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INVERSE RINGS AND INVERSE SEMIGROUPS OF RING HOMOMORPHISMS

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Abstract. In this paper, a ring is an inverse ring if its multiplicative semigroup is an inverse semigroup. We show that an inverse ring contains no nilpotent elements except 0 and that it is a subring of a subdirect product of skew fields.

Let $R = Z_n$. Let $(H(R), \circ)$ be the semigroup of ring homomorphisms(under composition) on R. We show that $H(R)$ is a commutative inverse semigroup and it is of order 2^n and that each of its elements has order 2 or less.

We show that the set of regular-ring homomorphisms on $Z_p[x]$, where p is a prime, is an inverse semigroup.

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1. Introduction

An inverse semigroup is a semigroup S such that each element a has a unique inverse b such that $aba, bab = b$. S is regular if for all a in S, there is b in S such that $aba = a$. Let R be a ring. One can ask the question: What are the rings R such that the multiplicative semigroup $(R, .)$ is an inverse semigroup?. Let us call such a ring as an *inverse ring*. If we

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just require that $(R, .)$ is a regular semigroup we get the so called *Von Neumann rings*. We cite the following theorem (See Ribenboim [3] p.134)

Theorem(A) A ring R is a Von Neumann ring whose only nilpotent element is 0 if and only if R is isomorphic to a ring C which is a subdirect product of skew fields $(B_i)_{i\in I}$, such that if $(b_i)_{i\in I} \in C$ then there exists $(b'_i)_{i\in I} \in C$, verifying that $b'_i = b_i^{-1}$ when $b_i \neq 0$.

2. Preliminaries

Most facts used about inverse semigroups can be found, for example, in [2]. Facts about Von Neumann regular rings can be found in [3].

3. Main results

Now we prove the following

Proposition 3.1. *Let R be an inverse ring. Then R cannot contain a non zero nilpotent element.*

Proof. Let $a \neq 0$ be an element such that $a^2 = 0$. We have $(a + aa^{-1})^2 = (a + aa^{-1})(a + aa^{-1}) = a + aa^{-1}$. Thus $(a + aa^{-1})$ is an idempotent. Since the idempotents commute we have $(a + aa^{-1})aa^{-1} = aa^{-1}(a + aa^{-1})$. Thus $aa^{-1} = a + aa^{-1}$. But then $a = 0$ and this is a contradiction.
This completes the proof.

Corollary 3.2. *A ring R is an inverse ring if and only if R is isomorphic to a ring C which is a subdirect product of skew fields $(B_i)_{i\in I}$, such that if $(b_i)_{i\in} I \in C$ then there exists $(b'_i)_{i\in I} \in C$, verifying that $b'_i = b_i^{-1}$ when $b_i \neq 0$*

Let R be the ring Z_m for some m. Then as it is well-known $R = \oplus_{i=1}^{n} Z_{p_j^{m_j}}, p_j$ are primes.

As a result the ring $R = Z_{p^m},\ m > 1$ is not an inverse ring because there are nonzero nilpotent elements. We notice that $ch(R)$ is not square-free.

Remark 3.3. Let $(R, .)$ be an inverse ring whose characteristic is finite. Then $ch(R)$ is square-free. Otherwise R would contain a non zero nilpotent element which is not the case.

We will show in an example 3.5 that this is not sufficient (even if the ring is finite and commutative) to give an inverse ring.

Proposition 3.4. *If R is an inverse ring such that every element has a finite number of exponents. Then the multiplicative semigroup $(R, .)$ is a union of cyclic semigroups.*

Proof. To show this assume that there is a nonzero element a such that $a^k = a^m, 1 < k < m$. Let m, k be the first such pair. Let $aba = a$. Then $a^m.b = a^k.b$ and so by regularity, $a^{m-1} = a^{k-1}$ and this contradicts the choice of m and k.
This completes the proof.

Example 3.5. There is a finite commutative ring R with 1 whose characteristic is square-free which is not an inverse ring. Consider Z_p, p is a prime integer. Let R be the ring of all 3 by 3 matrices with entries from Z_p of the form $\begin{bmatrix} 0 & m & k \\ 0 & 0 & m \\ 0 & 0 & 0 \end{bmatrix}, m, k \in Z_p$. This is a finite commutative ring whose characteristic is p and whose elements are nilpotent and hence it is not an inverse ring.

4. Inverse Semigroups of Ring Homomorphisms

Let R be a ring. Then the set of R-homorphisms, $Hom(R)$, under composition is a semigroup. One might ask under what conditions on R the semigroup $Hom(R)$ is an inverse semigroup?

Proposition 4.1. *Let $n = p_1^{e_1}...p_k^{e_k}$,p_i are distinct primes and consider the ring $R = Z_n$. Then (1) the number of idempotent elements in is 2^n. (2) The set $H(R)$of all ring homomorphisms on M is a commutative inverse semigroup under composition ; It is of order 2^n and each of its non identity elements is of order 2.*

Proof. First let R be the ring Z_{p^n}. If $f : R \longrightarrow R$ is a ring homomorphism then there is an element a in R such that $f(x) = ax$ and then $a^2 = a$. The only such a is 0 or 1. For $a^2 - a = 0 \ mod \ p^n$ implies that p^n divides $a(a-1)$. Thus p divides a or $p-1$ but not both since it is a prime. But then p^n divides a or $a-1$ although both of a and $a-1$ are less than p^n. Thus either $a = 1$ or $a = 0$. Thus the only ring homomorphisms on R are the zero and the identity mappings. In the general case R is a finite direct product of n rings of the form Z_{p^m} and thus its elements can be considered as $n-$tuples $a = (a_1, ..., a_n)$. Since R is cyclic under addition every ring homomorphism is of the form $f(x) = ax$ with $a^2 = (a_1^2, ..., a_n^2) = a = (a_1, ..., a_n)$. Since every a_i has only two choices 0 and 1 we have 2^n solutions as claimed. The rest of the proposition is clear. Thus the semigroup of ring homomorphisms $H(R)$ under mapping composition is a commutative semigroup of 2^n elements and such that $f^2 = fof = f$ for all f in $H(R)$. Thus $(H(R), \circ)$ is a commutative inverse semigroup of 2^n elements.
This completes the proof.

Example 4.2. Let R be the ring $Z \oplus Z$ and f be a ring homomorphism on R. Let $f(1,0) = (x,y)$ and $f(0,1) = (z,t)$. Then $(x,y)^2 = (x,y), (z,t)^2 = (z,t)$ and $(x+z, y+t)^2 = (x+z, y+t)$. Thus $x = 0$ or $1, y = 0$ or $1, z = 0$ or 1 and $t = 0$ or 1. We notice that $x + y = 0$ or 1 and $z + t = 0$ or 1. Thus $[f] = \begin{pmatrix} x & z \\ y & t \end{pmatrix}$. We have 16 options except 4 options. The exceptions are $\begin{pmatrix} 1 & 1 \\ x & y \end{pmatrix}$, $\begin{pmatrix} x & z \\ 1 & 1 \end{pmatrix}$, x, y, z, t in $\{0, 1\}$. Thus there are 7 exceptions and so we have 9 ring homomorphisms on $Z \oplus Z$. *Thus we have* $(H(R), \circ)$ *as a non-commutative inverse semigroup of 9 elements.* In this case $[f \circ g] = \begin{pmatrix} x & z \\ y & t \end{pmatrix} \begin{pmatrix} x' & z' \\ y' & t' \end{pmatrix}$ and so $\circ$ is not commutative. Why is it an inverse semigroup? I have simply tried all 9 cases and have found a unique inverse for each element; thus it is an inverse semigroup.

Example 4.3. If we go to the case $R = Z \oplus Z \oplus Z$ then we get a similar non-commutative inverse semigroup. Let f be a ring homomorphism and let $f(1,0,0) = (a,b,c), f(0,1,0) = (x,y,z)$ and $f(0,0,1) = (u,v,w)$. Since $(1,0,0), (0,1,0), (0,0,1)$ and the sum of any two or three of them are idempotents we see that

$$aa = a, bb = b, cc = c, xx = x, yy = y, zz = z, uu = u \ , \ vv = v, ww = w,$$

$$(a + x)(a + x) = a + x \ , \ (b + y)(b + y) = b + y, (c + z)(c + z) = c + z,$$

$$(a + x + u)(a + x + u) = (a + x + u), (b + y + v)(b + y + v) = b + y + v,$$

and $(c + z + w)(c + z + w) = c + z + w$. So we get a matrix $[f] = \begin{pmatrix} a & x & u \\ b & y & v \\ c & z & w \end{pmatrix}$. So we get 64 matrices. To see this we compute the number of ways of filling the first row. It is $1 + 3 = 4$. It is 1 if all entries are 0. The three come from filling in one entry of value 1. Similarly there are 4 ways of filling in the second row and four ways of filling the third row. Giving total of $4.4.4 = 64$.

Corollary 4.3. *In general there are $(n + 1)^n$ of ways of forming n by n matrix of the kind mentioned above: At most one entry of value 1 in each row. Thus for each $n > 1$ there is a non-commutative inverse semigroups of order $(n + 1)^n$.*

Remark 4.4. The above discussion can be generalized to the case of $R \otimes R \otimes ... \otimes R, k$ times; where R is the ring Z_n. There are m idempotent elements where m is the number of distinct factors of n of the form power of a prime. Then the ring homomorphisms we are looking for can be identified with all k by k matrices. Each matrix has at most one entry filled with an idempotent in R. Thus the first row can be dealt with in $km + 1$ ways. Similarly the second and all other rows. Thus we have in total $(km + 1)^k$ matrices and hence there is a non commutative semigroup of order $(km + 1)^k$. These semigroups are not known if they are inverse semigroups or not except in very special cases, some of which are mentioned above.

One may go further to consider the following problem. Let R be a finite commutative ring whose additive group is cyclic of order n and whose characteristic is square free. Let S be a finite direct product of k copies of R. The problem is to find all ring homomorphisms on R and on S.

5. Regular-Ring Homomorphisms on $Z_p[x]$ Is An Inverse Semigroup

Let $R = Z_p[x], p$ is a prime. We are looking for all ring homomorphisms on R.

Remark 5.1. Any such homomorphism F satisfies $F(1.1) = F(1) = a = aa, a \in R$. The only such a is 0 or 1. If it is 0 then $F = 0$. If a is 1 then we look for $F(x)$ being the image of x ; i.e. some $f(x)$. We call $f(x)$ the core of F. Then $F(h(x)) = h(f(x))$. It is easy to show that this is a ring homomorphism, and conversely any ring homomorphism on $Z_p[x]$ takes this form. Let F, G be two non-zero ring homomorphisms on $Z_p[x]$ and let $F(x) = f(x)$ and $G(x) = g(x)$. Then $FoG(h(x)) = F(G(h(x))) = F(h(g(x))) = h(g(f(x))) = h(gof(x))$.

Corollary 5.2. *The set of ring homomorphisms $H(Z_p[x])$ under composition of mappings is isomorphic to the semigroup $(Z_p[x], o)_{op}$.*

The semigroup $(Z_p[x], o)_{op}$ is is not an inverse semigroup but only a non-commutative semigroup.

Proposition 5.3. *Let S be the set of all ring homomorphisms F on $Z_p[x]$ with core of degree 1 or less. Then S is an inverse semigroup. If T is the subsemigroup of S whose elements has core of degree 1 then T is the group of automorphisms on $Z_p[x]$.*

Proof. If we take all homomorphisms F such that its corresponding $f(x)$ is of degree 1 then we get a non Abelian group of all automorphisms on the ring $Z_p[x]$ and they correspond to set of $\{a+bx : b$ is a unit under the operation $(a+bx)o(c+dx) = a+bc+bdx$. Thus we have found the set of all ring automorphisms on $R = Z_p[x]$. If we take the set of all ring homomorphisms F on $Z_p[x]$ whose core f has degree 1 or less then we will get in addition to the set of automorphisms the homomorphisms F whose core $f = a$ and it follows that $F(g) = g(f(x)) = g(a)$ and so the homomorphism F in this case is the evaluation map at some a in Z_p. Let us take the homomorphism G whose core is $1/a$. Thus $F(f(x)) = f(1/a)$. Then $FoGoF(f(x)) = FoG(f(a)) = F(f(a)) = f(a) = F(f(x))$, $GoFoG(f(x)) = GoF(f(1/a)) = G(f(1/a)) = f(1/a) = G(f(x))$.Thus we have two cases of ring homomorhisms and in each case there is a unique generalized inverse. Thus the set of regular ring homomorphisms on $Z_p[x]$ under composition is a finite non commutative inverse semigroup.

This completes the proof.

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