γ is a maximal parasite-free subopuntoid subgraph. If Γ has more than one host, then every host is a lobe of Γ. In addition, the unique reduced lobe path between any two hosts consists entirely of lobes that are hosts.*

Proof. The proof follows from the properties of finite trees and is similar to [1, Lemma 6.2]. □

Lemma 4.24. *Let $\mathscr{D}$ be a finite-lobe opuntoid automaton with a host Σ. If $\mathscr{D}'$ is obtained from $\mathscr{D}$ by Construction 4.17 then Σ is also a host of $\mathscr{D}'$.*

Proof. We adopt the notation of Construction 4.17. The automaton $\mathscr{D}$ is embedded into $\mathscr{D}'$, from Lemma 4.18. Let v'' denote its image in $\mathscr{D}'$. From the proof of Lemma 4.18, for any loop $v'' \rightarrow^{y} v''$ in $\Delta_j(v'')$ we have $y \geq f$ in S_j, for some $f \in E(U)$, where f labels a loop at v'' in $\Delta_j(v'')$. Thus $\Delta_j(v'')$ feeds off $\Delta_i(v'')$. If $\Delta_i(v'')$ is a lobe of Σ then, since $\Delta_j(v'')$ feeds off $\Delta_i(v'')$, it follows that Σ is a host of $\mathscr{D}'$. Otherwise, the lobe $\Delta_i(v'')$ feeds off some lobe Δ of Σ. In which case, the lobe $\Delta_j(v'')$ also feeds off Δ and so Σ is a host of $\mathscr{D}'$. □

Lemma 4.25. *Let Γ be a complete opuntoid graph with a host Σ. Let Γ' be a subopuntoid subgraph of Γ containing Σ and let $v \in V(\Gamma')$. Then the direct limit $\mathscr{E}$ of the directed system of all automata obtained from (v, Γ', v), by finitely many applications of Construction 4.17, is isomorphic to (v, Γ, v).*

Proof. If $v_1 \in V(\Gamma')$ is a bud of $\Delta_i(v_1)$ in Γ', for $i \in \{1,2\}$, then we can apply Construction 4.17 to (v,Γ',v), resulting in an automaton (v',Γ'',v'). Let v_1'' denote the image of v_1 in Γ'' and put $j = 3-i$. From Lemma 4.18, the automaton (v,Γ',v) is a subopuntoid subautomaton of (v',Γ'',v'). From the workings of Lemma 4.18, the language $L[(v_1'',\Delta_j(v_1''),v_1'')]$ consists of all words y such that $y \geq f$ in S_j, for some $f \in E(U)$ labeling a loop $v_1 \to^f v_1$ in $\Delta_i(v_1)$.

Since Γ is complete, there is a lobe Δ of Γ, colored by j, containing v_1. Since the lobe graph of Γ is a tree and Γ' contains the host Σ, the lobe Δ feeds off $\Delta_i(v_1)$. Since Γ has the equality property, we then have $L[(v_1,\Delta,v_1)] = L[(v_1'',\Delta_j(v_1''),v_1'')]$. Since the lobes are deterministic, we have $(v_1,\Delta,v_1) \cong (v_1'',\Delta_j(v_1''),v_1'')$. Thus (v',Γ'',v') is a subopuntoid subautomaton of (v,Γ,v).

Conversely, any lobe of $\Gamma\backslash\Gamma'$ that is adjacent to a lobe of Γ' must feed off the lobe of Γ' and so, by the equality property, must share a vertex that is a bud of Γ'. It now follows that $\mathscr{E} \cong (v,\Gamma,v)$. □

Theorem 4.26. *Let U denote a lower bounded inverse subsemigroup of inverse semigroups S_1 and S_2. Then the Schützenberger automata of $\langle S_1 *_U S_2\rangle$ are complete opuntoid with a host.*

Proof. Steps (i), (ii), (iii), (iv), (v) of Algorithm 4.20 result in a finite-lobe opuntoid automaton $\mathscr{D}$. By Lemma 4.22, the automaton $\mathscr{D}$ has a host Σ. Then, by step (vi), the direct limit $\mathscr{E}$ of the directed system of all automata obtained from $\mathscr{D}$, by finite applications of Construction 4.17, is complete opuntoid that is closed, relative to $\langle S_1 *_U S_2\rangle$.

The automaton $\mathscr{D}$ is embedded as a subopuntoid subautomaton into every automaton in the directed system, by Lemma 4.18. Let Δ be a lobe of $\mathscr{E}$ that is not a lobe of $\mathscr{D}$. There is some automaton $\mathscr{D}'$ in the directed system such that Δ is a lobe of $\mathscr{D}'$. By Lemma 4.24, the subgraph Σ is a host of $\mathscr{D}'$. Thus Δ feeds off some lobe of Σ and so Σ is a host of $\mathscr{E}$. Hence every Schützenberger automaton of $S_1 *_U S_2$ is complete opuntoid with a host. □

Lemma 4.27. *A host of $\mathscr{A}(S_1 *_U S_2, w)$ having more than one lobe is the assimilated form of the (underlying graph of the) direct limit of the directed system of all obtained from $\mathscr{A}(S_1 * S_2, w')$, by finite applications of Construction 4.5, for some word w'. A host of $\mathscr{A}(S_1 *_U S_2, w)$ with precisely one lobe is isomorphic to a Schützenberger graph of $\langle S_1\rangle$ or $\langle S_2\rangle$.*

Proof. From Corollary 4.14, there exists a word w' such that $w = w'$ in $S_1 *_U S_2$ and the direct limit $\mathscr{B}$ of the directed system of all automata obtained from $\mathscr{A} = \mathscr{A}(S_1 * S_2, w')$, by finitely many applications of Construction 4.5, has the separation property, where $\mathscr{A}$ has at most as many lobes as $\mathscr{A}(S_1 * S_2, w)$. Put $\mathscr{A} = (\alpha_1, \Gamma_1, \beta_1)$, $\mathscr{B} = (\alpha_2, \Gamma_2, \beta_2)$ and then let $\mathscr{D} = (\alpha_4, \Gamma_4, \beta_4)$ denote the assimilated form of $\mathscr{B}$.

If $\gamma_1, \delta_1 \in V(\Gamma_1)$ then we have $(\gamma_1, \Gamma_1, \delta_1) \cong \mathscr{A}(S_1 * S_2, w'')$, where $w'' \mathscr{D} w'$ in $S_1 *_U S_2$. From Lemma 4.7, the direct limit of the directed system of all automata obtained from $(\gamma_1, \Gamma_1, \delta_1)$, by finite applications of Construction 4.5, is isomorphic to $(\gamma_2, \Gamma_2, \delta_2)$, letting γ_2 and δ_2 denote the respective images of γ_1 and δ_1 in Γ_2. Thus, for the purpose of describing the hosts of $\mathscr{A}(S_1 *_U S_2, w)$, we can assume that α_1 and β_1 are any vertices of Γ_1. From Lemma 4.8, we can assume that the homomorphism from $\mathscr{A}$ into $\mathscr{B}$ induces an isomorphism of the lobe trees. From Corollary 4.11, we can assume that no application of Construction 4.9 can be applied to $\mathscr{B}$, for any choice of α_2 and β_2.

If Γ_4 has a parasite then there is an intersection v of Γ_4 such that $\Delta_i(v)$ is an extremal lobe, for some $i \in \{1,2\}$, and for any loop $v \rightarrow^y v$ in $\Delta_i(v)$, there exists a loop $v \rightarrow^f v$ in $\Delta_i(v)$, where $f \in E(U)$ and $y \geq f$ in S_i. Let v' denote the unique intersection of $\mathscr{B}$ that is a preimage of v and put $j = 3 - i$. Since $\mathscr{B}$ has the equality property, for any loop $v' \rightarrow^y v'$ in $\Delta_i(v')$, there exists a loop $v' \rightarrow^f v'$ in $\Delta_j(v')$, where $f \in E(U)$ and $y \geq f$ in S_i. Thus we can apply Construction 4.9 to $\mathscr{B}$, a contradiction. Hence Γ_4 is parasite-free. Then, from the proof of Theorem 4.26, the graph Γ_4 is a host of $\mathscr{A}(S_1 *_U S_2, w)$.

If Γ_4 has precisely one lobe then $\Gamma_4 \cong \Gamma_2 \cong \Gamma_1 \cong S\Gamma(S_i, s)$, for some $s \in S_i$ and $i \in \{1,2\}$. Suppose Δ is a host of $\mathscr{A}(S_1 *_U S_2, w)$ that is adjacent to Γ_4. Let v be an intersection common to Γ_4 and Δ. We have $(v, \Gamma_4, v) \cong \mathscr{A}(S_i, e)$, for some $e \in E(S_i)$ and $i \in \{1,2\}$. Since Γ_4 feeds off Δ, we have $e \in E(U)$ and $(v, \Delta, v) \cong \mathscr{A}(S_j, e)$, where $j = 3 - i$. The unique reduced lobe path between any two hosts consists of lobes that feed off each other, by Lemma 4.23. Thus if $\mathscr{A}(S_1 *_U S_2, w)$ has more than one host then every host is isomorphic to $S\Gamma(S_1, f)$ or $S\Gamma(S_2, f)$, for some $f \in E(U)$. □

We prove some results on homomorphisms of Schützenberger graphs.

Lemma 4.28. *Let Γ and Γ' be complete opuntoid graphs that have hosts and let Σ be a host of Γ. Then every homomorphism $\Sigma \rightarrow \Gamma'$ extends (uniquely) to a homomorphism $\Gamma \rightarrow \Gamma'$.*

Proof. Let $v \in V(\Sigma)$. The automaton (v, Γ, v) is the direct limit of the directed system of all automata obtained from (v, Σ, v), by finitely many applications of Construction 4.17, by Lemma 4.25. If

$y \in L[(v,\Gamma,v)]$ then $y \in L[(v,\Gamma'',v)]$ for some (v,Γ'',v) in the directed system for (v,Γ,v). From the proof of Lemma 4.18, if $y \in L[(v,\Gamma'',v)]$ then $y \geq z$ in $S_1 *_U S_2$, for some $z \in L[(v,\Sigma,v)]$.

Let $\psi : \Sigma \to \Gamma'$ be a homomorphism and put $v' = (v)\psi$. Then $L[(v,\Sigma,v)] \subseteq L[(v',\Gamma',v')]$. From Lemma 4.19, (v',Γ',v') is closed, relative to $\langle S_1 *_U S_2 \rangle$. Thus $L[(v,\Gamma,v)] \subseteq L[(v',\Gamma',v')]$, by Result 2.12. We have a homomorphism $\pi : \Gamma \to \Gamma'$ uniquely extending ψ, by Result 2.2. □

Corollary 4.29. *Let Γ and Γ' be complete opuntoid graphs that have hosts and let Σ be any host of Γ. Then every isomorphism from Σ onto some host of Γ' extends (uniquely) to an isomorphism of Γ onto Γ'.*

Proof. The proof is immediate from Lemma 4.28 □

Notation 4.30. Let Γ be an opuntoid graph. Then we let $AUT(\Gamma)$ denote the *automorphism group* and $END(\Gamma)$ denote the *endomorphism monoid* of Γ.

Lemma 4.31. *If Γ is a finite-lobe opuntoid graph then the group $AUT(\Gamma)$ is embedded into the automorphism group of some lobe of Γ.*

Proof. The proof is similar to [2, Lemma 7]. An automorphism of Γ induces an automorphism of the lobe tree of Γ. Every automorphism of a finite tree must stabilize some vertex. Thus every automorphism of Γ must induce an automorphism of some lobe Δ. It follows that $AUT(\Gamma)$ is isomorphic to a subgroup of $AUT(\Delta)$. □

Lemma 4.32. *Let Γ be a complete opuntoid graph that has a host. Let Γ' be the subgraph that consists of the lobes of every host of Γ. Then Γ' is a subopuntoid subgraph of Γ and $AUT(\Gamma)$ is isomorphic to $AUT(\Gamma')$. Thus if Γ has a finite number of hosts then $AUT(\Gamma)$ is embedded into the automorphism group of some lobe.*

Proof. The proof is similar to [2, Lemma 8]. If Γ has precisely one host then Γ' is this host. If Γ has more than one host then, by Lemma 4.23, every host is a lobe of Γ and the unique reduced lobe path between any two hosts consists entirely of lobes that are hosts. Thus Γ' is a subopuntoid subgraph of Γ. Since automorphisms of Γ map hosts onto hosts, every automorphism of Γ must induce an automorphism of Γ'. Thus we have a homomorphism from $AUT(\Gamma)$ into $AUT(\Gamma')$. Any automorphism of Γ' extends uniquely to an automorphism of Γ, by Corollary 4.29. It follows that $AUT(\Gamma) \cong AUT(\Gamma')$. The last statement is immediate from Lemma 4.31. □

Lemma 4.33. *Let Γ be a finite-lobe opuntoid graph. If $END(\Delta) = AUT(\Delta)$ for every lobe Δ of Γ then $END(\Gamma) = AUT(\Gamma)$.*

Proof. The proof is by induction on the number of lobes of Γ and is similar to that of [2, Lemma 9]. The result is immediately true if Γ has two lobes. Assume the result is true for less than N lobes, for some $N > 2$, and suppose Γ has N lobes. The induction hypothesis implies that any endomorphism ψ_1 of Γ induces an automorphism ψ_2 of the lobe tree of Γ. Since ψ_2 must have finite order, it follows that ψ_1 must be an automorphism. □

Notation 4.34. By Lemma 4.23, we can associate a number with an opuntoid graph Γ that has a host, by defining $n(\Gamma)$ to be the number of lobes in any host. Either Γ has one host, in which case $n(\Gamma) \geq 1$, or every host of Γ is a lobe, in which case $n(\Gamma) = 1$. If Δ is an extremal lobe of an opuntoid graph Γ then we let $\Gamma \backslash \Delta$ denote the subopuntoid subgraph of Γ consisting of all the lobes of Γ, except for Δ.

Lemma 4.35. *Suppose $END(\Gamma) = AUT(\Gamma)$ for any finite-lobe opuntoid graph Γ or for any complete opuntoid graph Γ with a host and $n(\Gamma) = 1$. Then $END(\Gamma) = AUT(\Gamma)$ for any complete opuntoid graph Γ with a host.*

Proof. Let Γ denote a complete opuntoid graph with a host Σ. The result is proved by induction on $n(\Gamma)$ and is similar to that given in [2, Proposition 1]. Assume that $END(\Gamma) = AUT(\Gamma)$ if $n(\Gamma) < N$, for some $N \geq 2$.

Now suppose Γ is a complete opuntoid graph with a host Σ and $n(\Gamma) = N$. Let $\alpha \in END(\Gamma)$. We show that the map α induces an endomorphism β of Σ. Since Σ is a finite-lobe opuntoid graph, we will have $\beta \in AUT(\Sigma)$. Since any two homomorphisms of a deterministic inverse word graph that agree on a vertex are equal, the map α is then the unique automorphism of Γ that extends β, by Corollary 4.29. Let Σ_1 denote the minimal subopuntoid subgraph of Γ containing $(\Sigma)\alpha$. Let Σ_2 denote the minimal subopuntoid subgraph of Γ containing Σ and $(\Sigma)\alpha$. We suppose $\Sigma_1 \nsubseteq \Sigma$ and reach a contradiction.

Suppose there are no lobes common to Σ and Σ_1. Then Σ_2 is the union of $\Sigma \cup \Sigma_1$ and a reduced lobe path from a lobe of Σ to a lobe of Σ_1. Since Σ has at least two extremal lobes, there exists an extremal lobe Δ of Σ that does not belong to Σ_1 and is also extremal in Σ_2.

Suppose there exist lobes that are common to Σ and Σ_1. Then $\Sigma_2 = \Sigma \cup \Sigma_1$. If every extremal lobe of Σ belongs to Σ_1 then the reduced lobe path between any two extremal lobes of Σ also belongs to Σ_1. In which case, we have $\Sigma \subseteq \Sigma_1$. Since Σ_1 has at most as many lobes as Σ, we would then have $\Sigma = \Sigma_1$, a contradiction. Thus there is an extremal lobe Δ of Σ that does not belong to Σ_1 and is extremal in Σ_2.

Now $\Sigma\backslash\Delta$ and $\Sigma_2\backslash\Delta$ are both finite-lobe subopuntoid subgraphs of Γ. Also the graphs $\Sigma\backslash\Delta$ and Σ_1 are subopuntoid subgraphs of $\Sigma_2\backslash\Delta$. By Lemma 4.22, every finite-lobe opuntoid graph has a host. Thus let Σ_3 denote a host of $\Sigma\backslash\Delta$. Suppose Δ^* is a lobe of $\Sigma_2\backslash\Delta$ that is not in $\Sigma\backslash\Delta$. Since Σ is a host of Γ, there is a reduced lobe path $\Delta_1, \Delta_2, \ldots, \Delta_n = \Delta^*$ in Σ_2, where Δ_1 is the only lobe of Σ and Δ_{k+1} feeds off Δ_k, for all k. If $\Delta_1 = \Delta$ then Δ would not be extremal in Σ_2, a contradiction. Thus Δ_1 is a lobe of $\Sigma\backslash\Delta$. Since every lobe of $\Sigma\backslash\Delta$ is in Σ_3 or feeds off a lobe of Σ_3, it follows that Σ_3 is a host of $\Sigma_2\backslash\Delta$.

We have a homomorphism $\Sigma\backslash\Delta \to \Sigma_2\backslash\Delta$ induced by α. Let x denote any vertex of $\Sigma_2\backslash\Delta$. From Lemma 4.19, we have a directed system of all automata obtained from $(x, \Sigma_2\backslash\Delta, x)$ by finite applications of Construction 4.17. The direct limit (x', Γ', x') is a complete opuntoid automaton and it follows, from Lemma 4.24, that Σ_3 is a host of Γ'. The map $\Sigma\backslash\Delta \to \Sigma_2\backslash\Delta$, induced by α, defines a homomorphism $\Sigma_3 \to \Gamma'$ that extends uniquely to an endomorphism β of Γ', by Lemma 4.28. Since Σ_3 has fewer than N lobes, the endomorphism β must be an automorphism, by the induction hypothesis. Thus α maps $\Sigma\backslash\Delta$ isomorphically onto a subopuntoid subgraph of Σ_1.

Let Δ' denote the unique lobe of Σ that is adjacent to Δ. Since the map α is one-one on $\Sigma\backslash\Delta$, the intersections common to Δ and Δ' cannot be mapped to intersections of $(\Sigma\backslash\Delta)\alpha$. Thus Δ cannot be mapped, under α, into a lobe of $(\Sigma\backslash\Delta)\alpha$. Therefore Δ is mapped, under α, into an extremal lobe Δ_1 of Σ_1. Now Δ_1 is external to $(\Sigma_3)\alpha$. Since Σ_3 is mapped isomorphically, under β, onto some host of Γ', it follows that Δ_1 feeds off some lobe of $(\Sigma_3)\beta$. Thus Δ_1 feeds off some lobe of $(\Sigma_3)\alpha$. Hence Δ_1 must feed off the unique lobe $(\Delta')\alpha$ of Σ_1 to which it is adjacent.

Let y be a vertex that is common to Δ and Δ'. Then $(y)\alpha$ is common to Δ_1 and $(\Delta')\alpha$. Let Δ and Δ_1 have color $i \in \{1,2\}$. If we have a loop $(y)\alpha \to^s (y)\alpha$ in Δ_1 then we have a loop $(y)\alpha \to^f (y)\alpha$ in Δ_1, for some $f \in E(U)$ with $s \geq f$ in S_i. Since Γ has the equality property, we also have a loop $(y)\alpha \to^f (y)\alpha$ in $(\Delta')\alpha$. Since β is an automorphism, we then have a loop $y \to^f y$ in Δ'. Again since Γ has the equality property, we have a loop $y \to^f y$ in Δ. Then, since Δ is closed, relative to $\langle S_i \rangle$, we have a loop $y \to^s y$ in Δ. It now follows that $(y, \Delta, y) \cong ((y)\alpha, \Delta_1, (y)\alpha)$. We reach a contradiction, since this implies that the extremal lobe Δ is a parasite of the host Σ. We conclude that $\Sigma_1 \subseteq \Sigma$ and the proof of the lemma is complete. □

Lemma 4.36. *Suppose the host of every Schützenberger graph of $\langle S_1 *_U S_2\rangle$ has lobes isomorphic to Schützenberger graphs of $\langle S_1\rangle$ and $\langle S_2\rangle$. Suppose $END(\Gamma) = AUT(\Gamma)$ for every Schützenberger graph Γ of $\langle S_1 *_U S_2\rangle$ with $n(\Gamma) = 1$. Then we have $END(\Gamma) = AUT(\Gamma)$ for every Schützenberger graph Γ of $\langle S_1 *_U S_2\rangle$.*

Proof. Let Γ denote a Schützenberger graph of $\langle S_1 *_U S_2\rangle$ with host Σ. The result is proved by induction on $n(\Gamma)$ and is similar to that of Lemma 4.35. Assume that $END(\Gamma) = AUT(\Gamma)$ if $n(\Gamma) < N$, for some $N \geq 2$. Adopt the notation of Lemma 4.35.

By assumption, the lobes of Σ are isomorphic to Schützenberger graphs of $\langle S_1\rangle$ or $\langle S_2\rangle$. It follows that Σ_3 approximates a Schützenberger graph of $\langle S_1 *_U S_2\rangle$. Hence Γ' is a Schützenberger graph of $\langle S_1 *_U S_2\rangle$, with host Σ_3, by Algorithm 4.20 and Lemma 4.24. Then, by the induction hypothesis, the endomorphism $\beta : \Gamma' \to \Gamma'$ is an automorphism. The proof that $\Sigma_1 \subseteq \Sigma$ now follows as in Lemma 4.35.

By assumption, we have $END(\Gamma_1) = AUT(\Gamma_1)$, for any Schützenberger graph Γ_1 of $\langle S_1 *_U S_2\rangle$, where any host has precisely one lobe. By Lemma 4.28, any endomorphism of a host extends to an endomorphism of Γ_1. Thus $END(\Delta^*) = AUT(\Delta^*)$, for any Schützenberger graph Δ^* of $\langle S_1\rangle$ or $\langle S_2\rangle$. Now $END(\Sigma) = AUT(\Sigma)$, by Lemma 4.33. The restriction of α to Σ extends uniquely to an automorphism of Γ, by Corollary 4.29. Hence $\alpha \in AUT(\Gamma)$ and so $END(\Gamma) = AUT(\Gamma)$. □

Notation 4.37. As defined in [2, Section 4], for $f, g \in E(U)$, we write $f \prec_i g$ if $f \mathscr{D} h \leq g$ in S_i, for some $h \in E(S_i)$, for $i = 1, 2$. Then let $\prec$ denote the transitive closure of $\prec_1$ and $\prec_2$.

An inverse semigroup is *completely semisimple* if two distinct idempotents in any $\mathscr{D}$-class are not comparable, under the natural partial order. From [2, Lemma 10], an inverse semigroup is completely semisimple if and only if the endomorphism monoid and the automorphism group coincide, for every Schützenberger automaton. The following result is true for any amalgamated free product $S_1 *_U S_2$, not just when U is lower bounded in S_1 and S_2.

Lemma 4.38. *If $S_1 *_U S_2$ is completely semisimple then S_1 and S_2 are both completely semisimple with $\prec \cap \succ_1 \subseteq \prec_1$ and $\prec \cap \succ_2 \subseteq \prec_2$, for any amalgam $[S_1, S_2; U]$ of inverse semigroups.*

Proof. Since the semigroups S_1 and S_2 are embedded into $S_1 *_U S_2$ they must be completely semisimple. Let $f, g \in E(U)$ with $f \prec g$ and $f \succ_1 g$. Then there exist idempotents $f_1, f_2, \ldots, f_n \in E(U)$, for some $n \geq 2$, where $f_1 = f$, $f_n = g$ and $f_k \prec_i f_{k+1}$, for some $i \in \{1,2\}$, for each $1 \leq k \leq n-1$. Thus we have $f_k = s_k s_k^{-1}$ and $s_k^{-1} s_k \leq f_{k+1}$ in S_i, for some $s_k \in S_i$ and some $i \in \{1,2\}$. Also, we have $f_n = s_n s_n^{-1}$ and $s_n^{-1} s_n \leq f_1$ in S_1, for some $s_n \in S_1$. Putting $t = s_1 s_2 \cdots s_{n-1} s_n$, we have $f_1 \mathscr{R} t \mathscr{L} t^{-1} t \leq s_n^{-1} s_n \leq f_1$ in $S_1 *_U S_2$. Thus $t^{-1} t = s_n^{-1} s_n = f_1$, since $S_1 *_U S_2$ is completely semisimple. Hence $f_1 \mathscr{D} f_n$ in S_1. Therefore $\prec \cap \succ_1 \subseteq \prec_1$. Similarly, we have $\prec \cap \succ_2 \subseteq \prec_2$. □

From Lemma 4.27, a host of a Schützenberger graph of $\langle S_1 *_U S_2 \rangle$ with precisely one lobe is isomorphic to a Schützenberger graph of $\langle S_1 \rangle$ or $\langle S_2 \rangle$. The following result generalises [2, Theorem 4].

Theorem 4.39. *Let U denote a lower bounded inverse subsemigroup of two completely semisimple inverse semigroups S_1 and S_2. Suppose the host of any Schützenberger graph of $\langle S_1 *_U S_2 \rangle$ has lobes isomorphic to Schützenberger graphs of $\langle S_1 \rangle$ or $\langle S_2 \rangle$. Then $S_1 *_U S_2$ is completely semisimple if and only if $\prec \cap \succ_1 \subseteq \prec_1$ and $\prec \cap \succ_2 \subseteq \prec_2$.*

Proof. If $S_1 *_U S_2$ is completely semisimple then $\prec \cap \succ_1 \subseteq \prec_1$ and $\prec \cap \succ_2 \subseteq \prec_2$, by Lemma 4.38. Suppose we have $\prec \cap \succ_1 \subseteq \prec_1$ and $\prec \cap \succ_2 \subseteq \prec_2$. We show that $END(\Gamma) = AUT(\Gamma)$, for any Schützenberger graph Γ of $\langle S_1 *_U S_2 \rangle$ with $n(\Gamma) = 1$. Then we will have $END(\Gamma) = AUT(\Gamma)$, for any Schützenberger graph Γ of $\langle S_1 *_U S_2 \rangle$, by Lemma 4.36.

Let M denote a lobe of Γ that is also a host. Let $\alpha \in END(\Gamma)$ and let M' denote the lobe of Γ containing $(M)\alpha$. If $M' = M$ then α restricts to an endomorphism β on M. The lobe M is isomorphic to a Schützenberger graph of $\langle S_1 \rangle$ or $\langle S_2 \rangle$. Since S_1 and S_2 are completely semisimple, we then have $\beta \in AUT(M)$. Then α is the unique automorphism of Γ that extends β, by Corollary 4.29.

Suppose $M' \neq M$. There is a non-trivial reduced lobe path $M = \Delta^{(1)}, \Delta^{(2)}, \ldots, \Delta^{(n-1)}, \Delta^{(n)} = M'$ in Γ. Let v_k denote some intersection common to $\Delta^{(k)}$ to $\Delta^{(k+1)}$, for $1 \leq k \leq n-1$. Put $v_n = (v_1)\alpha$. Since M is a host of Γ, the lobe $\Delta^{(k+1)}$ feeds off $\Delta^{(k)}$, for $1 \leq k \leq n-1$. Let $v_k \rightarrow^{s_k} v_{k+1}$ be a path in $\Delta^{(k+1)}$, for $1 \leq k \leq n-1$. Let $\Delta^{(1)}$ and $\Delta^{(n)}$ be colored by $i \in \{1,2\}$ and put $j = 3 - i$. We have $(v_1, \Delta^{(1)}, v_1) \cong \mathscr{A}(S_i, e)$, for some $e \in E(S_i)$, and we have a loop $v_n \rightarrow^{e} v_n$ in $\Delta^{(n)}$.

Since $\Delta^{(n)}$ feeds of $\Delta^{(n-1)}$, there is a loop $v_{n-1} \rightarrow^{f_{n-1}} v_{n-1}$ in $\Delta^{(n)}$, for some $f_{n-1} \in E(U)$ with $f_{n-1} \leq s_{n-1} e s_{n-1}^{-1}$ in S_i. Next, since $\Delta^{(n-1)}$ feeds of $\Delta^{(n-2)}$, there is a loop $v_{n-2} \rightarrow^{f_{n-2}} v_{n-2}$ in $\Delta^{(n-1)}$, for some $f_{n-2} \in E(U)$ with $f_{n-2} \leq s_{n-2} f_{n-1} s_{n-2}^{-1}$ in S_j. Continuing in this manner, we obtain idempotents $f_1, f_2, \ldots, f_{n-2} \in E(U)$ with $f_k \leq s_k f_{k+1} s_k^{-1}$ in the factor, S_1 or S_2, that

s_k belongs to, and so $f_k \mathscr{R} f_k s_k f_{k+1} \mathscr{L} f_{k+1} s_k^{-1} f_k s_k \leq f_{k+1}$, for $1 \leq k \leq n-2$. Also, we have $f_{n-1} \mathscr{R} f_{n-1} s_{n-1} e \mathscr{L} e s_{n-1}^{-1} f_{n-1} s_{n-1} \leq e \leq f_1$ in S_i.

Now $f_1 \prec_j f_2 \prec_i \cdots \prec_j f_{n-1} \prec_i f_1$ and so $f_1 \succ_j f_2 \succ_i \cdots \succ_j f_{n-1} \succ_i f_1$, since $\prec \cap \succ_i \subseteq \prec_i$ and $\prec \cap \succ_i \subseteq \prec_i$. Then, for $1 \leq k \leq n-2$, we have $f_k \mathscr{D} s_k^{-1} f_k s_k = f_{k+1}$ in the factor, S_1 or S_2, that s_k belongs to, and $f_{n-1} \mathscr{D} s_{n-1}^{-1} f_{n-1} s_{n-1} = e = f_1$ in S_i, since S_1 and S_2 are completely semisimple. Since $e = f_1$ and $\Delta^{(2)}$ feeds off $\Delta^{(1)}$, it follows that $(v_1, \Delta^{(2)}, v_1) \cong \mathscr{A}(S_j, f_1)$ and $\Delta^{(2)}$ is also a host of Γ. Next, since $e_j(v_2) = s_1^{-1} f_1 s_1 = f_2$ and $\Delta^{(3)}$ feeds off $\Delta^{(2)}$, we have $(v_2, \Delta^{(3)}, v_2) \cong \mathscr{A}(S_i, f_2)$ and $\Delta^{(3)}$ is also a host of Γ. We may continue in this manner and thus obtain $(v_{n-1}, \Delta^{(n)}, v_{n-1}) \cong \mathscr{A}(S_i, f_{n-1})$ and $\Delta^{(n)}$ is also a host of Γ. It then follows that $e_i(v_n) = s_{n-1}^{-1} f_{n-1} s_{n-1} = e$.

Hence α induces an isomorphism β from $\Delta^{(1)}$ onto $\Delta^{(n)}$. Since any two homomorphisms of an inverse word graph that agree on a vertex are equal, the map α is the unique automorphism of Γ that extends β, by Corollary 4.29. Therefore $END(\Gamma) = AUT(\Gamma)$, as required. □

Theorem 4.40. *Let U denote a lower bounded inverse subsemigroup of finitely presented inverse semigroups S_1 and S_2. Then $\langle S_1 *_U S_2 \rangle$ has decidable word problem if the following hold, where (ii)-(vi) relate to Algorithm 4.20:*

(i) *The presentations $\langle S_1 \rangle$ and $\langle S_2 \rangle$ have decidable word problem.*

(ii) *It is decidable whether or not Construction 4.5 needs to be applied and having decidable language is preserved by this construction.*

(iii) *Having decidable language is preserved by taking the direct limit of all automata obtained by finitely many applications of Construction 4.5.*

(iv) *It is decidable whether or not Construction 4.12 needs to be applied and having decidable language is preserved by this construction.*

(v) *Having decidable language is preserved by taking the assimilated form.*

(vi) *It is decidable whether or not any vertex is a bud, given a path from the initial root to the vertex, and having decidable language is preserved by Construction 4.17.*

Proof. Let w be a word where the subwords w_k alternate between a word over the generators of S_1 and a word over the generators of S_2. Condition (i) implies that $\mathscr{A}(S_1 * S_2, w)$ has decidable language, by Result 2.17. Conditions (i)-(v) imply that steps (i)-(v) of Algorithm 4.20 result in a finite-lobe subopuntoid subautomaton $(\alpha_0, \Gamma_0, \beta_0)$ of $\mathscr{A}(S_1 *_U S_2, w)$, with decidable language. Let $z_1 z_2 \cdots z_n$ be a word, where the subwords z_k alternate between a word over the generators of S_1 and a word over the generators of S_2.

It is decidable whether or not the vertex α_0 is an intersection, since S_1 and S_2 are finitely presented, and it is decidable whether or not α_0 is a bud, by condition (vi). If α_0 is an intersection or a non-intersection that is not a bud of Γ_0, then put $(\alpha_1, \Gamma_1, \beta_1) = (\alpha_0, \Gamma_0, \beta_0)$. If α_0 is a bud of Γ_0 then we can apply Construction 4.17 to obtain an automaton $(\alpha_1, \Gamma_1, \beta_1)$, such that α_1 is an intersection, with decidable language, by condition (vii). By Lemmas 4.18 and 4.19, the automaton $(\alpha_1, \Gamma_1, \beta_1)$ is embedded into $\mathscr{A}(S_1 *_U S_2, w)$ as a subopuntoid subautomaton.

Now $(\alpha_1, \Gamma_1, \alpha_1)$ has decidable language, since $w \in L[(\alpha_1, \Gamma_1, \beta_1)]$. Thus it is decidable whether or not we have a path $\alpha_1 \rightarrow^{z_1} v_1$ in Γ_1. If we do not have a path $\alpha_1 \rightarrow^{z_1} v_1$ in Γ_1 then $z_1 z_2 \cdots z_n$ cannot be in $L[\mathscr{A}(S_1 *_U S_2, w)]$. Suppose we have a path $\alpha_1 \rightarrow^{z_1} v_1$ in Γ_1. It is decidable whether or not v_1 is an intersection, since S_1 and S_2 are finitely presented, and it is decidable whether or not v_1 is a bud, from condition (vi). If v_1 is an intersection of Γ_1 then put $(\alpha_2, \Gamma_2, \beta_2) = (\alpha_1, \Gamma_1, \beta_1)$. If v_1 is a non-intersection that is not a bud of Γ_1 then $z_1 z_2 \cdots z_n$ cannot be a word in $L[\mathscr{A}(S_1 *_U S_2, w)]$. If v_1 is a bud of Γ_1 then we can apply Construction 4.17 to obtain $(\alpha_2, \Gamma_2, \beta_2)$, such that the image of v_1 is an intersection, with decidable language, by condition (vi). Then $(\alpha_2, \Gamma_2, \beta_2)$ is embedded into $\mathscr{A}(S_1 *_U S_2, w)$ as a subopuntoid subautomaton, from Lemmas 4.18 and 4.19.

Now $(\alpha_2, \Gamma_2, \alpha_2)$ has decidable language, since $w \in L[(\alpha_2, \Gamma_2, \beta_2)]$. Thus it is decidable whether or not we have a path $\alpha_2 \rightarrow^{z_1 z_2} v_2$ in Γ_2. If we do not have a path $\alpha_2 \rightarrow^{z_1 z_2} v_2$ in Γ_2 then $z_1 z_2 \cdots z_n$ cannot be in $L[\mathscr{A}(S_1 *_U S_2, w)]$. Suppose we have a path $\alpha_2 \rightarrow^{z_1 z_2} v_2$ in Γ_2. It is decidable whether or not v_2 is an intersection, since S_1 and S_2 are finitely presented, and it is decidable whether or not v_2 is a bud, from condition (vi). If v_2 is an intersection of Γ_2 then put $(\alpha_3, \Gamma_3, \beta_3) = (\alpha_2, \Gamma_2, \beta_2)$. If v_2 is a non-intersection that is not a bud of Γ_2 then $z_1 z_2 \cdots z_n$ cannot be a word in $L[\mathscr{A}(S_1 *_U S_2, w)]$. If v_2 is a bud of Γ_2 then we can apply Construction 4.17 to obtain $(\alpha_3, \Gamma_3, \beta_3)$, such that the image of v_2 is an intersection, with decidable language, by condition (vii). Then $(\alpha_3, \Gamma_3, \beta_3)$ is embedded into $\mathscr{A}(S_1 *_U S_2, w)$ as a subopuntoid subautomaton, from Lemmas 4.18 and 4.19.

We can continue in this manner. If we do not have a path $\alpha_k \rightarrow^{z_1 z_2 \cdots z_k} v_k$ in Γ_k, for some k, then $z_1 z_2 \cdots z_n$ cannot be a word in $L[\mathscr{A}(S_1 *_U S_2, w)]$. Suppose we have a path $\alpha_n \rightarrow^{z_1 z_2 \cdots z_n} v_n$

in Γ_n. Then $z_1 z_2 \cdots z_n$ is a word in $L[\mathscr{A}(S_1 *_U S_2, w)]$ if and only if $v_n = \beta_n$, since $(\alpha_n, \Gamma_n, \beta_n)$ is embedded into $\mathscr{A}(S_1 *_U S_2, w)$. Thus, since $L[(\alpha_n, \Gamma_n, \beta_n)]$ is decidable, it is decidable whether or not $z_1 z_2 \cdots z_n$ is a word in $L[\mathscr{A}(S_1 *_U S_2, w)]$. Hence the automaton $\mathscr{A}(S_1 *_U S_2, w)$ has decidable language. Therefore, from Result 2.4, the word problem for $\langle S_1 *_U S_2 \rangle$ is decidable. □

Cherubini, Meakin and Piochi [7] proved that the amalgamated free product of finite (and finitely presented) inverse semigroups has decidable word problem. The following overlaps with this result.

Corollary 4.41. *Let $[S_1, S_2; U]$ be an amalgam of inverse semigroups where S_1 and S_2 have finite presentations with decidable word problems and U is finite and lower bounded in S_1 and S_2. Then $S_1 *_U S_2$ has decidable word problem.*

Proof. The result is proved by showing that conditions (i)-(vi) of Theorem 4.40 hold. Let $w = w_1 w_2 \cdots w_n$ be a word, where the w_k alternate between words over the generators of S_1 and words over the generators of S_2. Now $\mathscr{A} = (\alpha_1, \Gamma_1, \beta_1) = \mathscr{A}(S_1 * S_2, w)$ has decidable language. We have paths $\alpha_1 \rightarrow^{w_1 w_2 \cdots w_k} v_k$ in $\mathscr{A}$, for $1 \leq k \leq n$, where the vertices $v_1, v_2, \ldots, v_{n-1}$ are the only possible intersections of $\mathscr{A}$. Since we have a path $\alpha_1 \rightarrow^{w_1 w_2 \cdots w_k} v_k$, the automaton (v_k, Γ_1, v_k) has decidable language, for $1 \leq k \leq n-1$. Thus it is decidable whether or not $w_1(f)$ or $w_2(f)$ labels a loop at v_k, for each of the finitely many $f \in E(U)$, for $1 \leq k \leq n-1$. Hence it is decidable whether or not Construction 4.5 needs to be applied to $\mathscr{A}$.

If Construction 4.5 is applied to $\mathscr{A}$ then, by Results 2.5, 2.9 and 2.15, or using a proof similar to [1, Lemma 2.2], the resulting automaton is an automaton of $\langle S_1 * S_2 \rangle$, with decidable language. Since $E(U)$ is finite, there are finitely many automata in the directed system of all automata obtained from $\mathscr{A}$, by finitely many applications of Construction 4.5. Hence the direct limit of this directed system is also an automaton of $\langle S_1 * S_2 \rangle$, with decidable language.

As we have paths $\alpha_1 \rightarrow^{w_1 w_2 \cdots w_k} v_k$, the automaton (v_j, Γ_1, v_k) has decidable language, for all $1 \leq j, k \leq n-1$. Thus it is decidable whether or not $w_1(u)$ or $w_2(u)$ labels a path from v_j to v_k, for any of the finitely many $u \in U$ and $1 \leq j, k \leq n-1$. Hence it is decidable whether or not Construction 4.12 needs to be applied to $\mathscr{A}$. If Construction 4.12 is applied to $\mathscr{A}$ then, by Results 2.5, 2.9 and 2.15, or using a proof similar to [1, Lemma 3.6], it follows that the resulting automaton is an automaton of $\langle S_1 * S_2 \rangle$, with decidable language.

If $\mathscr{B} = (\alpha_2, \Gamma_2, \beta_2)$ is the assimilated form of $\mathscr{A}$ then $\mathscr{B} = \mathscr{A}/\eta$, where η is the V-equivalence generated by the related pairs. We have a path $\alpha_2 \to^z \beta_2$ in Γ_2 if and only if there are paths $x_1 \to^{z_1} y_1$, $x_2 \to^{z_2} y_2$, …, $x_m \to^{z_m} y_m$ in Γ, where $m \geq 1$, $\alpha_1 \eta x_1$, $y_1 \eta x_2$, …, $y_{m-1} \eta x_m$, $y_m \eta \beta_1$ and $z_1 z_2 \cdots z_m = z$. In which case, $a_1 z_1 a_2 z_2 \cdots a_m z_m a_{m+1} \in L[\mathscr{A}]$, where $a_k = (w_1(u))^{-1} \cdot w_2(u)$ or $a_k = (w_2(u))^{-1} \cdot w_1(u)$, for some $u \in U$, for each k. There are finitely many ways to write a given word z as a concatenation $z_1 z_2 \cdots z_m$ and U is finite. Thus it is decidable whether or not a given word z belongs to $L[\mathscr{B}]$, by testing whether or not $a_1 z_1 a_2 z_2 \cdots a_m z_m a_{m+1} \in L[\mathscr{A}]$, for the finitely many such expressions, as above. Hence $\mathscr{B}$ has decidable language.

Given a path $\alpha_2 \to^z v$ in $\mathscr{B}$, the automaton (v, Γ_2, v) has decidable language. Since U is finite, it is then decidable whether or not the vertex v is a bud. If v is a bud belonging to a lobe colored by $i \in \{1,2\}$, then Construction 4.17 can be performed by sewing on $\mathscr{A}(S_j, f)$ at v, where f is the least idempotent of $E(U)$ labeling a loop at v, and assimilating the two lobes containing the image of v. If $y \in L[(v, \Gamma_2, v) \times \mathscr{A}(S_j, f)]$ then y can be expressed as a concatenation of words alternating from $L[(v, \Gamma_2, v)]$ and $L[\mathscr{A}(S_j, f)]$. Since (v, Γ_2, v) and $\mathscr{A}(S_j, f)$ have decidable languages and there are finite ways to write a given word y as a concatenation of subwords, the automaton $(v, \Gamma_2, v) \times \mathscr{A}(S_j, f)$ has decidable language. The automaton obtained by assimilating the two lobes containing the image of v has decidable language, using a proof similar to that for the assimilated form. It follows that the automaton resulting from Construction 4.17 has decidable language. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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