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STRONGLY RPP SEMIGROUPS ENDOWED WITH SOME NATURAL PARTIAL ORDERS†

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Abstract. We study strongly rpp semigroups endowed with some natural partial orders. We first consider some natural partial orders on a strongly rpp semigroup, namely the natural partial orders $\leq_{sl}, \leq_{sr}$ and $\leq_s$. Then we give some characterization theorems for the strongly rpp semigroups under the above natural partial orders. In particular, we also determine when will a super rpp semigroups be compatible with the natural partial order $\leq_{sl}$. Some known results on natural partial orders on semigroups previously given by K. S. S. Nambooripad, M. V. Lawson, T. S. Blyth, Gracida M. S. Gomes, X. J. Guo and Y. F. Luo, and X. J. Guo and K. P. Shum are enriched and strengthened.

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1. Introduction

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We call a semigroup S *right principal projective*, in short, *rpp*, if for any $a \in S$, the right principal ideal aS^1, regarded as an S^1-system, is projective. Dually, we define the *left principal projective semigroup* (in short, *lpp semigroup*). For any two elements a, b in S, we say a and b are $\mathcal{L}^*$*-related* if and only if they are $\mathcal{L}$-related in some oversemigroup of S. The relation $\mathcal{R}^*$ can be dually defined. Equivalently, a semigroup S is called rpp [3]if and only if each $\mathcal{L}^*$-class of S contains at least one idempotent. According to J. B. Fountain [3], a semigroup S is said to be *abundant* if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class of S contains at least one idempotent of S. We also call a semigroup *adequate* if it is an abundant semigroup whose set of idempotents forms a semilattice. It is also easy to verify that all inverse semigroups are adequate and all regular semigroups are abundant.

It is well known that the class of completely regular semigroups plays an important role in the theory of semigroups. Many results on completely regular semigroups have been collected in the monograph [21]. As an analogue of completely regular semigroups in the range of right principal projective semigroups, Y. Q. Guo, K. P. Shum and P. Y. Zhu [15] introduced the concept of strongly rpp semigroups. In fact, a *strongly rpp semigroups* is an rpp semigroup S in which for any element a of S, there is exactly one idempotent $a^\diamond$ in S such that $a\mathcal{L}^*a^\diamond$ and $a^\diamond a = a$. In [8], we called a strongly rpp semigroup S *super rpp* if $\overline{\mathcal{R}}$ is a left congruence on S. The class of strongly rpp semigroups and its special subclasses (that is, the super rpp semigroups) has been extensively investigated by many authors, for instance, see [4], [5]-[9], [10, 11, 14, 15, 22] and others.

It is well known that the natural partial orders are useful in semigroup theory. In particular, K. S. S. Nambooripad [19] first investigated the natural partial order on regular semigroups and he proved the following well-known theorem in 1980.

Theorem 1.1. [19] *Let S be a regular semigroup. Then the natural partial order $\leq$ on S is compatible with the semigroup multiplication if and only if S is a locally inverse semigroup.*

Later on, M. V. Lawson [17] defined the natural partial order $\leq$ on an abundant semigroup which are generalizations of Nambooripad order on a regular semigroup and he deduced the following interesting theorem.

Theorem 1.2. [17] *Let S be a concordant semigroup. Then $\leq$ is compatible with the multiplication of the semigroup S if and only if S is a locally type-A semigroup.*

We noticed that X. J. Guo and Y. F. Luo first pointed out in [10] that in any abundant semigroup with compatible natural partial order $\leq$ is an idempotent connected semigroup, that is, an IC semigroup, and furthermore, they established the Theorem 1.2 without the assumption that S is a concordant semigroup. In [13], X. J. Guo and K. P. Shum have further studied the natural partial order on an rpp semigroup. In the known monograph [21], Petrich and Reilly considered the natural partial orders on completely regular semigroups. It is well known that strongly rpp semigroups (super rpp semigroups) are analogue of completely regular semigroups within the range of rpp semigroups. It is natural to ask can we generalize and establish simlar results of X. J. Guo and K. P. Shum given in [12, 13] concerning the Lawson partial order on left cyber semigroups to rpp semigroups? In this paper, we will concentrate on this question.

2. Preliminaries

Throughout this paper, we follow the terminologies and notations used in [3] and [16].

We begin by giving some elementary facts about the relation $\mathcal{R}^*$ are given in the following lemmas, and the duals for the relation $\mathcal{L}^*$.

Lemma 2.1. *Let S be a semigroup and $a, b \in S$. Then the following conditions are equivalent:*

(1) $a\mathcal{R}^*b$.

(2) *For all $x, y \in S^1$, $xa = ya$ if and only if $xb = yb$.*

The following Lemma is an easy consequence of Lemma 2.1 due to [2].

Lemma 2.2. *Let S be a semigroup and $a, e^2 = e \in S$. Then the following conditions are equivalent:*

(1) $a\mathcal{R}^*e$.

(2) *$ea = a$ and for all $x, y \in S^1$, $xa = ya$ implies that $xe = ye$.*

It is well known that $\mathcal{R}^*$ is a left congruence on S while $\mathcal{L}^*$ is a right congruence on S. In general, we have $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. But when a and b are regular elements of S, $a\mathcal{R}\ (\mathcal{L})\ b$ if and only if $a\mathcal{R}^*\ (\mathcal{L}^*)\ b$. In particular, when S is a regular semigroup, we always have $\mathcal{L} = \mathcal{L}^*$ and $\mathcal{R} = \mathcal{R}^*$. For the sake of convenience, we use $E(S)$ to denote the set of idempotents of S; a^* to denote an idempotent which is $\mathcal{L}^*$-related to a and $a^\dagger$ to denote an idempotent which is $\mathcal{R}^*$-related to a. Usually, we call the relations $\mathcal{L}^*, R^*, H^*, D^*$ and $\mathcal{J}^*$ the Green's $*$-relations. If $\mathcal{K}$ is one of the $\mathcal{L}^*, R^*, H^*, D^*$ and $\mathcal{J}^*$ relations, then we denote the $\mathcal{K}$-class of S containing a by K_a.

We now call a left ideal L of S a *left $*$-ideal* if $L = \bigcup_{a\in L} L_a^*$. The right $*$-ideal can be dually defined. For $a \in S$, we denote by $L^*(a)$ the smallest left $*$-ideal of S containing a and by $R^*(a)$ the smallest right $*$-ideal of S containing a.

By using the left (right) $*$-ideals, we now characterize the relations $\mathcal{R}^*$ and $\mathcal{L}^*$ in the following Lemma:

Lemma 2.3. [3] *For the elements a, b of a semigroup S, we have*

(1) *$a\mathcal{L}^*b$ if and only if $L^*(a) = L^*(b)$;*

(2) *$a\mathcal{R}^*b$ if and only if $R^*(a) = R^*(b)$.*

Moreover, we have the following proposition.

Proposition 2.4. *For any elements $a, x \in S$, we have $L^*(xa) \subseteq L^*(a)$.*

Proof. By definition, we have $S^1a \subseteq L^*(a)$ and $xa \in L^*(a)$. But $L^*(xa)$ is the smallest left $*$-ideal containing xa. Hence, it follows that $L^*(xa) \subseteq L^*(a)$.

To study the properties of rpp semigroups, Y. Q. Guo, K. P. Shum and P. Y. Zhu [15] first introduced a kind of Green's relations $\mathcal{L}^{(l)}, \mathcal{R}^{(l)}, \mathcal{H}^{(l)}, \mathcal{D}^{(l)}$ and $\mathcal{J}^{(l)}$ on a semigroup S. Indeed, this kind of Green's relations is a mixture of the usual Green's relations and the Green's $*$-relations. They are $\mathcal{L}^{(l)}, \mathcal{R}^{(l)}, \mathcal{H}^{(l)}, \mathcal{D}^{(l)}$ and $\mathcal{J}^{(l)}$. In fact, $\mathcal{L}^{(l)} = \mathcal{L}^*, \mathcal{R}^{(l)} = \mathcal{R}, \mathcal{H}^{(l)} = \mathcal{L}^{(l)} \cap \mathcal{R}^{(l)}, \mathcal{D}^{(l)} = \mathcal{L}^{(l)} \vee \mathcal{R}^{(l)}$ and

$$a\mathcal{J}^{(l)}b \text{ if and only if } J^{(l)}(a) = J^{(l)}(b),$$

where $J^{(l)}(x)$ is the smallest ideal of S containing x and which is a left $*$-ideal.

Also, we have the following Lemma.

Lemma 2.5. [7] *The following statements hold for a semigroup S:*

(1) $\mathcal{D}^{(l)} = \mathcal{L}^{(l)} \circ \mathcal{R}^{(l)} = \mathcal{R}^{(l)} \circ \mathcal{L}^{(l)}$.

(2) *Each $\mathcal{D}^{(l)}$-class contains at most one regular $\mathcal{D}$-class.*

For strongly rpp semigroups, we have the following Lemmas.

Lemma 2.6. [7] *Let S be a strongly rpp semigroup and $a, b \in S$. If $a\mathcal{D}^{(l)}b$, then $(ab)^{\diamond} = (a^{\diamond}b^{\diamond})^{\diamond}$.*

Lemma 2.7. [7] *Let S be a strongly rpp semigroup and $a \in S$.Then the following statements hold:*

(1) *If a is regular, then $a\mathcal{H}a^{\diamond}$.*

(2) *$D_a^{(l)}$ (the $\mathcal{D}^{(l)}$-class of S containing a) is a $\mathcal{D}^{(l)}$-simple strongly rpp semigroup.*

By a *super rpp semigroup* [8], we mean a strongly rpp semigroup in which $\mathcal{D}^{(l)}$ is a semilattice congruence. In the following Lemma, we characterize the super rpp semigroups.

Lemma 2.8. [8] *Let S be a strongly rpp semigroup. Then the following statements are equivalent:*

(1) *S is a super rpp semigroup.*

(2) $\mathcal{J}^{(l)} = \mathcal{D}^{(l)}$.

(3) *$\mathcal{D}^{(l)}$ is a semilattice congruence.*

Let S be an rpp semigroup. As in [17], we define the following partial order: for any $a, b \in S$,

> $a \leq_l b \Leftrightarrow L_a^* \leq L_b^*$ (that is, $L^*(a) \subseteq L^*(b)$) and $a = be$ for some $e \in E(S) \cap L_a^*$.

Dually, we define the order $\leq_r$ on an lpp semigroup. If S is an abundant semigroup, then we define $\leq$ on S as $\leq_l \cap \leq_r$. We now call the partial orders $\leq_l, \leq_r$ and $\leq$ the *Lawson orders.* We now give an alternative description for the Lawson order in terms of the idempotents in S.

For rpp semigroups, we have the following Lemma.

Lemma 2.9. [17] *Let x, y be elements of an rpp semigroup S. Then $x \leqslant_l y$ if and only if for each idempotent $y^* \in L_y^*$ there exists an idempotent $x^* \in L_x^*$ such that $x^* \omega y^*$ and $x = yx^*$.*

3. Definitions

Let S be a strongly rpp semigroup and $a, b \in S$. Define

$$a \leq_{sl} b \Leftrightarrow a^\diamond \omega b^\diamond \text{ and } a = ba^\diamond;$$

$$a \leq_{sr} b \Leftrightarrow a^\diamond \omega b^\diamond \text{ and } a = a^\diamond b;$$

$$a \leq_s b \Leftrightarrow a \leq_{sl} b \text{ and } a \leq_{sr} b.$$

Proposition 3.1. *$\leq_{sl}, \leq_{sr}$ and $\leq_s$ are partial orders on S. Moreover, the restrictions of $\leq_{sl}, \leq_{sr}$ and $\leq_s$ to $E(S)$ coincide with ω on $E(S)$.*

Proof. We only prove the case $\leq_{sl}$. Because the other cases can be similarly proved and their proofs are therefore omitted. The reflexivity of $\leq_{sl}$ is obvious because S is a strongly rpp semigroup. Now suppose that $a \leq_{sl} b$ and $b \leq_{sl} a$. Then $a = ba^\diamond, a^\diamond \omega b^\diamond$ and $b = ab^\diamond, b^\diamond \omega a^\diamond$. It follows that $a^\diamond = b^\diamond$. Thus $a = ba^\diamond = bb^\diamond = b$. This means that $\leq_{sl}$ is anti-symmetric.

It remains to prove that $\leq_{sl}$ is transitive. For this purpose, we let $a \leq_{sl} b$ and $b \leq_{sl} c$. Then we have $a = ba^\diamond, a^\diamond \omega b^\diamond$ and $b = cb^\diamond, b^\diamond \omega c^\diamond$. Hence, we have

$$a^\diamond \omega c^\diamond \text{ and } a = ba^\diamond = (cb^\diamond)a^\diamond = c(b^\diamond a^\diamond) = ca^\diamond,$$

and whence $a \leq_{sl} c$. This shows that $\leq_{sl}$ is transitive, as required.

Assume that $e, f \in E(S)$ and $e \leq_{sl} f$. Since S is a strongly rpp semigroup, $e = e^\diamond \omega f^\diamond = f$, and whence $\leq_{sl}|_{E(S)} \subseteq \omega$. On the other hand, if $e \omega f$, then $e = fe$. This implies $e \leq_{sl} f$ and so we deduce that $\omega \subseteq \leq_{sl}|_{E(S)}$. Thus $\leq_{sl}$ coincides with ω on $E(S)$. This completes the proof.

Proposition 3.2. *Let $a, b \in S$ and $a \leq_{sl} b$. Then the following statements hold:*

(1) *If b is an idempotent of S, then a is an idempotent.*

(2) *If b is a regular element of S, then a is a regular element of S.*

Proof. (1) Assume that $a \leq_{sl} b$ and b is an idempotent of S. Then $a^\diamond \omega b^\diamond$ $(= b)$ and $a = ba^\diamond = a^\diamond$. Thus, a is an idempotent.

(2) Since $a \leq_{sl} b$, $a^\diamond \omega b^\diamond$ and $a = ba^\diamond$. If b is a regular element of S, then there exists an element $x \in S$ such that $b = bxb$, and hence, it follows that $ba^\diamond = (bxb)a^\diamond = bx(ba^\diamond) = bxa$, and hence by $b\mathcal{L}^*b^\diamond$, we have $b^\diamond xa = b^\diamond a^\diamond$ $(= a^\diamond$ since $a^\diamond \omega b^\diamond)$. This shows that $a = aa^\diamond = ab^\diamond xa$. In other words, a is a regular element of S. This completes the proof.

Similar to Proposition 3.2, we have the following propositions for $\leq_{sr}$.

Proposition 3.3. *Let $a, b \in S$ and $a \leq_{sr} b$. Then the following statements hold:*

(1) *If b is an idempotent of S, then a is an idempotent.*

(2) *If b is a regular element of S, then a is a regular element of S.*

Proof. (1) The proof of this part is similar to Proposition 3.2(1).

(2) Since $a \leq_{sr} b$, $a^\diamond \omega b^\diamond$ and $a = a^\diamond b$. If b is a regular element of S, then by Lemma 2.7, $b\mathcal{H}b^\diamond$, and hence $b\mathcal{R}b^\diamond$. Since b is a regular element of S, there exists an element $x \in S$ such that $b = bxb$ and so $b^\diamond = bxb^\diamond$. Thus $a^\diamond = b^\diamond a^\diamond = bxb^\diamond a^\diamond = bxa^\diamond$ (since $a^\diamond \omega b^\diamond$). Hence $a = a^\diamond a^\diamond a = a^\diamond bxa^\diamond a = axa$. In other words, a is a regular element of S. This completes the proof.

Let ρ be a given relation on a semigroup T with a partial order $\prec$. Then ρ is said to *satisfy the ρ-majorization* condition if for any $a, b, c \in T$, both $b \prec a, c \prec a$ and $b\rho c$ can imply $b = c$. Hence by Lemma 2.3 and Proposition 2.4, we can easily verify that $\leq_{sl} \subseteq \leq_l$ for a strongly rpp semigroup. However, it is still not yet known when $\leq_{sl} = \leq_l$. In this aspect, we now prove a weaker result.

Proposition 3.4. *Let S be a strongly rpp semigroup. If S satisfies $\mathcal{L}$-majorization for idempotents, then $\leq_{sl} = \leq_l$.*

Proof. It is obvious that $\leq_{sl} \subseteq \leq_l$. Conversely, if $x \leqslant_l y$, then by Lemma 2.9. For any $y^\diamond \in L^*_y$, there exists an idempotent $x^* \in L^*_x$ such that $x^* \omega y^\diamond$ and $x = yx^*$, so that $x = yy^\diamond x^* = yx^* y^\diamond = xy^\diamond$. Hence, by $x\mathcal{L}^*x^\diamond$, we have $x^\diamond = x^\diamond y^\diamond$. Now, we have

$$(y^\diamond x^\diamond)^2 = y^\diamond (x^\diamond y^\diamond) x^\diamond = y^\diamond x^\diamond x^\diamond = y^\diamond x^\diamond \in E(S)$$

and $x^\diamond(y^\diamond x^\diamond) = (x^\diamond y^\diamond)x^\diamond = x^\diamond x^\diamond = x^\diamond$. It follows that $y^\diamond x^\diamond \mathcal{L} x^\diamond$. On the other hand, we have

$$y^\diamond x^\diamond x = y^\diamond x = y^\diamond y x^* = yx^* = x.$$

This shows that $y^\diamond x^\diamond = x^\diamond$ since S is a strongly rpp semigroup. Again associating with the foregoing equality: $x^\diamond = x^\diamond y^\diamond$, we have proved that $x^\diamond \leqslant y^\diamond$. By hypothesis: S satisfies $\mathcal{L}$-majorization for idempotents, and $x^\diamond \mathcal{L} x^*$, we have $x^\diamond = x^*$. Hence $x = yx^* = yx^\diamond$ and $x^\diamond \omega y^\diamond$. It follows that $x \leq_{sl} y$, and whence $\leq_l \subseteq \leq_{sl}$. Consequently, $\leq_{sl} = \leq_l$.

Proposition 3.5. *Let S be a strongly rpp semigroup. If $ea = ae$ for any $a \in S$ and $e \in \omega(a^\diamond)$, then $\leq_{sr} = \leq_{sl} = \leq_l$.*

Proof. If $a \leq_{sr} b$, then $a^\diamond \omega b^\diamond$ and $a = a^\diamond b$, hence by hypothesis, $a = a^\diamond b = ba^\diamond$, whence $a \leq_{sl} b$. So, we have $\leq_{sr} \subseteq \leq_{sl}$; similarly we deduce that $\leq_{sl} \subseteq \leq_{sr}$. Thus $\leq_{sr} = \leq_{sl}$. Obviously, $\leq_{sl} \subseteq \leq_l$. Conversely, if $a \leq_l b$, then by Lemma 2.9, there exists an idempotent $a^* \in L^*_a$ such that $a^* \omega b^\diamond$ and $a = ba^*$. By hypothesis, $a^* a = a^* b a^* = ba^* = a$ and $a^* = a^\diamond$ since S is a strongly rpp semigroup. Associating with $a = ba^* = ba^\diamond$, we can obtain that $a \leq_{sl} b$. Whence we have $\leq_l \subseteq \leq_{sl}$. Consequently, $\leq_{sr} = \leq_{sl} = \leq_l$. This completes the proof.

It is well known that for a regular semigroup, we always have $\leq_l = \leq_r$. Naturally, one would ask whether $\leq_{sl} = \leq_{sr}$ lies in a strongly rpp semigroup or not? This question is still an open question.

The following example was due to [22], which illustrates that $\leq_{sr} \neq \leq_{sl}$ holds in general in a strongly rpp semigroup.

Example 3.6. Let N be the set of non-negative integers and $S = \{(m,n) \in N \times N \mid m \geq n\}$. Define the following operation "$*$" on S by

$$(m,n) * (p,q) = (m - n + max\{n,p\}, q - p + max\{n,p\}).$$

Then, it can be easily verified that S, under the multiplication "$*$", is a semigroup. Now, by [22], we can easily verify that S is a strongly rpp semigroup, but $\leq_{sr} \neq \leq_{sl}$ in S. In fact, consider the elements $a = (3,2)$ and $b = (2,1)$, then we have $a^\diamond = (3,2)^\diamond = (2,2)$

and $b^\diamond = (2,1)^\diamond = (1,1)$, it is easy to verify that $a^\diamond \omega b^\diamond$ and $a = (3,2) = (2,1)(2,2) = ba^\diamond$. Hence, $a \leq_{sl} b$ but $a^\diamond b = (2,2)(2,1) = (2,1) \neq a$. In other words, $a \not\leq_{sr} b$.

4. Compatibility with the Multiplication of a semigroup

In this section, T is a semigroup endowed with a partial order $\prec$ *right (left) compatible with multiplication for* $\prec$ if for any $a, b, c \in T$, $a \prec b$ implies $ac \prec bc$ ($ca \prec cb$). Moreover, the semigroup T is called *compatible with multiplication for* $\prec$ when T is left and right compatible with multiplication. We now consider when will a strongly rpp semigroup be (left; right) compatible with multiplication ?

For a strongly rpp semigroup S, we have the following theorem.

Theorem 4.1. *The following statements are equivalent for a strongly rpp semigroup S:*

(1) *S is left compatible with the multiplication for the order $\leq_{sl}$.*

(2) *For any $a, b, c \in S$, If $a^\diamond \omega b^\diamond$ then $(cba^\diamond)^\diamond = (cb)^\diamond a^\diamond = (cb)^\diamond a^\diamond (cb)^\diamond$.*

Proof. (1) $\Rightarrow$ (2) Assume that S is left compatible with multiplication for $\leq_{sl}$. Let $a, b, c \in S$ and $a^\diamond \omega b^\diamond$. Now, by hypothesis, we have $cba^\diamond \leq_{sl} cbb^\diamond = cb$, and thereby, we have $cba^\diamond = cb(cba^\diamond)^\diamond$ and $(cba^\diamond)^\diamond \omega (cb)^\diamond$.By $cb\mathcal{L}^{(l)}(cb)^\diamond$, the first equality implies $(cb)^\diamond a^\diamond = (cb)^\diamond (cba^\diamond)^\diamond$. Again by the second formula, we deduce $(cb)^\diamond (cba^\diamond)^\diamond = (cba^\diamond)^\diamond = (cba^\diamond)^\diamond (cb)^\diamond$. Thus, we have the following equality:

$$(cb)^\diamond a^\diamond = (cb)^\diamond (cba^\diamond)^\diamond = (cba^\diamond)^\diamond = (cba^\diamond)^\diamond (cb)^\diamond = (cb)^\diamond a^\diamond (cb)^\diamond,$$

that is, $(cba^\diamond)^\diamond = (cb)^\diamond a^\diamond = (cb)^\diamond a^\diamond (cb)^\diamond$.

(2) $\Rightarrow$ (1) Let $a, b \in S$ with $a \leq_{sl} b$. Then $a = ba^\diamond$ and $a^\diamond \omega b^\diamond$. The first equality implies $ca = cba^\diamond = cb((cb)^\diamond a^\diamond) = cb(cb)^\diamond a^\diamond (cb)^\diamond$ since, by hypothesis, $(cb)^\diamond a^\diamond = (cb)^\diamond a^\diamond (cb)^\diamond$. On the other hand, since $cb\mathcal{L}^*(cb)^\diamond$, we have $ca = cba^\diamond (cb)^\diamond \mathcal{L}^* (cb)^\diamond a^\diamond (cb)^\diamond$. Also, $(cb)^\diamond a^\diamond (cb)^\diamond ca = (cba^\diamond)^\diamond cba^\diamond = cba^\diamond = ca$. By the uniqueness of $x^\diamond$, we deduce that $(ca)^\diamond = (cb)^\diamond a^\diamond (cb)^\diamond$ and so $(ca)^\diamond \omega (cb)^\diamond$. Now, we have proved

$$ca \leq_{sl} cb$$

and whence S is left compatible with multiplication for $\leq_{sl}$.

The following theorems give some descriptions of the strongly rpp semigroups.

Theorem 4.2. *The following statements are equivalent for a strongly rpp semigroup S:*

(1) *S is right compatible with multiplication for $\leq_{sl}$.*

(2) *For any $a, b, c \in S$, if $a \leq_{sl} b$, then $a^\diamond c \leq_{sl} b^\diamond c$ and $(bc)^\diamond(a^\diamond c)^\diamond = (ac)^\diamond$.*

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds. Let $a, b, c \in S$ and $a \leq_{sl} b$. Then, by definition, $a^\diamond \omega b^\diamond$ and $a = ba^\diamond$. Also, by hypothesis, we have $a^\diamond c \leq_{sl} b^\diamond c$, and so $a^\diamond c = b^\diamond c(a^\diamond c)^\diamond$ and $(a^\diamond c)^\diamond \omega (b^\diamond c)^\diamond$. On the other hand, by $a \leq_{sl} b$ again, we have $ac \leq_{sl} bc$ and whence, we deduce that $(ac)^\diamond \omega (bc)^\diamond$. Compute

$$bc(ac)^\diamond = ac = b(a^\diamond c) = b(b^\diamond c(a^\diamond c)^\diamond) = bc(a^\diamond c)^\diamond.$$

Then, it follows that $(ac)^\diamond = (bc)^\diamond(ac)^\diamond = (bc)^\diamond(a^\diamond c)^\diamond$ since $bc\mathcal{L}^*(bc)^\diamond$ and $(ac)^\diamond \omega (bc)^\diamond$.

(2) $\Rightarrow$ (1) Let $a, b \in S$ with $a \leq_{sl} b$. Then $a = ba^\diamond$ and $a^\diamond \omega b^\diamond$, so by our hypothesis, we deduce that

$$ac = ba^\diamond c = bb^\diamond c(a^\diamond c)^\diamond = bc(a^\diamond c)^\diamond = bc(bc)^\diamond(a^\diamond c)^\diamond = bc(ac)^\diamond.$$

On the other hand, since $\mathcal{L}^*$ is a right congruence, we have $(b^\diamond c)^\diamond \mathcal{L}^* b^\diamond c \mathcal{L}^* bc \mathcal{L}^* (bc)^\diamond$; similarly, $(a^\diamond c)^\diamond \mathcal{L}^* (ac)^\diamond$. Also, by hypothesis, we have $a^\diamond c \leq_{sl} b^\diamond c$, and hence, we obtain $(a^\diamond c)^\diamond \omega (b^\diamond c)^\diamond$ and so $(a^\diamond c)^\diamond (b^\diamond c)^\diamond = (a^\diamond c)^\diamond$. Now, we have

$$\begin{aligned}(ac)^\diamond &= (ac)^\diamond(a^\diamond c)^\diamond = (ac)^\diamond((a^\diamond c)^\diamond(b^\diamond c)^\diamond)\\ &= (ac)^\diamond(a^\diamond c)^\diamond((b^\diamond c)^\diamond(bc)^\diamond) = (ac)^\diamond((a^\diamond c)^\diamond(b^\diamond c)^\diamond)(bc)^\diamond\\ &= (ac)^\diamond(bc)^\diamond\end{aligned}$$

and further by hypothesis, we have $(ac)^\diamond = (bc)^\diamond(a^\diamond c)^\diamond = (bc)^\diamond(ac)^\diamond$. Therefore, $(ac)^\diamond \omega (bc)^\diamond$. Consequently, $ac \leq_{sl} bc$.

Theorem 4.3. *The following statements are equivalent for a strongly rpp semigroup S:*

(1) *S is right compatible with multiplication for $\leq_{sr}$.*

(2) *For any $a, b, c \in S$, if $a^\diamond \omega b^\diamond$, then $a^\diamond bc = (a^\diamond bc)^\diamond bc$ and $(a^\diamond bc)^\diamond \omega (bc)^\diamond$.*

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds. Let $a, b, c \in S$ with $a^{\diamond}\omega b^{\diamond}$. Then $a^{\diamond} \leq_{sr} b^{\diamond}$, whence by hypothesis, $a^{\diamond}bc \leq_{sr} b^{\diamond}bc = bc$, and so we deduce that $a^{\diamond}bc = (a^{\diamond}bc)^{\diamond}bc$ and $(a^{\diamond}bc)^{\diamond}\omega(bc)^{\diamond}$.

(2) $\Rightarrow$ (1) Let $a, b \in S$ with $a \leq_{sr} b$. Then $a = a^{\diamond}b$ and $a^{\diamond}\omega b^{\diamond}$, and so we further deduce that

$$ac = a^{\diamond}bc = (a^{\diamond}bc)^{\diamond}bc = (ac)^{\diamond}bc.$$

On the other hand, by hypothesis, we have $(a^{\diamond}bc)^{\diamond}\omega(bc)^{\diamond}$ and further we get $(ac)^{\diamond}\omega(bc)^{\diamond}$ since $a = a^{\diamond}b$. Now, $ac \leq_{sr} bc$ and this shows that S is right compatible with multiplication for $\leq_{sr}$. This completes the proof.

We now call a semigroup S a *locally $\mathcal{P}$ semigroup* if for any $e \in E(S)$, eSe has the property $\mathcal{P}$.

In the following proposition, we discuss the strongly rpp semigroups whose orders are compatible with the multiplication of the semigroup.

proposition 4.4. *Let S be a strongly rpp semigroup. If S is compatible with multiplication for $\leq_{sl}$, then S has the following properties:*

(1) *S satisfies both $\mathcal{L}^*$-majorization and $\overline{\mathcal{R}}$-majorization.*

(2) *S is a locally right adequate semigroup.*

Proof. (1) Firstly, we verify that S satisfies both $\mathcal{L}$-majorization and $\mathcal{R}$-majorization for idempotents in S. To see this fact. We first let $e, f, g \in E(S)$ such that $f \leq e$ and $g \leq e$, then by our hypothesis, we have

$$fg \leqslant_{sl} eg = g \text{ and } gf \leqslant_{sl} ge = g,$$

and by Proposition 3.2, $fg, gf \in E(S)$, $fg = g(fg)$ and $(gf)g = gf$. Thus $fg = gf$. We consider the following two cases:

- If $f\mathcal{L}g$, then $f = fg = gf = g$. This shows that S satisfies the $\mathcal{L}$-majorization for idempotents.
- If $f\mathcal{R}g$, then $f = gf = fg = g$ and whence, S satisfies the $\mathcal{R}$-majorization for idempotents.

Now let $a, b, c \in S$ be such that $b \leqslant_{sl} a$ and $c \leqslant_{sl} a$. Then $b = ab^\diamond$, $b^\diamond \leqslant a^\diamond$, $c = ac^\diamond$ and $c^\diamond \leqslant a^\diamond$. We need consider the following two cases:

- If $b\mathcal{L}^*c$, then $b^\diamond\mathcal{L}^*b\mathcal{L}^*c\mathcal{L}^*c^\diamond$. Hence, by the foregoing proof, $b^\diamond = c^\diamond$, thus $b = ab^\diamond = ac^\diamond = c$.
- If $b\overline{\mathcal{R}}c$, then $b^\diamond\mathcal{R}c^\diamond$. In this case, by the foregoing proof, we have $b^\diamond = c^\diamond$, and whence $b = ab^\diamond = ac^\diamond = c$.

Therefore, S satisfies both $\mathcal{L}^*$- and $\overline{\mathcal{R}}$-majorization.

(2) It is well known that if S is rpp, then eSe is also rpp for any $e \in E(S)$. Hence, we only need to verify that $E(eSe)$ is a semilattice, for all $e \in E(S)$. Now let $g, h \in E(eSe)$. Then $g\omega e$ and $h\omega e$. By hypothesis, $gh \leq_{sl} e$, hence by Proposition 3.2, $gh \in E(eSe)$, thereby $E(eSe)$ is a band. If $g\mathcal{D}h$ in the semigroup $E(eSe)$, then there exists $f \in E(S)$ such that $g\mathcal{L}f\mathcal{R}h$. By the property of f, we have $f = fg = hf$, so that

$$f = fge = fe = hf = ehf = ef \in E(eSe).$$

Thus $f\omega e$. By (1), S satisfies $\mathcal{L}^*$-majorization. By this property, $f\omega e, g\omega e$ and $f\mathcal{L}g$ imply that $f = g$; similarly, $f = h$. Therefore $g = h$, in other words, the relation $\mathcal{D}$ on $E(eSe)$ is the identity relation. Note that for a band $E(S)$, the relation $\mathcal{D}$ is a semilattice congruence on $E(S)$, and $\mathcal{D} = \mathcal{J}$. Hence, by $ef = e(fe)f, fe = f(ef)e$, we have $ef\mathcal{D}fe$. Thus $ef = fe$ and whence $E(eSe)$ is a semilattice. This completes the proof.

Let S be a strongly rpp semigroup. We now define the following relations on S. For $a, b \in S$, we define

$$a\overline{\mathcal{R}}b \text{ if and only if } a^\diamond\mathcal{R}b^\diamond.$$

Obviously, $\overline{\mathcal{R}}$ is an equivalent relation on S. In what follows, we define $\overline{\mathcal{H}} = \overline{\mathcal{R}} \cap \mathcal{L}^*$; equivalently, $a\overline{\mathcal{H}}b$ if and only if $a^\diamond = b^\diamond$. In [8], it has been pointed out that $\mathcal{D}^{(l)} = \overline{\mathcal{R}} \circ \mathcal{L}^*$ and that a strongly rpp semigroup is a super rpp semigroup if and only if $\mathcal{D}^{(l)}$ is a semilattice congruence.

By a *C-rpp semigroup*, we mean a rpp semigroup whose idempotents are central. Equivalently, a semigroup is C-rpp if and only if it is a semilattice of left cancellative monoids. Obviously, a C-rpp semigroup is Clifford if and only if it is regular.

We now characterize the super rpp semigroup whose order $\leq_{sl}$ is compatible with the multiplication of the semigroup. This theorem is the main theorem of the paper.

Theorem 4.5. *Let S be a super rpp semigroup. Then S is compatible with multiplication for the order $\leq_{sl}$ if and only if S is a locally C-rpp semigroup.*

Proof. ($\Rightarrow$) Suppose that S is compatible with multiplication for $\leq_{sl}$. Then, for any $e \in E(S)$, $a \in eSe$, $f \in E(eSe)$ and so $f \leq_{sl} e$, and by hypothesis, we deduce that $fa \leq_{sl} ea = a$ and we have $fa = a(fa)^\diamond$ with $(fa)^\diamond \in \omega(a^\diamond)$ and $af \leq_{sl} ae = a$, and hence, we further deduce that $af = a(af)^\diamond$ with $(af)^\diamond \in \omega(a^\diamond)$. By Lemmas 2.6 and 2.8, we see that S is a semilattice Y of $\mathcal{D}^{(l)}$-simple strongly rpp semigroups $D_\alpha^{(l)}$ with $\alpha \in Y$. Now, it is clear that $(af)^\diamond, (fa)^\diamond \in D_\alpha^{(l)}$ for some $\alpha \in Y$. This leads to $(af)^\diamond \mathcal{D} (fa)^\diamond$. Hence, we have shown that there exists an element $x \in S$ such that$(af)^\diamond \mathcal{L} x \mathcal{R} (fa)^\diamond$. Since $(af)^\diamond, (fa)^\diamond$ are regular elements. We therefore prove that x is a regular element of S. Now, by Lemma 2.7, we deduce that $x \mathcal{H} x^\diamond$, it follows that

$$(af)^\diamond \mathcal{L} x^\diamond \mathcal{R} (fa)^\diamond \text{ and } x^\diamond = x^\diamond (af)^\diamond = x^\diamond (af)^\diamond a^\diamond = x^\diamond a^\diamond;$$

and similarly $x^\diamond = a^\diamond x^\diamond$. Thus $x^\diamond \omega a^\diamond$. But by Proposition 4.4(1), S satisfies $\mathcal{L}^*$-majorization of S and $\overline{\mathcal{R}}$-majorization of S. Now, $(af)^\diamond = x^\diamond$ by $\mathcal{L}^*$-majorization and $x^\diamond = (fa)^\diamond$ by $\overline{\mathcal{R}}$-majorization. Thus $(af)^\diamond = (fa)^\diamond$. Together with $fa = a(fa)^\diamond$ and $af = a(af)^\diamond$, we obtain $af = fa$. This shows that $E(eSe)$ is in the center of eSe. It is well known that if S is rpp, then eSe is rpp for all $e \in E(S)$. Consequently, we have proved that S is a locally C-rpp semigroup.

($\Leftarrow$) Assume that S is a locally C-rpp semigroup. Then, we make the following claim:

S satisfies $\mathcal{L}$-majorization for idempotents.

Indeed, if $e, f, g \in E(S)$ such that $f \leq e$ and $g \leq e$, then $f, g \in E(eSe)$. If, in addition, $f \mathcal{L} g$, then $f = fg = gf = g$ since eSe is a C-rpp semigroup. Thus S satisfies $\mathcal{L}$-majorization for idempotents.

Now let $a, b \in S$ and $a \leqslant_{sl} b$, then $a = ba^\diamond$ and $a^\diamond \in \omega(b^\diamond)$, and so $ca = cba^\diamond$, for any $c \in S$. Since $cb = cbb^\diamond$ and by $cb \mathcal{L}^{(l)} (cb)^\diamond$, we get $(cb)^\diamond = (cb)^\diamond b^\diamond$. This implies

$$\left(b^\diamond (cb)^\diamond\right)^2 = b^\diamond ((cb)^\diamond b^\diamond) (cb)^\diamond = b^\diamond (cb)^\diamond (cb)^\diamond = b^\diamond (cb)^\diamond \in E(b^\diamond S b^\diamond).$$

Now, we compute

$$\left(b^{\diamond}\left(cb\right)^{\diamond}\right)\left(cb\right)^{\diamond}=b^{\diamond}\left(cb\right)^{\diamond}$$

and

$$\left(cb\right)^{\diamond}\left(b^{\diamond}\left(cb\right)^{\diamond}\right)=\left(\left(cb\right)^{\diamond}b^{\diamond}\right)\left(cb\right)^{\diamond}=\left(cb\right)^{\diamond}\left(cb\right)^{\diamond}=\left(cb\right)^{\diamond}.$$

Thus $b^{\diamond}\left(cb\right)^{\diamond}\mathcal{L}\left(cb\right)^{\diamond}\mathcal{L}^{(l)}cb$. Also, by $a^{\diamond}\omega b^{\diamond}$, we have $a^{\diamond}\in E(b^{\diamond}Sb^{\diamond})$. On the other hand, because S is a locally C-rpp semigroup, $E(b^{\diamond}Sb^{\diamond})$ is a semilattice. Therefore we have

- $b^{\diamond}\left(cb\right)^{\diamond}a^{\diamond}\in E(b^{\diamond}Sb^{\diamond})$;
- $(b^{\diamond}\left(cb\right)^{\diamond})a^{\diamond}=a^{\diamond}(b^{\diamond}\left(cb\right)^{\diamond})$;
- $ca=cba^{\diamond}=cb\left(cb\right)^{\diamond}a^{\diamond}=cb(b^{\diamond}\left(cb\right)^{\diamond}a^{\diamond})$; and
- $ca=cba^{\diamond}\mathcal{L}^{(l)}b^{\diamond}\left(cb\right)^{\diamond}a^{\diamond}$ since $\mathcal{L}^{(l)}$ is a right congruence.

In addition, by Lemma 2.3 and Proposition 2.4, we deduce the following equality.

$$L^{*}\left(ca\right)=L^{*}\left(b^{\diamond}\left(cb\right)^{\diamond}a^{\diamond}\right)=L^{*}\left(a^{\diamond}b^{\diamond}\left(cb\right)^{\diamond}\right)\subseteq L^{*}\left(\left(cb\right)^{\diamond}\right)=L^{*}(cb).$$

This shows that $ca\leqslant_{l}cb$. Now, by Proposition 3.4, we obtain $ca\leqslant_{sl}cb$, and whence S is left compatible with multiplication for $\leq_{sl}$.

By $a=ba^{\diamond}$, we get $ac=ba^{\diamond}c$, for any $c\in S$. Note that S is a super rpp semigroup, we observe that $\overline{\mathcal{R}}$ is a left congruence and $\mathcal{D}^{(l)}$ is a semilattice congruence on S and so, we have

$$(b^{\diamond}c)^{\diamond}\overline{\mathcal{R}}b^{\diamond}c=b^{\diamond}b^{\diamond}c\overline{\mathcal{R}}b^{\diamond}(b^{\diamond}c)^{\diamond}\overline{\mathcal{R}}(b^{\diamond}c)^{\diamond}(b^{\diamond}c)^{\diamond}\overline{\mathcal{R}}(b^{\diamond}c)^{\diamond}b^{\diamond}(b^{\diamond}c)^{\diamond}.$$

It follows that $(b^{\diamond}(b^{\diamond}c)^{\diamond})^{\diamond}\mathcal{R}(b^{\diamond}c)^{\diamond}\mathcal{R}((b^{\diamond}c)^{\diamond}b^{\diamond}(b^{\diamond}c)^{\diamond})^{\diamond}$ and also $(b^{\diamond}c)^{\diamond}\mathcal{D}^{(l)}b^{\diamond}(b^{\diamond}c)^{\diamond}\mathcal{D}^{(l)}(b^{\diamond}c)^{\diamond}b^{\diamond}(b^{\diamond}c)^{\diamond}$.

Now, by Lemma 2.6, we further deduce the followings:

$$\begin{aligned}(b^{\diamond}(b^{\diamond}c)^{\diamond})^{\diamond} &= (b^{\diamond}c)^{\diamond}(b^{\diamond}(b^{\diamond}c)^{\diamond})^{\diamond}\\ &= ((b^{\diamond}c)^{\diamond}b^{\diamond}(b^{\diamond}c)^{\diamond})^{\diamond}\\ &= ((b^{\diamond}c)^{\diamond}b^{\diamond}(b^{\diamond}c)^{\diamond}(b^{\diamond}c)^{\diamond})^{\diamond}\\ &= (((b^{\diamond}c)^{\diamond}b^{\diamond}(b^{\diamond}c)^{\diamond})^{\diamond}(b^{\diamond}c)^{\diamond})^{\diamond}\\ &= ((b^{\diamond}c)^{\diamond})^{\diamond}=(b^{\diamond}c)^{\diamond},\end{aligned}$$

Accordingly, we further obtain the following equalities:

$$b^{\diamond}\left(b^{\diamond}c\right)^{\diamond}=\left(b^{\diamond}\left(b^{\diamond}c\right)^{\diamond}\right)^{\diamond}b^{\diamond}\left(b^{\diamond}c\right)^{\diamond}=\left(b^{\diamond}c\right)^{\diamond}b^{\diamond}\left(b^{\diamond}c\right)^{\diamond}.$$

and hence, we have the following equalities:

$$\left(b^{\diamond}\left(b^{\diamond}c\right)^{\diamond}\right)^{2}=b^{\diamond}(\left(b^{\diamond}c\right)^{\diamond}b^{\diamond}\left(b^{\diamond}c\right)^{\diamond})=b^{\diamond}(b^{\diamond}\left(b^{\diamond}c\right)^{\diamond})=b^{\diamond}\left(b^{\diamond}c\right)^{\diamond}\in E(S),$$

and $(b^{\diamond}c)^{\diamond}=b^{\diamond}(b^{\diamond}c)^{\diamond}$, so that we have

$$((b^{\diamond}c)^{\diamond}b^{\diamond})^{2}=(b^{\diamond}c)^{\diamond}(b^{\diamond}(b^{\diamond}c)^{\diamond})b^{\diamond}=(b^{\diamond}c)^{\diamond}(b^{\diamond}c)^{\diamond}b^{\diamond}=(b^{\diamond}c)^{\diamond}b^{\diamond}\in E(S)$$

and furthermore, we have $\left(b^{\diamond}c\right)^{\diamond}b^{\diamond}\omega b^{\diamond}$. Thus $a^{\diamond},\left(b^{\diamond}c\right)^{\diamond}b^{\diamond}\in E(b^{\diamond}Sb^{\diamond})$, and thereby

$$\begin{aligned}((b^{\diamond}c)^{\diamond}b^{\diamond}a^{\diamond}(b^{\diamond}c)^{\diamond})^{2} &= ((b^{\diamond}c)^{\diamond}b^{\diamond}a^{\diamond})((b^{\diamond}c)^{\diamond}(b^{\diamond}c)^{\diamond}b^{\diamond}a^{\diamond})(b^{\diamond}c)^{\diamond}\\ &= \left(a^{\diamond}(b^{\diamond}c)^{\diamond}b^{\diamond}\right)\left(a^{\diamond}(b^{\diamond}c)^{\diamond}b^{\diamond}\right)(b^{\diamond}c)^{\diamond}\\ &= (a^{\diamond}(b^{\diamond}c)^{\diamond}b^{\diamond})(b^{\diamond}c)^{\diamond}\\ &= (b^{\diamond}c)^{\diamond}b^{\diamond}a^{\diamond}(b^{\diamond}c)^{\diamond}\end{aligned}$$

since $b^{\diamond}Sb^{\diamond}$ is a C-rpp semigroup. Hence, it follows that $(b^{\diamond}c)^{\diamond}b^{\diamond}a^{\diamond}(b^{\diamond}c)^{\diamond}\omega(b^{\diamond}c)^{\diamond}$.

Denote $f=(b^{\diamond}c)^{\diamond}b^{\diamond}a^{\diamond}(b^{\diamond}c)^{\diamond}$. Obviously, $b^{\diamond}c\in(b^{\diamond}c)^{\diamond}S(b^{\diamond}c)^{\diamond}$, $f\in E((b^{\diamond}c)^{\diamond}S(b^{\diamond}c)^{\diamond})$. But S is a locally C-rpp semigroup, we conclude that $(b^{\diamond}c)^{\diamond}S(b^{\diamond}c)^{\diamond}$ is a C-rpp semigroup. Thus

$$a^{\diamond}c=a^{\diamond}b^{\diamond}c=a^{\diamond}(b^{\diamond}c)^{\diamond}b^{\diamond}(b^{\diamond}c)=(b^{\diamond}c)^{\diamond}b^{\diamond}a^{\diamond}(b^{\diamond}c)^{\diamond}\bullet b^{\diamond}c=f(b^{\diamond}c)=(b^{\diamond}c)f,$$

and whence $ac=ba^{\diamond}c=b(b^{\diamond}c)f=bcf$. And hence $ac\mathcal{L}^{*}a^{\diamond}c=(b^{\diamond}c)f\mathcal{L}^{*}(b^{\diamond}c)^{\diamond}f=f$ since $\mathcal{L}^{*}$ is a right congruence on S so that

$$L^{*}\left(ac\right)=L^{*}\left(f\right)=L^{*}\left(f\left(b^{\diamond}c\right)^{\diamond}\right)\subseteq L^{*}((b^{\diamond}c)^{\diamond})=L^{*}\left(b^{\diamond}c\right)=L^{*}\left(bc\right).$$

Consequently, we have proved $ac\leqslant_{l}bc$ and by Proposition 3.4, we have $ac\leqslant_{sl}bc$. Thus, we have shown that S is compatible with multiplication for $\leq_{sl}$. This completes the proof.

Finally,we noticed that in [17], M. V. Lawson has pointed out that in a regular semigroup S, $\leq_{l}=\leq=\leq_{r}$. Now let S be a completely regular semigroup. Then S becomes a super rpp semigroup. Let us turn to the proof of our Theorem 4.5. If S is compatible for $\leq$, then by Propositions 4.4 and 3.4, $\leq=\leq_{sl}$. Now by Theorem 4.5, S is a locally C-rpp semigroup, that is, eSe is a C-rpp semigroup for any $e\in E(S)$. Since S is regular, it is not difficult to verify that eSe is regular. Hence eSe is a C-rpp regular semigroup, in other words, eSe is a Clifford semigroup. Therefore S is a locally Clifford semigroup.

Conversely, if S is a locally Clifford semigroup, then by the proof of Theorem 4.5, $\leq_{sl}=\leq_l$, and so $\leq_{sl}=\leq$. Now by Theorem 4.5, S is compatible for $\leq$.

In closing this paper, we consider completely regular semigroups. Finally, we give a characterization for the locally Clifford semigroups in the following corollary.

Corollary 4.6. *Let S be a completely regular semigroup. Then S is compatible with multiplication for $\leq$ if and only if S is a locally Clifford semigroup.*

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