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ON THE DECOMPOSITION OF COMPLETELY REGULAR ORDERED SEMIGROUPS INTO UNION OF LEFT AND RIGHT SIMPLE ORDERED SEMIGROUPS

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Abstract: We give sufficient conditions (Rs - condition and Ls - condition) which a completely regular ordered semigroup S must satisfy so that the set $G_e = \{\alpha \in S \mid \alpha \in (eS] \cap (Se],\ e \in (\alpha S] \cap (S\alpha]\}$, $e \in E(S)$, is (maximal under the inclusion relation) right and left simple subsemigroup of S, where $E(S)$ is the set of elements of S for which $e \le e^2$. We prove that $S = \bigcup_{e \in E(S)} G_e$ and thus S is decomposed into a union of (disjoint) right and left simple semigroups. In addition every $\mathcal{N}$ - class of a completely regular ordered semigroup satisfying Rs - condition and Ls - condition is a union of right and left simple subsemigroups of S, where $\mathcal{N}$ is the least complete semilattice congruence on S. Finally we prove that the previous decompositions into unions of right and left simple semigroups characterize equivalently an ordered semigroup satisfying both Rs - condition and Ls - condition in order to be a completely regular ordered semigroup.

Keywords: ordered semigroup; completely (right, left) ordered semigroup; right (left) simple ordered semigroup; least complete semilattice congruence; union of left and right simple ordered semigroups.

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1. INTRODUCTION

Let e be an idempotent of a semigroup S. Then the set $\{\alpha \in S \mid \alpha \in eS \cap Se,\ e \in \alpha S \cap S\alpha\}$ is the greatest subgroup of S having e as its identity [2, I.4.11 PROPOSITION]. The family *F* of the above sets for all idempotents of S plays a fundamental role in the decomposition of completely regular semigroups into union of groups since every completely regular semigroup is union of the sets of *F* (which means that every completely regular semigroup is union of its maximal subgroups [2, Chapter IV]). The aim of the paper is to study "similar concepts" in case of ordered semigroups. Since a semigroup without order is a group if and only if it is left and right simple [1, §1.1, §1.11], it is natural to consider left and right simple ordered semigroups instead of ordered groups (we cannot prove properties of groups such as the existence of unit or the existence of reverse element based on order relation). In the paper we give conditions a completely regular ordered semigroup must satisfy so that it can be decomposed into union of left and right simple ordered semigroups.

2. PRELIMINARIES

An ordered semigroup $(S,\cdot,\leq)$ is a semigroup $(S,\cdot)$ endowed with an order relation $\leq$ which is compatible with the operation (that is, $\alpha \leq b$ implies $c\cdot\alpha \leq c\cdot b$ and $\alpha\cdot c \leq b\cdot c$ for all $c \in S$). Throughout the following paper we always refer to ordered semigroups. We define $E(S)=\{e \in S \mid e \leq e^2\}$ and $(A]=\{x \in S \mid x \leq y \text{ for some } y \in S\}$ for A a nonempty subset of S. We say that S is

- left regular [5, 7, 9] if $\alpha \in (S\alpha^2]$ for every $\alpha \in S$
- right regular [4, 7] if $\alpha \in (\alpha^2 S]$ for every $\alpha \in S$
- completely regular [7, 10] if $\alpha \in (\alpha^2 S\alpha^2]$ for every $\alpha \in S$.

Since $(\alpha^2 S\alpha^2] \subseteq (S\alpha^2] \cap (\alpha^2 S]$ then every completely regular ordered semigroup is both right regular and left regular ordered semigroup.

Also S is

- left simple [6, 8] if $S \subseteq (S\alpha]$ for every $\alpha \in S$
- right simple if $S \subseteq (\alpha S]$ for every $\alpha \in S$.

A nonempty subset of S is a filter of S [3] if

i) for $\alpha, b \in S$ we have

$$\alpha b \in T \Leftrightarrow \alpha, b \in T$$

ii) for $\alpha \in T$ and $b \in S$ such that $\alpha \le b$ we have $b \in T$.

For $\alpha \in S$ we denote by $N(\alpha)$ the least filter of S containing α. The relation $\mathcal{N}$ on S defined as follows [3]

$$\alpha \mathcal{N} b \Leftrightarrow N(\alpha) = N(b)$$

is the least complete semilattice congruence on S (a relation σ on S is a complete semilattice congruence on S [3] if it is a congruence on S having two more properties:

I) for $\alpha, b \in S$ such that $\alpha \le b$ it holds $(\alpha, \alpha b) \in \sigma$

II) $(\alpha b, b\alpha) \in \sigma$ for every $\alpha, b \in S$).

In [3] one can find a detailed study about $\mathcal{N}$ which plays an important role in the decomposition of ordered semigroups as well as in the decomposition of semigroups without order [2, Chapter II]. We may refer here to the systematic study on decomposition of ordered semigroups given by the author and N. Kehayopulu through many joint papers concerning complete semilattices of ordered semigroups of various type.

3. MAIN RESULTS

Definition 1: Let $(S, \cdot, \le)$ be an ordered semigroup and $e \in E(S)$. Then

i) $RG_e = \{\alpha \in S \mid \alpha \in (eS],\ e \in (\alpha S]\}$

ii) $LG_e = \{\alpha \in S \mid \alpha \in (Se],\ e \in (S\alpha]\}$

iii) $G_e = LG_e \cap RG_e$ ($= \{\alpha \in S \mid \alpha \in (eS] \cap (Se],\ e \in (\alpha S] \cap (S\alpha]\}$). □

Since for $e \in E(S)$ we clearly have $e \in LG_e \cap RG_e = G_e$, then RG_e, LG_e, G_e are nonempty subsets of S. Also for $e, f \in E(S)$ we immediately have $f \in RG_e$ (resp. $f \in LG_e$) if and only if $e \in RG_f$ (resp. $e \in LG_f$) and hence $f \in G_e$ if and only if $e \in G_f$.

As we mentioned in Introduction, if S is a semigroup without order and e is an idempotent of S, then the set $\{\alpha \in S \mid \alpha \in eS \cap Se,\ e \in \alpha S \cap S\alpha\}$ is a subgroup of S having e as its identity which means that e is the unique idempotent of the previous set. We can prove it as following:

Let f be an idempotent of S such that $f \in \{\alpha \in S \mid \alpha \in eS \cap Se,\ e \in \alpha S \cap S\alpha\}$. Thus $e \in Sf$ and $f \in eS$ and hence $e = sf$ and $f = et$ for some $s, t \in S$. Consequently

$$f = et = e^2 t = e(et) = ef = (sf)f = sf^2 = sf = e$$

This implies that $e = f = f^2 \in fS$.

From the above we observe that if e, f are idempotents of S with $e \in Sf$ and $f \in eS$ then it follows that $e \in fS$. Similarly we show that if e, f are idempotents of S such that $e \in fS$ and $f \in Se$, then $e \in Sf$. But, in general, an ordered semigroup does not meet such properties for $e, f \in E(S)$. So, it is natural to assume that "similar" properties should also be kept true in the case of ordered semigroups for the elements of the set $E(S)$. Therefore, since clearly a semigroup without order is an ordered semigroup with order relation the equality relation, we give the following

Definition 2: We say that an ordered semigroup S satisfies

i) Rs - condition if for $e, f \in E(S)$ such that $e \in (Sf]$ and $f \in (eS]$ we have $e \in (fS]$.

ii) Ls - condition if for $e, f \in E(S)$ such that $e \in (fS]$ and $f \in (Se]$ we have $e \in (Sf]$.

□

By Definition 1i), ii) we immediately get that S satisfies

- Rs - condition if and only if for $e, f \in E(S)$ such that $e \in (Sf]$ and $f \in (eS]$ we have $f \in RG_e$ (equivalently $e \in RG_f$)

- Ls - condition if and only if for $e,f \in E(S)$ such that $e \in (fS]$ and $f \in (Se]$ we have $f \in LG_e$ (equivalently $e \in LG_f$).

We will next prove that Rs - condition and Ls - condition together are sufficient conditions for a completely regular ordered semigroup to be (equivalently) a union of right and left simple ordered semigroups.

Proposition 3: Let $(S,\cdot,\le)$ be an ordered semigroup and $e,f \in E(S)$ such that $RG_e \cap RG_f \ne \varnothing$ (resp. $LG_e \cap LG_f \ne \varnothing$). Then $RG_e = RG_f$ (resp. $LG_e = LG_f$).

Proof: Since $RG_e \cap RG_f \ne \varnothing$, then there exists $\alpha \in S$ such that $\alpha \in RG_e \cap RG_f$. Consequently, by Definition 1i), $\alpha \le ex$, $e \le \alpha y$, $\alpha \le fz$ and $f \le \alpha w$ for some $x,y,z,w \in S$. Therefore $f \le \alpha w \le exw$. Now let $b \in RG_e$. We will prove that $b \in RG_f$. Indeed:

Since $b \in RG_e$ then, by Definition 1i), $b \le eu$ and $e \le bv$.

- $b \le eu \le (\alpha y)u = \alpha(yu) \le (fz)(yu) = f(zyu)$
- $f \le \alpha w \le (ex)w = e(xw) \le (bv)(xw) = b(vxw)$

Thus (Definition 1i)) $b \in RG_f$ and so we proved that $RG_e \subseteq RG_f$. Similarly $RG_f \subseteq RG_e$.

□

Corollary 4: Let $(S,\cdot,\le)$ be an ordered semigroup and $e,f \in E(S)$ such that $G_e \cap G_f \ne \varnothing$. Then $G_e = G_f$.

Proof: Since $G_e \cap G_f \ne \varnothing$ and

$$G_e \cap G_f = (RG_e \cap LG_e) \cap (RG_f \cap LG_f) = (RG_e \cap RG_f) \cap (LG_e \cap LG_f)$$

we have $RG_e \cap RG_f \ne \varnothing$ and $LG_e \cap LG_f \ne \varnothing$. Then (Proposition 3) $RG_e = RG_f$ and $LG_e = LG_f$. Consequently (Definition 1iii)) $G_e = G_f$. □

Remark 5: For $e,f \in E(S)$, by Proposition 3 and Corollary 4, we have $f \in RG_e$ **(**resp. $f \in LG_e$ or $f \in G_e$) if and only if $RG_f = RG_e$ (resp. $LG_f = LG_e$ or $G_f = G_e$). □

Proposition 6: Let $(S,\cdot,\leq)$ be a right (resp. left) regular ordered semigroup and $e\in E(S)$. Then RG_e (resp. LG_e) is a subsemigroup of S containing e.

Proof: As we mentioned above RG_e is a nonempty subset of S (clearly $e\in RG_e$). Now let $\alpha,b\in RG_e$. Then (Definition 1i)) $\alpha,b\in(eS]$ and $e\in(\alpha S]\cap(bS]$. Therefore there exist $x,y,z,w\in S$ such that $\alpha\leq ex$, $b\leq ey$, $e\leq\alpha z$ and $e\leq bw$. Since S is a right regular ordered semigroup and $\alpha\in S$, we have $\alpha\in(\alpha^2 S]$. Thus $\alpha\leq\alpha^2 u$ for some $u\in S$. Consequently

$$e\leq\alpha z\leq\alpha^2 uz=\alpha\alpha uz\leq\alpha(ex)uz=\alpha e(xuz)\leq\alpha(bw)(xuz)=\alpha b(wxuz)$$

Hence $e\in(\alpha bS]$. We also have $\alpha b\in(eS]$ because $\alpha b\leq exb$. Therefore (Definition 1i)) $\alpha b\in RG_e$. □

By Proposition 6 and Definition 1 we have the following

Corollary 7: Let $(S,\cdot,\leq)$ be a right and left regular ordered semigroup and $e\in E(S)$. Then G_e is a subsemigroup of S containing e. □

Theorem 8: Let $(S,\cdot,\leq)$ be a right and left regular ordered semigroup and $\alpha\in S$. Then there exists $e\in\alpha S\alpha\cap E(S)$ such that $\alpha^2\leq\alpha^2 e$, $\alpha^2\leq e\alpha^2$ and $\alpha\in G_e$.

Proof: Since S is right and left regular ordered semigroup then there exist $x,y\in S$ such that $\alpha\leq\alpha^2 x$ and $\alpha\leq y\alpha^2$. Now we have

$$y^3\alpha\leq y^3\alpha^2 x=(y^3\alpha)\alpha x\leq(y^3\alpha)(\alpha^2 x)x=(y^3\alpha^2)\alpha x^2\leq(y^3\alpha^2)(\alpha^2 x)x^2=y^3\alpha^4 x^3$$

Consequently $\alpha y^3\alpha^2=\alpha(y^3\alpha)\alpha\leq\alpha(y^3\alpha^4 x^3)\alpha=\alpha y^3\alpha^4 x^3\alpha$.

Similarly $\alpha^2 x^3\alpha\leq\alpha y^3\alpha^4 x^3\alpha$. Therefore

$$\alpha y^3\alpha^4 x^3\alpha=(\alpha y^3\alpha^2)(\alpha^2 x^3\alpha)\leq(\alpha y^3\alpha^4 x^3\alpha)(\alpha y^3\alpha^4 x^3\alpha)$$

Then for $e=\alpha y^3\alpha^4 x^3\alpha$ we have $e\in E(S)\cap\alpha S\alpha$. We will show that $\alpha^2\leq\alpha^2 e$. Indeed:

$$\alpha^2 e=\alpha^2\left(\alpha y^3\alpha^4x^3\alpha\right)=\left(\alpha^3y^2\right)\left(y\alpha^2\right)\left(\alpha^2x^3\alpha\right)\geq\left(\alpha^3y^2\right)\alpha\left(\alpha^2x^3\alpha\right)=$$
$$=\left(\alpha^3y\right)\left(y\alpha^2\right)\left(\alpha x^3\alpha\right)\geq\left(\alpha^3y\right)\alpha\left(\alpha x^3\alpha\right)=\alpha^3\left(y\alpha^2\right)\left(x^3\alpha\right)\geq\alpha^3\alpha\left(x^3\alpha\right)=$$
$$=\alpha^2\left(\alpha^2x\right)\left(x^2\alpha\right)\geq\alpha^2\alpha\left(x^2\alpha\right)=\alpha\left(\alpha^2x\right)\left(x\alpha\right)\geq\alpha\alpha\left(x\alpha\right)=\left(\alpha^2x\right)\alpha\geq\alpha\alpha=\alpha^2$$

Similarly $\alpha^2\leq e\alpha^2$. Also

$$\alpha\leq y\alpha^2\leq y\left(\alpha^2e\right)=\left(y\alpha^2\right)e$$

so that $\alpha\in(Se]$ and since $e\in S\alpha\subseteq(S\alpha]$ we have (Definition 1ii)) that $\alpha\in LG_e$. Similarly $\alpha\in RG_e$. Therefore (Definition 1iii)) $\alpha\in G_e$. □

Immediately, by Theorem 8 and Definition 1, it follows

Corollary 9: Let $(S,\cdot,\leq)$ be a right and left regular ordered semigroup. Then

$$S=\bigcup_{e\in E(S)}G_e=\bigcup_{e\in E(S)}RG_e=\bigcup_{e\in E(S)}LG_e$$ □

Proposition 10: Let $(S,\cdot,\leq)$ be an ordered semigroup, $e\in E(S)$ and $\alpha\in RG_e$ (or $\alpha\in LG_e$). Then $e\in(\alpha)_{\mathcal{N}}$.

Proof: Since $\alpha\in RG_e$ then we have $\alpha\leq ex$ and $e\leq\alpha y$ for some $x,y\in S$. Since $N(\alpha)$ is a filter of S containing α, $\alpha\leq ex$ implies $ex\in N(\alpha)$ and this in turn implies $e\in N(\alpha)$. Since $N(\alpha)$ is a filter of S and $N(e)$ is the least filter of S containing e, $e\in N(\alpha)$ implies $N(e)\subseteq N(\alpha)$. Similarly $N(\alpha)\subseteq N(e)$. Thus $N(\alpha)=N(e)$ which means $(e,\alpha)\in\mathcal{N}$. Thus $e\in(\alpha)_{\mathcal{N}}$. □

Since (Definition 1iii)) $G_e\subseteq RG_e$ then, by Proposition 10, we immediately have the following

Corollary 11: Let $(S,\cdot,\leq)$ be an ordered semigroup, $e\in E(S)$ and $\alpha\in G_e$. Then $e\in(\alpha)_{\mathcal{N}}$.

□

Proposition 12: Let $(S,\cdot,\leq)$ be a right and left regular ordered semigroup and $\alpha\in S$. Then $(\alpha)_{\mathcal{N}}=\bigcup_{e\in E(S)\cap(\alpha)_{\mathcal{N}}}RG_e=\bigcup_{e\in E(S)\cap(\alpha)_{\mathcal{N}}}LG_e=\bigcup_{e\in E(S)\cap(\alpha)_{\mathcal{N}}}G_e$.

Proof:

- $(\alpha)_{\mathcal{N}} \subseteq \bigcup_{e \in E(S) \cap (\alpha)_{\mathcal{N}}} RG_e$: Let $b \in (\alpha)_{\mathcal{N}}$. By Corollary 9, $b \in RG_e$ for some $e \in E(S)$. It suffices to show $e \in (\alpha)_{\mathcal{N}}$. Since $b \in RG_e$ then, by Proposition 10, $e \in (b)_{\mathcal{N}}$. Since $b \in (\alpha)_{\mathcal{N}}$ we have $(b)_{\mathcal{N}} = (\alpha)_{\mathcal{N}}$. Consequently $e \in (\alpha)_{\mathcal{N}}$ and so we obtain $e \in E(S) \cap (\alpha)_{\mathcal{N}}$.
- $\bigcup_{e \in E(S) \cap (\alpha)_{\mathcal{N}}} RG_e \subseteq (\alpha)_{\mathcal{N}}$: Let $b \in RG_e$ for some $e \in E(S) \cap (\alpha)_{\mathcal{N}}$. We will prove that $b \in (\alpha)_{\mathcal{N}}$. Since $b \in RG_e$ then, by Proposition 10, $e \in (b)_{\mathcal{N}}$ which implies that $b \in (e)_{\mathcal{N}}$. Since $e \in (\alpha)_{\mathcal{N}}$ we have $(e)_{\mathcal{N}} = (\alpha)_{\mathcal{N}}$ and thus $b \in (\alpha)_{\mathcal{N}}$.

From the above we conclude that $(\alpha)_{\mathcal{N}} = \bigcup_{e \in E(S) \cap (\alpha)_{\mathcal{N}}} RG_e$. Similarly we prove that

$(\alpha)_{\mathcal{N}} = \bigcup_{e \in E(S) \cap (\alpha)_{\mathcal{N}}} LG_e$ and $(\alpha)_{\mathcal{N}} = \bigcup_{e \in E(S) \cap (\alpha)_{\mathcal{N}}} G_e$. □

Proposition 13: Let $(S, \cdot, \leq)$ be a right and left regular ordered semigroup. The following are equivalent:

i) S satisfies Rs - condition

ii) For $\alpha \in S$, $e \in E(S)$ such that $e \in (S\alpha]$ and $\alpha \in (eS]$ we have $\alpha \in RG_e$.

Proof:

i) ⇒ ii) Let $\alpha \in S$, $e \in E(S)$ such that $e \in (S\alpha]$ and $\alpha \in (eS]$. By Corollary 9 there exists $f \in E(S)$ such that $\alpha \in G_f = LG_f \cap RG_f$. Thus $\alpha \in LG_e \cap LG_f$ which means that $LG_e \cap LG_f \neq \varnothing$. Therefore, by Proposition 3, $LG_e = LG_f$. Then, since $f \in LG_f$, we have $f \in LG_e$ and so (Definition 1ii)) $e \in (Sf]$. Also, since $\alpha \in RG_f$, then (Definition 1i)) $f \in (\alpha S]$. Therefore $f \leq \alpha y$ for some $y \in S$. Since $\alpha \in (eS]$ then $\alpha \leq ex$ for some $x \in S$. Consequently

$$f \leq \alpha y \leq (ex)y = e(xy)$$

and thus $f \in (eS]$. So we showed that $e \in (Sf]$ and $f \in (eS]$. But S satisfies Rs - condition. Therefore (Definition 2i)) $e \in (fS]$ and, since $f \in (eS]$, then (Definition 1i)) we have $f \in RG_e$. It follows, by Remark 5, $RG_f = RG_e$ and hence $\alpha \in RG_e$.

ii)$\Rightarrow$ **i)** Let $e, f \in E(S)$ such that $e \in (Sf]$ and $f \in (eS]$. By hypothesis we have $f \in RG_e$ from which it follows (Definition 1i)) that $e \in (fS]$. So we proved that S satisfies Rs - condition.

□

In a similar way we can prove the next

Proposition 14: Let $(S, \cdot, \leq)$ be a right and left regular ordered semigroup. The following are equivalent:

i) S satisfies Ls - condition

ii) For $\alpha \in S$, $e \in E(S)$ such that $e \in (\alpha S]$ and $\alpha \in (Se]$ it holds $\alpha \in LG_e$. □

By Definition 1i), ii), Proposition 13 and Proposition 14 we immediately have the following

Corollary 15: Let $(S, \cdot, \leq)$ be a right and left regular ordered semigroup satisfying Rs - condition (resp. Ls - condition) and $\alpha \in S$, $e \in E(S)$ such that $\alpha \in LG_e \cap (eS]$ (resp. $\alpha \in RG_e \cap (Se]$). Then $\alpha \in RG_e$ (resp. $\alpha \in LG_e$). □

Proposition 16: Let $(S, \cdot, \leq)$ be a completely regular ordered semigroup satisfying Rs - condition, $e \in E(S)$, $\alpha \in RG_e$ and $b \in LG_e \cap (eS]$. Then there exists $c \in G_e$ such that $b \leq \alpha c$.

Proof: Since $b \in LG_e \cap (eS]$ and $\alpha \in RG_e$ then (Definitions 1i), 1ii)) there exist $x, y, z, w, u \in S$ such that $b \leq xe$, $e \leq yb$, $b \leq ez$, $e \leq \alpha w$ and $\alpha \leq eu$. Since S is completely regular and $\alpha, b \in S$ then there exist $s, t \in S$ such that $\alpha \leq \alpha^2 s \alpha^2$ and $b \leq b^2 t b^2$. Consequently

$$\begin{aligned} b &\leq b^2 t b^2 = b\left(btb^2\right) \leq (ez)\left(btb^2\right) = e\left(zbtb^2\right) \leq (\alpha w)\left(zbtb^2\right) = \alpha\left(wzbtb^2\right) \leq \\ &\leq \left(\alpha^2 s \alpha^2\right)\left(wzbtb^2\right) = \alpha\left(\alpha s \alpha^2 wzbtb^2\right) \end{aligned}$$

Therefore, for $c = \alpha s \alpha^2 wzbtb^2$, it holds $b \leq \alpha c$. Since

$$e \leq yb \leq y(\alpha c) = (y\alpha)c$$

(i.e. $e \in (Sc]$) and

$$c = \alpha s\alpha^2 wzbtb^2 = (\alpha s\alpha^2 wzbtb)b \le (\alpha s\alpha^2 wzbtb)(xe) = (\alpha s\alpha^2 wzbtbx)e$$

(i.e. $c \in (Se]$), it follows (Definition 1i)) that $c \in LG_e$. Moreover

$$c = \alpha s\alpha^2 wzbtb^2 = \alpha(s\alpha^2 wzbtb^2) \le (eu)(s\alpha^2 wzbtb^2) = e(us\alpha^2 wzbtb^2)$$

so that $c \in (eS]$. Therefore, since $e \in (Sc]$, $c \in (eS]$ and S satisfies Rs - condition we have, by Proposition 13 i)$\Rightarrow$ii), $c \in RG_e$. Thus $c \in LG_e \cap RG_e = G_e$ and so the proof is complete. □

By symmetry we have the following

Proposition 17: Let $(S,\cdot,\le)$ be a completely regular ordered semigroup satisfying Ls - condition, $e \in E(S)$, $\alpha \in LG_e$ and $b \in RG_e \cap (Se]$. Then there exists $c \in G_e$ such that $b \le c\alpha$. □

Theorem 18: Let $(S,\cdot,\le)$ be a completely regular ordered semigroup satisfying Rs - condition (resp. Ls - condition) and $e \in E(S)$. Then G_e is a right (resp. left) simple subsemigroup of S.

Proof: By Corollary 7 we get that G_e is a subsemigroup of S. Let $\alpha, b \in G_e$. It suffices to show that $b \le \alpha c$ for some $c \in G_e$. Since $\alpha \in G_e$ then (Definition 1iii)) $\alpha \in RG_e$. Also, since $b \in G_e$, then (Definitions 1iii), 1i)) $b \in LG_e \cap (eS]$. By Proposition 16, there exists $c \in G_e$ such that $b \le \alpha c$. □

By Theorem 18 we immediately obtain the next

Theorem 19: Let $(S,\cdot,\le)$ be a completely regular ordered semigroup satisfying both Rs - condition and Ls - condition . Then G_e is a right and left simple subsemigroup of S for every $e \in E(S)$. □

Proposition 20: Let $(S,\cdot,\le)$ be an ordered semigroup, $e \in E(S)$ and T be a right (resp. left) simple subsemigroup of S containing e. Then $T \subseteq RG_e$ (resp. $T \subseteq LG_e$).

Proof: Let $\alpha \in T$. Since $e \in T$ and T is a right simple subsemigroup of S we have $\alpha \le ex$, $e \le \alpha y$ for some $x, y \in T \subseteq S$. Hence $\alpha \in (eS]$ and $e \in (\alpha S]$ and thus (Definition 1i)) $\alpha \in RG_e$.

□

Combining Proposition 20 and Definition 1iii) we have the following

Corollary 21: Let $(S,\cdot,\le)$ be an ordered semigroup, $e \in E(S)$ and T be a right and left simple subsemigroup of S containing e. Then $T \subseteq G_e$. □

Theorem 22: Let $(S,\cdot,\le)$ be a completely regular ordered semigroup satisfying both Rs - condition and Ls - condition and $e \in E(S)$. Then

i) G_e is the greatest (under the inclusion relation) right and left simple subsemigroup of S containing e.

ii) G_e is maximal (under the inclusion relation) right and left simple subsemigroup of S.

Proof:

i) By Theorem 19, G_e is a right and left simple subsemigroup of S. Clearly $e \in G_e$. Let now T be a right and left simple subsemigroup of S containing e. By Corollary 21, it follows immediately that $T \subseteq G_e$ and so the proof is complete.

ii) By Theorem 19, G_e is a right and left simple subsemigroup of S (containing e). Let H be a right and left simple subsemigroup of S such that $G_e \subseteq H$. We will prove $G_e = H$. Indeed:

Since $G_e \subseteq H$ and $e \in G_e$, we have $e \in H$. Therefore H is a right and left simple subsemigroup of S containing e. By Corollary 21, it follows that $H \subseteq G_e$ and thus $G_e = H$. □

Proposition 23: Let $(S,\cdot,\le)$ be a right (resp. left) simple ordered semigroup. Then S is right (resp. left) regular.

Proof: Let $\alpha \in S$. Since $\alpha, \alpha^2 \in S$ and S is right simple, we have $\alpha \le \alpha^2 x$ for some $x \in S$. Thus S is right regular. □

Proposition 24: Let $(S,\cdot,\leq)$ be an ordered semigroup and T be a right and left simple subsemigroup of S. Then there exists $e \in E(S)$ such that $T \subseteq G_e$.

Proof: By Proposition 23 we have that T is a right and left regular subsemigroup of S. Then, by Theorem 7, we have $E(T) \neq \varnothing$. Thus there exists $e \in T$ such that $e \in E(T)$. Since clearly $E(T) \subseteq E(S)$, we have $e \in E(S)$. Therefore T is a right and left simple subsemigroup of S containing $e \in E(S)$. Then, by Corollary 21, it follows that $T \subseteq G_e$. □

Proposition 25: Let $(S,\cdot,\leq)$ be a completely regular ordered semigroup satisfying both Rs - condition and Ls - condition and T be a subset of S. The following are equivalent:

i) T is maximal (under the inclusion relation) right and left simple subsemigroup of S.

ii) $T = G_e$ for some $e \in E(S)$.

Proof: Since the implication **ii)**$\Rightarrow$ **i)** follows immediately from Theorem 22ii), we will prove only the implication **i)**$\Rightarrow$ **ii)**.

i)$\Rightarrow$ **ii)** Since, by hypothesis, T is a right and left simple subsemigroup of S then, by Proposition 24, there exists $e \in E(S)$ such that $T \subseteq G_e$ which implies $T = G_e$ because, by hypothesis, T is maximal right and left simple subsemigroup of S and G_e is (Theorem 19) a right and left simple subsemigroup of S. □

If S is a completely regular ordered semigroup S satisfying both Rs - condition and Ls - condition, then, by Theorem 22ii) and Proposition 25i)$\Rightarrow$ii) , we immediately obtain that the set $\{G_e \mid e \in E(S)\}$ is the set of (all) maximal right and left simple subsemigroups of S. So, by Corollary 9, we have the next

Theorem 26: A completely regular ordered semigroup satisfying both Rs - condition and Ls - condition is union of its maximal right and left simple subsemigroups of S. □

From the above and Corollary 4, the union mentioned in Theorem 26 is a union of disjoint sets.

Proposition 27: Let $(S,\cdot,\leq)$ be an ordered semigroup which is a union of right and left simple subsemigroups of S. Then S is completely regular.

Proof: Let $\alpha \in S$. By hypothesis, there exists T right and left simple subsemigroup of S such that $\alpha \in T$. Since $\alpha, \alpha^2 \in T$ and T is right simple subsemigroup of S, we have $\alpha \leq \alpha^2 x$ for some $x \in T$. Since $x, \alpha^2 \in T$ and T is left simple subsemigroup of S, we have $x \leq y\alpha^2$ for some $y \in T$. Consequently

$$\alpha \leq \alpha^2 x \leq \alpha^2 y \alpha^2$$

Thus $\alpha \in (\alpha^2 S \alpha^2]$. □

By Proposition 12, Corollary 4 and Theorem 19, we clearly have the following

Theorem 28: Let $(S,\cdot,\leq)$ be a completely regular ordered semigroup satisfying both Rs - condition and Ls - condition. Then every $\mathcal{N}$ - class of S is union of disjoint right and left simple subsemigroups of S. □

Now, summarizing all the above, we have the next fundamental Theorem about the decomposition of a completely regular ordered semigroup into union of right and left simple subsemigroups of it (which equivalently characterizes an ordered semigroup having both Rs - condition and Ls – condition in order to be completely regular).

Theorem 29: The following conditions on an ordered semigroup S satisfying both Rs - condition and Ls - condition are equivalent:

i) S is completely regular

ii) Every $\mathcal{N}$ - class of S is a union of (disjoint) right and left simple subsemigroups of S

iii) S is a union of (disjoint) right and left simple subsemigroups of S.

Proof:

i) $\Rightarrow$ **ii)** It follows immediately from Proposition 12 and Theorem 19.

ii) $\Rightarrow$ **iii)** It is clear since $\mathcal{N}$ is a (complete semilattice) congruence on S.

iii) $\Rightarrow$ **i)** It follows immediately from Proposition 27. □

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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