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OPERATORS OF EXPONENTIAL TYPE AND THE ABSTRACT CAUCHY PROBLEM

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Abstract. In this paper, we introduce closed operators of exponential type, and use it to study the solution of the homogeneous abstract Cauchy problem of the first order, usual and fractional.

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1. INTRODUCTION

Let X be a Banach space and $I=[0,\infty)$. Let $C(I)$ be the Banach space of all bounded continuous real valued functions defined on I ,

and let $C(I,X)$ be the set of all bounded continuous function from I to X .

Now , the first order nonhomogeneous Abstract Cauchy Problem is

$$\left\{\begin{array}{c} u'(t)=Au(t)+f(t) \\ u(0)=x_\circ \end{array}\right\}.....................(1)$$

Here, u is a differentiable function from I to X, and A is a densely defined closed linear operators on X. Such equation appears in many applications in physics and applied sciences.

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The solution of such equation depends mainly on the operator A.

We will discuss in this paper the solution of (1),when $f=0$.Further, and we will discuss equation (1),when the derivative is replaced by fractional derivative.

So let us recall some basics of the conformable derivative.

In [3], the authors gave a new definition of fractional derivative which is a natural extension to the usual first derivative as follows:

Given a function $f:[0,\infty)\longrightarrow\mathbb{R}$. Then for all $t>0$, $\alpha\in(0,1)$, let

$$D_{\alpha}(f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon},$$

$D_{\alpha}f$ is called the conformable fractional derivative of f of order α.

Let $f^{(\alpha)}(t)$ stands for $D_{\alpha}(f)(t)$. Hence $f^{(\alpha)}(t)=\lim\limits_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$.

If f is $\alpha-$ differentiable in some $(0,b)$, $b>0$, and $\lim\limits_{t\to 0^{+}}f^{(\alpha)}(t)$ exists, then let

$$f^{(\alpha)}(0)=\lim_{t\to 0^{+}}f^{(\alpha)}(t).$$

The conformable derivative satisfies all the classical properties of the usual first derivative.

Further, according to this derivative, the following statements are true, see [3].

1. $D_{\alpha}(t^{p})=pt^{p-\alpha}$ for all $p\in\mathbb{R}$,
2. $D_{\alpha}(\sin\frac{1}{\alpha}t^{\alpha})=\cos\frac{1}{\alpha}t^{\alpha}$,
3. $D_{\alpha}(\cos\frac{1}{\alpha}t^{\alpha})=-\sin\frac{1}{\alpha}t^{\alpha}$,
4. $D_{\alpha}(e^{\frac{1}{\alpha}t^{\alpha}})=e^{\frac{1}{\alpha}t^{\alpha}}$.

The $\alpha-$fractional integral of a function f starting from $a\geq 0$ is :

$$I_{\alpha}^{a}(f)(t)=I_{1}^{a}(t^{\alpha-1}f)=\int_{a}^{t}\frac{f(x)}{x^{1-\alpha}}\,dx,$$

In this paper we will study the Abstract Cauchy Problem :

$$\left\{\begin{array}{c} u^{(\alpha)}(t)=Au(t)\\ u(0)=x_{\circ}\end{array}\right\}\text{....................}(2)$$

We refer [1] and [3] for more on conformable fractional derivative, and to [4] for the theory of semigroups of operators and the Abstract Cauchy Problem.

2. Exponential Type Operators

In this section we introduce a class of operators to be called of exponential type.

Definition 2.1. Let $A : Dom(A) \subseteq X \to X$, be a densely defined linear operator. The operator A is called of exponential type if

(i) The operator $B(t) = e^{tA}$ exists and well defined for all $x \in Dom(A)$, in the sense: $B(t)x = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x$ converges absolutely for all $x \in Dom(A)$. That is $\sum_{n=0}^{\infty} \frac{t^n}{n!} \|A^n x\| < \infty$.

(ii) $C(x,A) = \{x, Ax, A^2x, ...\} \subseteq Dom(A)$.

Examples. (1) Clearly every bounded linear operator on a Banach space X is of exponential type.

(2) Consider the operator $T : \ell^2 \to \ell^2$ defined by $T(\delta_n) = n\, \delta_1$. Clearly T is densely defined. Further:

$$T(a_1\delta_1 + ...a_k\ \delta_k) = (\sum_{n=1}^{k} ia_i)\delta_1$$

Hence $T^2x = Tx$. In fact $T^n x = Tx$, for any x which can be written as a finite linear combination of the basis elements $\{\delta_1, ...\delta_n, ...\}$

Hence

$$\|A^n x\| \leq \frac{k(k+1)}{2} \sup_{1 \leq j \leq k} |a_j|$$

Now

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \|A^n x\| \leq \sup_{1 \leq j \leq k} |a_j| \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{k(k+1)}{2} < \infty$$

Thus, T is of exponential type.

Theorem 2.1. Let $A : Dom(A) \subseteq X \to X$ be of exponential type. Then $\frac{d}{dt}B(t)x = \frac{d}{dt}e^{tA}x = Ae^{tA}x = AB(t)x$ for $x \in Dom(A)$

Proof. $B(t)x = B(t)x = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x$. Then using classical tools we get

$$\begin{aligned} \frac{d}{dt}B(t)x &= \sum_{n=1}^{\infty} n \frac{t^{n-1}}{n!} A^n x \\ &= \sum_{n=1}^{\infty} A \frac{t^{n-1}}{(n-1)!} A^{n-1} x \end{aligned}$$

$$= A\sum_{n=1}^{\infty}\frac{t^{n-1}}{(n-1)!}A^{n-1}x$$
$$= A\sum_{n=0}^{\infty}\frac{t^{n}}{n!}A^{n}x$$
$$= AB(t)x$$

This ends the proof.

Theorem 2.2. Let $A : Dom(A) \subseteq X \to X$ be of exponential type, and $u : [0,\infty) \to X$ be differentiable.

Then $\left\{ \begin{array}{c} u'(t) = Au(t) \\ u(0) = x_\circ \end{array} \right\}$ has a unique solution.

Proof. It follows from Theorem 2.1 that $u_1(t) = e^{tA}x_0$ is a solution.

Assume if possible that u_2 is another solution. Then $u_2'(t) = Au_2(t)$. But then

$$u_2'(t) - u_1'(t) = A(u_2(t) - u_1(t))$$

But this implies that $u_2(t) - u_1(t) = e^{tA}y$. Since $u_2(0) - u_1(0) = x_0 - x_0 = 0,$ it follows that $y = 0$, and hence $u_2(t) = u_1(t)$.

3. Fractional Abstract Cauchy Problem

Let us write $D_\alpha(f)(t)$ for $f^{(\alpha)}(t) = \lim\limits_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$. In this section we are interested in discussing $\left\{ \begin{array}{c} u^{(\alpha)}(t) = Au(t) \\ u(0) = x_\circ \end{array} \right\}$.

Let us recall that, [2]:

$T : [0,\infty) \to L(X)$, the space of bounded linear operators on X, is called an α-fractional semigroup of operators if $T(0) = I$ and $T(s+t)^{\frac{1}{\alpha}} = T(s^{\frac{1}{\alpha}})T(t^{\frac{1}{\alpha}})$.

The generator of the semigroup $T(t)$ is just the $\alpha-$conformable derivative of $T(t)$ at $t = 0$.

We refer to [2] for more results on fractional semigroups of operators.

Now we have:

Theorem 3.1. Let $A : Dom(A) \subseteq X \to X$ be of exponential type. Then $T(t)x = e^{\frac{t^\alpha}{\alpha}A}x$ is an $\alpha-$fractional semigroup with A as the generator.

Proof. That $T(t)x = e^{\frac{t^\alpha}{\alpha}A}x$ is an $\alpha-$fractional semigroup is straight forward computations.

Consider $D_\alpha T(t)x = D_\alpha \sum\limits_{n=0}^{\infty} \frac{(\frac{t^\alpha}{\alpha})^n}{n!} A^n x.$

Using the same ideas in Theorem 2.1, we get

$$D_\alpha T(t)x = AT(t)x = e^{\frac{t^\alpha}{\alpha}A}x.............(2)$$

Note that we used $D_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$.

Taking the limit as $t \to 0$, we get $D_\alpha T(0)x = A$. That ends the proof.

Now we discuss the fractional Abstract Cauchy Problem

$$\left\{ \begin{array}{c} u^{(\alpha)}(t) = Au(t) \\ u(0) = x_\circ \end{array} \right\} ....(3)$$

Theorem 3.2. If A is of exponential type, then (3) has a unique solution.

Proof. By (2) in Theorem 3.1, we get $u_1(t) = e^{\frac{t^\alpha}{\alpha}A}x_0$ as a solution of (3).

Using the same idea as in Theorem 2.2, we get our result.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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