Available online at http://scik.org

J. Semigroup Theory Appl. 2023, 2023:1

https://doi.org/10.28919/jsta/8044

ISSN: 2051-2937

ON ω-COSETS AND PARTIAL RIGHT CONGRUENCES ON RHOTRIX AMPLE SEMIGROUPS

R. U. NDUBUISI[1,*], O. G. UDOAKA[2], R. B. ABUBAKAR[3], M. PATIL KAILASH[4], PRECIOUS C. ASHARA[1]

[1]Department of Mathematics, Federal University of Technology, Owerri, Nigeria

[2]Department of Mathematics, Akwa Ibom State University, Ikot Akpaden, Nigeria

[3]Department of Mathematics, Federal College of Education (T), Omoku, Nigeria

[4]Department of Mathematics, Dharmsinh Desai University, India

Abstract. This work presents some characteristics of ω-cosets of rhotrix ample subsemigroup which helps in studying rhotrix ample semigroups. Consequently, some partial right congruences on rhotrix ample semigroups are also considered.

Keywords: partial right congruences; ω-cosets; rhotrix ample subsemigroups; partial ordering.

2020 AMS Subject Classification: 20M10.

1. INTRODUCTION

Ajibade [1] introduced the concept of rhotrix as an object whose elements are arranged in a rhomboidal nature. This concept was indeed an extension of earlier works of [9] on matrix-tersions and matrix noitrets. Suppose R and Q are two rotrices such that

$$R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle, Q = \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$$ where $h(R)$ and $h(Q)$ are the hearts of these rhotrices.

*Corresponding author

E-mail addresses: u_ndubuisi@yahoo.com, rich.ndubuisi@futo.edu.ng

Received June 07, 2023

It follows from [1] that

$$R+Q=\left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{matrix} \right\rangle$$

and $$R\circ Q=\left\langle \begin{matrix} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{matrix} \right\rangle$$

Sani [5] gave an alternative method for multiplying rhotrices which is given by

$$R\circ Q=\left\langle \begin{matrix} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bj+ek & \end{matrix} \right\rangle.$$

The generalization of this alternative method of multiplication was later given by Sani [6] as follows;

$$R_n\circ Q_n=\langle a_{i_1j_1},c_{l_ik_1}\rangle\circ\langle b_{i_2j_2},d_{l_2k_2}\rangle=\langle\sum_{i_2j_{1=1}}^{t}\left(a_{i_1j_1}b_{i_2j_2}\right),\sum_{l_2k_1=1}^{t-1}\left(c_{l_ik_1}d_{l_2k_2}\right)\rangle\ ,t=\frac{n+1}{2},$$

where R_n and Q_n denote n-dimensional rhotrices (with n rows and n columns).

Mohammed [2] and Isere [4] obtained a new way of representing rhotices in a general form. Another method known as row-wise representation was also given by Chinedu [11]. In [3], some construction of rhotrix semigroup was obtained. The ample (formerly type A) version of rhotrix semigroup as well as its congruences was given by Ndubuisi et al [12]. Again in [13], Ndubuisi et al employed ω-cosets of rhotrix ample subsemigroups and obtained a more general form of representation for a rhotrix ample semigroup than the one given in [12]. This result is analogous to those of inverse semigroups (see [8] and [10]) and ample semigroups (see [14]).

The later representation of rhotrix ample semigroup given in [13] presented the importance of ω-cosets of rhotrix ample subsemigroups of a rhotrix ample semigroup as a useful tool in studying representations of rhotrix ample semigroups.

This paper is therefore aimed at the properties of ω-cosets of closed rhotrix ample subsemigroups. It also considers partial right congruences on rhotrix ample semigroups.

2. PRELIMINARIES

Some definitions and known results needed in this work are presented in this section. For the notation and terminologies not mentioned in this paper, the reader is referred to [1], [12] and [13]. Throughout our study, we will use the term semigroup S to refer to rhotrix ample while $E(S)$ denotes the semilattice of idempotents.

Definition 2.1. Suppose a, b be elements of a semigroup S, we define $a\,\mathcal{R}^*b$ if and only if for all $x, y \in S^1$, $xa = ya \Leftrightarrow xb = yb$. Dually we define the relation $\mathcal{L}^*$.

Let S be a semigroup and $a \in S$. The elements $a^{\dagger}$(resp. a^*) will denote an idempotent element in $\mathcal{R}^*$(resp. $\mathcal{L}^*$)-class R_a^* (resp. L_a^*).

Definition 2.2. A semigroup S with a semilattice of idempotents $E(S)$ is said to be an adequate semigroup if each $\mathcal{R}^*$-class and $\mathcal{L}^*$-class contain an idempotent. With $E(S)$ being a semilattice such an idempotent is unique.

Definition 2.3. A left adequate semigroup is said to be a left ample (formerly left type A) if for all $e \in E(S)$ and $a \in S$, $ae = (ae)^{\dagger}a$ (see [7]) and dually for right ample (formerly right type A) semigroups. A semigroup S is said to be an ample (formerly type A) semigroup if it is both left and right ample.

Theorem 2.4. [12] Let $R_n(\mathrm{F})$ be a set of all rhotrices of size n with entries from an arbitrary field F. For any A_n, $B_n \in R_n(\mathrm{F})$, define a binary operation $\circ$ on $R_n(\mathrm{F})$ by the rule:

$$A_n \circ B_n = \langle a_{i_1j_1}, c_{l_ik_1}\rangle \circ \langle b_{i_2j_2}, d_{l_2k_2}\rangle = \langle \sum_{i_2j_{1=1}}^{t} \left(a_{i_1j_1}b_{i_2j_2}\right), \sum_{l_2k_1=1}^{t-1} \left(c_{l_ik_1}d_{l_2k_2}\right)\rangle \;, t = \frac{n+1}{2},$$

where A_n and B_n denote n-dimensional rhotrices. Then $S = (R_n(\mathrm{F}), \circ)$ is a semigroup.

Lemma 2.5. [12] Let $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle \in S = (R_n(\mathrm{F}), \circ)$. Then we have

i)$\langle a_{ij}, c_{lk}\rangle\,\mathcal{R}^*\,\langle b_{ij}, d_{lk}\rangle$ if and only if $a_{ij}\mathcal{R}^*b_{ij}$ and $c_{lk}\,\mathcal{R}^*d_{lk}$.

ii) $\langle a_{ij}, c_{lk}\rangle\,\mathcal{L}^*\,\langle b_{ij}, d_{lk}\rangle$ if and only if $a_{ij}\mathcal{L}^*b_{ij}$ and $c_{lk}\,\mathcal{L}^*\,d_{lk}$.

iii) $\langle a_{ij}, c_{lk}\rangle\,\mathcal{H}^*\langle b_{ij}, d_{lk}\rangle$ if and only if $\langle a_{ij}, c_{lk}\rangle\,\mathcal{R}^*\,\langle b_{ij}, d_{lk}\rangle$ and $\langle a_{ij}, c_{lk}\rangle\,\mathcal{L}^*\,\langle b_{ij}, d_{lk}\rangle$.

Lemma 2.6.[12] Suppose $\langle a_{ij}, c_{lk}\rangle \epsilon\, S = (R_n(\mathrm{F}),\ \circ\,)$. Then $\langle a_{ij}, c_{lk}\rangle \epsilon\, E(S)$ if and only if $a_{ij}\ \epsilon\, E(\mathcal{M}_t(\mathrm{F}))$ and $c_{lk}\ \epsilon\, E(\mathcal{M}_{t-1}(\mathrm{F}))$.

Theorem 2.7. [12] $S = (R_n(\mathrm{F}),\ \circ\,)$ is an ample semigroup.

Remark 2.8. [12] It is important to note that $\langle a_{ij}, I_{lk}\rangle$ is the idempotent in the $\mathcal{R}^*$-class as well as the $\mathcal{L}^*$-class. For the sake of ambiguity, we will use $\langle a_{ij}, c_{lk}\rangle^{\dagger} = \langle a_{ij}, I_{lk}\rangle$ and $\langle a_{ij}, c_{lk}\rangle^{*} = \langle a_{ij}, I_{lk}\rangle$ to denote the respective idempotents.

3. Partial Order Relation on S

Now let S be a rhotrix ample semigroup with a semilattice $E(S)$ of idempotents. The partial order relation $\leq$ will be defined on S as follows:

$\langle a_{ij}, c_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle$ if and only if $\langle a_{ij}, c_{lk}\rangle = \langle I_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle$ for $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle\ \epsilon\ S$ and $\langle I_{ij}, c_{lk}\rangle\ \epsilon\ E(S)$.

To see that $\leq$ is an equivalence relation on S, we have that; for $\langle a_{ij}, c_{lk}\rangle\ \epsilon\ S$, $\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, I_{lk}\rangle\langle a_{ij}, c_{lk}\rangle$ which shows reflexivity.

It can be easily checked that $\leq$ is anti-symmetric and transitive.

A partial ordering $\leq$ on S is also defined as a relation on S such that for $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle\ \epsilon\ S$, $\langle a_{ij}, c_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle$ if and only if $\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle\langle a_{ij}, c_{lk}\rangle^{*}$.

More so, suppose $\langle a_{ij}, c_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle$ then $\langle a_{ij}, c_{lk}\rangle^{\dagger} \leq \langle b_{ij}, d_{lk}\rangle^{\dagger}$.

In the same way, if $\langle a_{ij}, c_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle$ then $\langle a_{ij}, c_{lk}\rangle^{*} \leq \langle b_{ij}, d_{lk}\rangle^{*}$.

The lemma below presents some properties of the partial order relation $\leq$ on a rhotrix ample semigroup S.

Lemma 3.1. Suppose S is a rhotrix ample semigroup such that $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle\ \epsilon\ S$. Then the following statements are equivalent.

i) $\langle a_{ij}, c_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle$

ii) $\langle a_{ij}, c_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle^{\dagger}\langle a_{ij}, c_{lk}\rangle$ and $\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^{*}$

iii) $\langle a_{ij}, c_{lk}\rangle = \left(\langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle\right)^{\dagger}\langle a_{ij}, c_{lk}\rangle$ and $\langle a_{ij}, c_{lk}\rangle\left(\langle b_{ij}, d_{lk}\rangle\langle a_{ij}, c_{lk}\rangle^{*}\right)^{*}$

iv) More so, $\langle a_{ij}, c_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle$ and $\langle u_{ij}, v_{lk}\rangle \leq \langle m_{ij}, n_{lk}\rangle$ implies that $\langle a_{ij}, c_{lk}\rangle\langle u_{ij}, v_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle\langle m_{ij}, n_{lk}\rangle$ for $\langle u_{ij}, v_{lk}\rangle, \langle m_{ij}, n_{lk}\rangle \in S$.

Proof. i) $\Longrightarrow$ ii) Let $\langle a_{ij}, c_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle$ so we have $\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle$,

$$\begin{aligned}\langle b_{ij}, d_{lk}\rangle^{\dagger}\langle a_{ij}, c_{lk}\rangle &= \langle b_{ij}, d_{lk}\rangle^{\dagger}\langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle \\ &= \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle \\ &= \langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle \\ &= \langle a_{ij}, c_{lk}\rangle.\end{aligned}$$

More so, we have that

$\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle$ implies that

$$\begin{aligned}\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^{*} &= \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^{*} \\ &= \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle \\ &= \langle a_{ij}, c_{lk}\rangle.\end{aligned}$$

iii) $\Longrightarrow$ i) is obvious

iv) Follows from the transitivity property of the partial order relation $\leq$ applied to the product given by $\langle a_{ij}, c_{lk}\rangle\langle u_{ij}, v_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle\langle m_{ij}, n_{lk}\rangle$.

Lemma 3.2. Let S be a rhotrix ample semigroup and let $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle \in S$, then there exist $\langle u_{ij}, v_{lk}\rangle \in S$ such that $\langle a_{ij}, c_{lk}\rangle \leq \langle u_{ij}, v_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle \leq \langle u_{ij}, v_{lk}\rangle$, also there exists $\langle q_{ij}, r_{lk}\rangle \in S$ such that $\langle q_{ij}, r_{lk}\rangle \leq \langle a_{ij}, c_{lk}\rangle$ and $\langle q_{ij}, r_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle$.

Proof. If $\langle a_{ij}, c_{lk}\rangle \leq \langle u_{ij}, v_{lk}\rangle$, $\langle b_{ij}, d_{lk}\rangle \leq \langle u_{ij}, v_{lk}\rangle \in S$ then with $\langle a_{ij}, c_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle u_{ij}, v_{lk}\rangle$, $\langle b_{ij}, d_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle^{\dagger}\langle u_{ij}, v_{lk}\rangle$. It follows that

$$\begin{aligned}\langle b_{ij}, d_{lk}\rangle^{\dagger}\langle a_{ij}, c_{lk}\rangle &= \langle b_{ij}, d_{lk}\rangle^{\dagger}\langle a_{ij}, c_{lk}\rangle^{\dagger}\langle u_{ij}, v_{lk}\rangle \\ &= \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle^{\dagger}\langle u_{ij}, v_{lk}\rangle \\ &= \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle\end{aligned}$$

So that if we let $\langle q_{ij}, r_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle^{\dagger}\langle a_{ij}, c_{lk}\rangle$ then we have that

$$\langle q_{ij}, r_{lk}\rangle^{\dagger} = \langle b_{ij}, d_{lk}\rangle^{\dagger}\langle a_{ij}, c_{lk}\rangle^{\dagger} \text{ and}$$

$$\langle q_{ij}, r_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle^{\dagger}\langle a_{ij}, c_{lk}\rangle^{\dagger}\langle a_{ij}, c_{lk}\rangle$$

$$= \langle q_{ij}, r_{lk}\rangle^{\dagger}\langle a_{ij}, c_{lk}\rangle.$$

Thus, $\langle q_{ij}, r_{lk}\rangle \leq \langle a_{ij}, c_{lk}\rangle$.

Similarly, $\langle q_{ij}, r_{lk}\rangle = \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle$

$$= \langle a_{ij}, c_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle$$

$$= \langle q_{ij}, r_{lk}\rangle^{\dagger}\langle b_{ij}, d_{lk}\rangle$$

which implies that $\langle q_{ij}, r_{lk}\rangle \leq \langle b_{ij}, d_{lk}\rangle$.

Lemma 3.3. Let S be a rhotrix ample semigroup and let $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle \epsilon S$. Suppose that $\langle e_{ij}, I_{lk}\rangle$ is the right unit of $\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle$ and $\langle f_{ij}, I_{lk}\rangle$ the right of $\langle b_{ij}, d_{lk}\rangle$ then $\langle e_{ij}, I_{lk}\rangle \leq \langle f_{ij}, I_{lk}\rangle$.

Proof. Let $\langle e_{ij}, I_{lk}\rangle$ be the right unit of $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle$ and $\langle f_{ij}, I_{lk}\rangle$ the right of $\langle b_{ij}, d_{lk}\rangle$ then we have that $\langle e_{ij}, I_{lk}\rangle = \left(\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\right)^{*}$ and $\langle f_{ij}, I_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle^{*}$.

We know that

$$\left(\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\right)^{*} = \left(\langle a_{ij}, c_{lk}\rangle^{*}\langle b_{ij}, d_{lk}\rangle\right)^{*},\ \langle a_{ij}, c_{lk}\rangle^{*}\langle b_{ij}, d_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle\left(\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\right)^{*}.$$

Consequently,

$$\langle e_{ij}, I_{lk}\rangle\langle f_{ij}, I_{lk}\rangle = \langle b_{ij}, d_{lk}\rangle^{*}\left(\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\right)^{*}$$

$$= \left(\langle b_{ij}, d_{lk}\rangle\left(\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^{*}\right)\right)^{*}$$

$$= \left(\langle b_{ij}, d_{lk}\rangle\left(\langle a_{ij}, c_{lk}\rangle^{*}\langle b_{ij}, d_{lk}\rangle\right)^{*}\right)^{*}$$

$$= \left(\langle a_{ij}, c_{lk}\rangle^{*}\langle b_{ij}, d_{lk}\rangle\right)^{*}$$

$$= \left(\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\right)^{*}$$

$$= \langle e_{ij}, I_{lk}\rangle.$$

It is known in [12] that idempotents commute on rhotrix ample semigroup, so we have that

$$\langle f_{ij}, I_{lk}\rangle\langle e_{ij}, I_{lk}\rangle = \langle e_{ij}, I_{lk}\rangle\langle f_{ij}, I_{lk}\rangle$$

$$= \langle e_{ij}, I_{lk}\rangle.$$ Thus $\langle e_{ij}, I_{lk}\rangle \leq \langle f_{ij}, I_{lk}\rangle$.

4. ω-COSETS OF A CLOSED SUBSEMIGROUP OF RHOTRIX AMPLE SEMIGROUP

Throughout this section, we shall denote the natural partial order of a rhotrix ample semigroup by ω. Now let $\langle x_{ij}, y_{lk}\rangle \epsilon S$ and put $\langle x_{ij}, y_{lk}\rangle\omega = \{\langle a_{ij}, c_{lk}\rangle \epsilon S : \langle x_{ij}, y_{lk}\rangle\, \omega\, \langle a_{ij}, c_{lk}\rangle\}$.

Let $K = \langle m_{ij}, n_{lk}\rangle$ be a subset of S such that the closure of K in S becomes

$$\langle m_{ij}, n_{lk}\rangle\omega = \{\langle a_{ij}, c_{lk}\rangle \epsilon S : \langle m_{ij}, n_{lk}\rangle\, \omega\, \langle a_{ij}, c_{lk}\rangle\}.$$

We state below some properties of the closure of K in S as adopted from [13].

Lemma 4.1. [13] Let $K = \langle m_{ij}, n_{lk}\rangle$ and $T = \langle s_{ij}, t_{lk}\rangle$ be subsets of a rhotrix ample semigroup S. Then the following is evident

i) $\langle m_{ij}, n_{lk}\rangle \subseteq \langle m_{ij}, n_{lk}\rangle\, \omega$

ii) $\langle m_{ij}, n_{lk}\rangle\, \omega \subseteq \langle s_{ij}, t_{lk}\rangle\, \omega$ if $\langle m_{ij}, n_{lk}\rangle \subseteq \langle s_{ij}, t_{lk}\rangle$

iii) $\left(\left(\langle m_{ij}, n_{lk}\rangle\omega\right)\langle a_{ij}, c_{lk}\rangle\right)\omega = \left(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\right)\omega$

Lemma 4.2. Let S be a rhotrix ample semigroup with $K = \langle m_{ij}, n_{lk}\rangle$ as its subsemigroup. Then

i) $\langle m_{ij}, n_{lk}\rangle\omega$ is closed ample subsemigroup of S

ii) $\left(\langle m_{ij}, n_{lk}\rangle\omega\right)\langle g_{ij}, h_{lk}\rangle = \langle m_{ij}, n_{lk}\rangle\omega$ for each $\langle g_{ij}, h_{lk}\rangle \epsilon\ S$

Proof. i) The proof is a routine check.

ii) The proof follows from Lemma 3.1 for each $\langle a_{ij}, c_{lk}\rangle \epsilon\ K$, we have

$\left(\langle m_{ij}, n_{lk}\rangle\omega\right)\langle a_{ij}, c_{lk}\rangle \subseteq \left(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\right) = \langle m_{ij}, n_{lk}\rangle\omega$ which completes the proof.

The following Lemma is true for a closed subsemigroup of rhotrix ample semigroups.

Lemma 4.3. Let $Q = \langle p_{ij}, q_{lk}\rangle$ be a closed subsemigroup of a rhotrix ample semigroup S. Suppose V is a closed subsemigroup of Q, then V is also closed in S.

Proof. Suppose $\langle t_{ij}, h_{lk}\rangle \epsilon\, Q$ is in $V\omega$ then for some $\langle s_{ij}, o_{lk}\rangle \epsilon\, V$ we have $\langle s_{ij}, o_{lk}\rangle$, $\langle t_{ij}, h_{lk}\rangle \epsilon\, \omega$. Since V is closed in Q then $V\omega = V$ which implies that $\langle t_{ij}, h_{lk}\rangle \epsilon\, V$.

From $\langle t_{ij}, h_{lk}\rangle \epsilon\, V$ and $V \subset Q \subset Q\omega$ then $\langle t_{ij}, h_{lk}\rangle\, \omega\, \langle x_{ij}, y_{lk}\rangle$ for $\langle x_{ij}, y_{lk}\rangle \epsilon\, Q\omega$, it follows from the transitivity of ω that $\left(\langle s_{ij}, o_{lk}\rangle, \langle x_{ij}, y_{lk}\rangle\right) \epsilon\, \omega$.

Thus $\langle x_{ij}, y_{lk}\rangle \epsilon\, V\omega$.

But we already know that $\left(\langle s_{ij}, o_{lk}\rangle, \langle x_{ij}, y_{lk}\rangle\right) \epsilon\ \omega$ implies that $\left(\langle s_{ij}, o_{lk}\rangle^{\dagger}, \langle x_{ij}, y_{lk}\rangle^{\dagger}\right) \epsilon\ \omega$ and $\left(\langle s_{ij}, o_{lk}\rangle^{*}, \langle x_{ij}, y_{lk}\rangle^{*}\right) \epsilon\ \omega$ so that $\langle s_{ij}, o_{lk}\rangle^{\dagger}, \langle s_{ij}, o_{lk}\rangle^{*} \epsilon\ V\omega = V$.

But $\langle x_{ij}, y_{lk}\rangle$, $\langle x_{ij}, y_{lk}\rangle^{\dagger}, \langle x_{ij}, y_{lk}\rangle^{*} \epsilon\ Q\omega$ and $Q\omega = Q$, thus $\langle x_{ij}, y_{lk}\rangle^{\dagger}, \langle x_{ij}, y_{lk}\rangle^{*} \epsilon\ V\omega = V$ which implies that V is closed in S.

5. PARTIAL RIGHT CONGRUENCES ON S

Now let K be a rhotrix ample semigroup of S and define a relation ρ^K on S in terms of ω-cosets of K by the rule:

$\langle a_{ij}, c_{lk}\rangle\ \rho^K\ \langle b_{ij}, d_{lk}\rangle$ if and only if $\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^{*} \epsilon \left(\langle m_{ij}, n_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\right) \epsilon\ \omega$ for some $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle \epsilon\ S$.

Lemma 5.1. ρ^K is a partial equivalence on S.

Proof. That ρ^K is reflexive is clear. It is known from the proof of lemma 4.3 that if $\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^{*} \epsilon \left(\langle m_{ij}, n_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\right)\omega$ then $\langle m_{ij}, n_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\ \omega\ \langle a_{ij}, c_{lk}\rangle$ and we have that $\langle a_{ij}, c_{lk}\rangle \epsilon \left(\langle m_{ij}, n_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\right)\omega$.

It also follows from lemma 4.3 that $\left(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\right)\omega = \left(\langle m_{ij}, n_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\right)\omega$, and that $\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^{*} \epsilon \left(\langle m_{ij}, n_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\right)\omega$ implies that $\langle b_{ij}, d_{lk}\rangle\langle a_{ij}, c_{lk}\rangle^{*} \epsilon \left(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\right)\omega$ so that we have $\langle a_{ij}, c_{lk}\rangle\ \rho^K\ \langle b_{ij}, d_{lk}\rangle$ if and only if $\langle b_{ij}, d_{lk}\rangle\ \rho^K\ \langle a_{ij}, c_{lk}\rangle$.

Hence ρ^K is a symmetric relation.

ρ^K is a transitive relation since from $\langle a_{ij}, c_{lk}\rangle\ \rho^K\ \langle b_{ij}, d_{lk}\rangle$ and $\langle b_{ij}, d_{lk}\rangle\ \rho^K\ \langle u_{ij}, v_{lk}\rangle$ it follows from lemma 4.3 that $\left(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\right)\omega = \left(\langle m_{ij}, n_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\right)\omega = \left(\langle m_{ij}, n_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\right)\omega$ so that $\langle a_{ij}, c_{lk}\rangle\langle u_{ij}, v_{lk}\rangle^{*} \epsilon \left(\langle m_{ij}, n_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\right)\omega$.

Thus ρ^K is a partial equivalence on S.

Remark 5.2. We note that if $\langle a_{ij}, c_{lk}\rangle^{\dagger} \notin K$, then obviously $\langle a_{ij}, c_{lk}\rangle \notin \left(\langle m_{ij}, n_{lk}\rangle\langle a_{ij}, c_{lk}\rangle\right)\omega$ which implies that $\langle a_{ij}, c_{lk}\rangle\ \rho^K\ \langle a_{ij}, c_{lk}\rangle$ may not hold for all $\langle a_{ij}, c_{lk}\rangle \epsilon\ S$.

Put $D_K^{*} = \left\{\langle x_{ij}, y_{lk}\rangle \epsilon\ S\ :\ \langle x_{ij}, y_{lk}\rangle^{\dagger} \epsilon\ K\right\}$. We call the set D_K^{*} the domain of the relation ρ^K.

We say that a partial equivalence ρ^K on S is a partial right congruence if and only if $\langle a_{ij}, c_{lk}\rangle, \langle b_{ij}, d_{lk}\rangle \epsilon S$ we have that

$\langle a_{ij}, c_{lk}\rangle \rho^K \langle b_{ij}, d_{lk}\rangle$ implies that $\left(\langle a_{ij}, c_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\right) \rho^K \left(\langle b_{ij}, d_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\right)$ is true for all $\langle x_{ij}, y_{lk}\rangle \epsilon S$ or both $\langle a_{ij}, c_{lk}\rangle\langle x_{ij}, y_{lk}\rangle$, $\langle b_{ij}, d_{lk}\rangle\langle x_{ij}, y_{lk}\rangle \epsilon S|dom\ \rho^K$.

The next lemma shows that ρ^K is a partial right congruence on S.

Lemma 5.3. ρ^K is a partial right congruence on S

Proof. Let $\langle a_{ij}, c_{lk}\rangle \rho^K \langle b_{ij}, d_{lk}\rangle$. So we have that $\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^* \epsilon \left(\langle m_{ij}, n_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\right)\omega$.

But $\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^* = \left(\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^*\right)^{\dagger}\langle a_{ij}, c_{lk}\rangle$ so that

$\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^*\langle p_{ij}, q_{lk}\rangle = \left(\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^*\right)^{\dagger} \langle a_{ij}, c_{lk}\rangle\langle p_{ij}, q_{lk}\rangle$.

Hence we have

$\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^*\langle p_{ij}, q_{lk}\rangle\ \omega\ \langle a_{ij}, c_{lk}\rangle\langle p_{ij}, q_{lk}\rangle$ and

$\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^*\langle p_{ij}, q_{lk}\rangle \epsilon \left(\langle m_{ij}, n_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\langle p_{ij}, q_{lk}\rangle\right)\omega$, so that

$\langle a_{ij}, c_{lk}\rangle\langle p_{ij}, q_{lk}\rangle \epsilon \left(\langle m_{ij}, n_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\langle p_{ij}, q_{lk}\rangle\right)\omega$. It is now obvious that when

$\left(\langle a_{ij}, c_{lk}\rangle\langle p_{ij}, q_{lk}\rangle\right)^{\dagger} \epsilon K$, then $\left(\langle m_{ij}, n_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\langle p_{ij}, q_{lk}\rangle\right)\omega$ is an ω-coset and then

$$\begin{aligned}\langle a_{ij}, c_{lk}\rangle\langle b_{ij}, d_{lk}\rangle^*\langle p_{ij}, q_{lk}\rangle &= \langle a_{ij}, c_{lk}\rangle\left(\langle b_{ij}, d_{lk}\rangle^*\langle p_{ij}, q_{lk}\rangle\right)\\ &= \langle a_{ij}, c_{lk}\rangle\langle p_{ij}, q_{lk}\rangle\left(\langle b_{ij}, d_{lk}\rangle^*\langle p_{ij}, q_{lk}\rangle\right)^*\\ &= \left(\langle a_{ij}, c_{lk}\rangle\langle p_{ij}, q_{lk}\rangle\right)\left(\langle b_{ij}, d_{lk}\rangle^*\langle p_{ij}, q_{lk}\rangle\right)^* \epsilon \left(\langle m_{ij}, n_{lk}\rangle\langle b_{ij}, d_{lk}\rangle\langle p_{ij}, q_{lk}\rangle\right)\omega.\end{aligned}$$

Obviously, $\langle a_{ij}, c_{lk}\rangle \rho^K \langle b_{ij}, d_{lk}\rangle$ implies that $\langle a_{ij}, c_{lk}\rangle\langle p_{ij}, q_{lk}\rangle \rho^K \langle b_{ij}, d_{lk}\rangle\langle p_{ij}, q_{lk}\rangle$ if

$\langle p_{ij}, q_{lk}\rangle \epsilon S$ is such that $\left(\langle b_{ij}, d_{lk}\rangle\langle p_{ij}, q_{lk}\rangle\right)^{\dagger} \epsilon K$.

Thus ρ^K is a right partial congruence on S.

Remark 5.4. It is important to note that the ω-coset $\left(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\right)\omega$ is a closed set since it is known that the closure of a set is closed.

Lemma 5.5. $\left(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\right)\omega$ is a congruence class modulo ρ^K .

Proof. It is known that $\left(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\right)\omega$ is an ω-coset for $\langle x_{ij}, y_{lk}\rangle \epsilon D_K^*$.

Let $\langle u_{ij}, v_{lk}\rangle \epsilon \left(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\right)\omega$, then $\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\ \omega\ \langle u_{ij}, v_{lk}\rangle$.

But we have that

$\big(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)^{\dagger}\omega\ \langle x_{ij}, y_{lk}\rangle^{\dagger}$, $\big(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)^{\dagger} = \big(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle^{\dagger}\big)^{\dagger}\omega\ \langle x_{ij}, y_{lk}\rangle^{\dagger}$

and since $\langle x_{ij}, y_{lk}\rangle^{\dagger} \epsilon\ K$, then $\big(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)^{\dagger} \epsilon\ K$.

Consequently, $\langle x_{ij}, y_{lk}\rangle^{\dagger} \epsilon\ K$ so that $\langle u_{ij}, v_{lk}\rangle \epsilon\ D_K^*$ and $\big(\langle m_{ij}, n_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\big)\omega$ is an ω-coset .

We have that $\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\ \omega\ \langle u_{ij}, v_{lk}\rangle$ implies that

$\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\langle x_{ij}, y_{lk}\rangle^* = \langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\ \omega\ \langle u_{ij}, v_{lk}\rangle\langle x_{ij}, y_{lk}\rangle^*$, so $\langle u_{ij}, v_{lk}\rangle\langle x_{ij}, y_{lk}\rangle^*$.

Also we have that $\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\ \omega\ \langle u_{ij}, v_{lk}\rangle$ implies that

$$\big(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)^* = \big(\langle m_{ij}, n_{lk}\rangle^*\langle x_{ij}, y_{lk}\rangle\big)^*\omega\ \langle u_{ij}, v_{lk}\rangle^*.$$

Together with $\langle x_{ij}, y_{lk}\rangle\big(\langle m_{ij}, n_{lk}\rangle^*\langle x_{ij}, y_{lk}\rangle\big)^* = \langle m_{ij}, n_{lk}\rangle^*\langle x_{ij}, y_{lk}\rangle\ \epsilon\ \big(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)\omega$ and $\langle m_{ij}, n_{lk}\rangle^*\langle x_{ij}, y_{lk}\rangle\ \omega\ \langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle^*$ then $\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle^*\ \epsilon\ \big(\langle m_{ij}, n_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\big)\omega$.

Consequently, it follows that

$\big(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)\omega = \big(\langle m_{ij}, n_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\big)\omega$ resulting in $\langle x_{ij}, y_{lk}\rangle\ \rho^K\ \langle u_{ij}, v_{lk}\rangle$.

Conversely, let $\langle x_{ij}, y_{lk}\rangle\ \rho^K\ \langle u_{ij}, v_{lk}\rangle$, then since $\langle u_{ij}, v_{lk}\rangle\langle x_{ij}, y_{lk}\rangle^*\ \epsilon\ \big(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)\omega$ and $\langle u_{ij}, v_{lk}\rangle\langle x_{ij}, y_{lk}\rangle^* = \big(\langle u_{ij}, v_{lk}\rangle\langle x_{ij}, y_{lk}\rangle^*\big)^{\dagger}\langle u_{ij}, v_{lk}\rangle\ \omega\ \langle u_{ij}, v_{lk}\rangle$ so we have that

$\langle u_{ij}, v_{lk}\rangle\ \epsilon\ \big(\langle m_{ij}, n_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\big)\omega$.

But $\langle u_{ij}, v_{lk}\rangle\ \epsilon\ \big(\langle m_{ij}, n_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\big)\omega$ implies that $\langle u_{ij}, v_{lk}\rangle^{\dagger}\ \epsilon\ K$ so obviously $\langle u_{ij}, v_{lk}\rangle\ \epsilon\ D_K^*$, and thus $\big(\langle m_{ij}, n_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)\omega$ is a ρ^K -class.

So far, we have proved the following theorem

Theorem 5.6. Let K be a closed subsemigroup of a rhotrix ample semigroup S. Define a relation ρ^K on S as follows: $\langle a_{ij}, c_{lk}\rangle\ \rho^K\ \langle b_{ij}, d_{lk}\rangle$ implies that

$\big(\langle a_{ij}, c_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)\ \rho^K\ \big(\langle b_{ij}, d_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)$.

Then ρ^K is a partial right congruence on S.

Remark 5.7. If however $\langle x_{ij}, y_{lk}\rangle\ \epsilon\ K$ then $\big(\langle m_{ij}, n_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\big)\omega = \langle m_{ij}, n_{lk}\rangle\omega$ showing that $\langle m_{ij}, n_{lk}\rangle\omega$ is also a ρ^K -class. It is obvious that K is the only ρ^K –class containing idempotents.

We will conclude this section by showing an interesting property of a partial right congruence on a rhotrix ample semigroup.

Now let σ be a partial right congruence on a rhotrix ample semigroup S. Then σ will be called a principal partial right congruence if it is also a right cancellation congruence such that σ-class is closed, and only one σ-class contains some idempotents.

Theorem 5.8. Let S be a rhotrix ample semigroup and let σ be a principal right congruence on S. A σ-class K containing idempotents is a closed rhotrix ample subsemigroup of S.

Proof. To prove that K is a rhotrix ample semigroup, we observe that if $\langle x_{ij}, y_{lk}\rangle, \langle u_{ij}, v_{lk}\rangle \in K$ then $\langle x_{ij}, y_{lk}\rangle^{\dagger}\langle u_{ij}, v_{lk}\rangle \in K$. Also $\langle x_{ij}, y_{lk}\rangle \,\sigma\, \langle x_{ij}, y_{lk}\rangle^{\dagger}$ implies $\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle \,\sigma\, \langle x_{ij}, y_{lk}\rangle^{\dagger}\langle u_{ij}, v_{lk}\rangle$, hence $\langle x_{ij}, y_{lk}\rangle, \langle u_{ij}, v_{lk}\rangle \in K$, so K must be a subsemigroup of S. Suppose $\langle e_{ij}, I_{lk}\rangle \in E(S)$ then we have that $\langle e_{ij}, I_{lk}\rangle \in K$, so every $\langle x_{ij}, y_{lk}\rangle \in K$, it follows that $\langle e_{ij}, I_{lk}\rangle\langle x_{ij}, y_{lk}\rangle, \big(\langle e_{ij}, I_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)^{\dagger}, \big(\langle e_{ij}, I_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)^{*}$ are in K and $\langle e_{ij}, I_{lk}\rangle\langle x_{ij}, y_{lk}\rangle = \langle x_{ij}, y_{lk}\rangle\big(\langle e_{ij}, I_{lk}\rangle\langle x_{ij}, y_{lk}\rangle\big)^{*}$ holds in K.

In the same manner, $\langle x_{ij}, y_{lk}\rangle\langle e_{ij}, I_{lk}\rangle =, \big(\langle x_{ij}, y_{lk}\rangle\langle e_{ij}, I_{lk}\rangle\big)^{\dagger}\langle x_{ij}, y_{lk}\rangle \in K$. Also for $\langle u_{ij}, v_{lk}\rangle \in K\omega$, then $\langle x_{ij}, y_{lk}\rangle \,\omega\, \langle u_{ij}, v_{lk}\rangle$ for some $\langle x_{ij}, y_{lk}\rangle \in K$.

But $\langle x_{ij}, y_{lk}\rangle^{\dagger} \,\omega\, \langle u_{ij}, v_{lk}\rangle^{\dagger}$, $\langle x_{ij}, y_{lk}\rangle^{*} \,\omega\, \langle u_{ij}, v_{lk}\rangle^{*}$ and $\langle u_{ij}, v_{lk}\rangle^{\dagger}, \langle u_{ij}, v_{lk}\rangle^{*} \in K$, so $\langle u_{ij}, v_{lk}\rangle \in K$. Thus, $K\omega \subseteq K$.

Consequently, suppose that $\langle u_{ij}, v_{lk}\rangle \in K\omega$ then we have $\langle f_{ij}, I_{lk}\rangle\langle u_{ij}, v_{lk}\rangle = \langle x_{ij}, y_{lk}\rangle \in K$, for $\langle x_{ij}, y_{lk}\rangle \in K, \langle f_{ij}, I_{lk}\rangle \in E(K)$. Now $\langle f_{ij}, I_{lk}\rangle\langle u_{ij}, v_{lk}\rangle = \langle u_{ij}, v_{lk}\rangle\big(\langle f_{ij}, I_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\big)^{*}$ and with $\langle f_{ij}, I_{lk}\rangle\langle u_{ij}, v_{lk}\rangle \in K$, $\big(\langle f_{ij}, I_{lk}\rangle\langle u_{ij}, v_{lk}\rangle\big)^{*} \in K$ then $\langle u_{ij}, v_{lk}\rangle \in K$. Thus, $K\omega = K$.

Now suppose $\langle x_{ij}, y_{lk}\rangle \,\sigma\, \langle u_{ij}, v_{lk}\rangle$ where $\langle x_{ij}, y_{lk}\rangle, \langle u_{ij}, v_{lk}\rangle \in S$. Evidently, $\langle x_{ij}, y_{lk}\rangle \in K\langle x_{ij}, y_{lk}\rangle \subseteq \big(K\langle x_{ij}, y_{lk}\rangle\big)\omega$ and $\langle u_{ij}, v_{lk}\rangle \in \big(K\langle x_{ij}, y_{lk}\rangle\big)\omega$ since $\langle x_{ij}, y_{lk}\rangle^{\dagger}$, $\langle u_{ij}, v_{lk}\rangle^{*} \in K$. But we have that $\langle x_{ij}, y_{lk}\rangle^{*} \,\sigma\, \langle u_{ij}, v_{lk}\rangle^{*}$. If $\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle^{*} \in \big(K\langle x_{ij}, y_{lk}\rangle\big)\omega$ then with $\langle u_{ij}, v_{lk}\rangle^{*} \in K$, we have that $\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle^{*} =$

$\left(\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle^*\right)^\dagger\langle x_{ij}, y_{lk}\rangle \epsilon K\langle x_{ij}, y_{lk}\rangle$ so that $\langle x_{ij}, y_{lk}\rangle\langle u_{ij}, v_{lk}\rangle^* \epsilon \left(K\langle x_{ij}, y_{lk}\rangle\right)\omega$ implying that $\left(K\langle u_{ij}, v_{lk}\rangle\right)\omega \subseteq \left(K\langle x_{ij}, y_{lk}\rangle\right)\omega$.

ACKNOWLEDGEMENTS

The authors would like to thank the anonymous reviewers for their constructive criticisms and suggestions.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

REFERENCES

[1] A.O. Ajibade, The concept of rhotrix in mathematical enrichment, Int. J. Math. Educ. Sci. Technol. 34 (2003), 175–179.

[2] A. Mohammed, Theoretical development and applications of rhotrices, Ph.D Thesis, Ahmadu Bello University, Zaria, (2011).

[3] A. Mohammed, M. Balarabe, A.T. Imam, On construction of rhotrix semigroup, J. Nigerian Assoc. Math. Phys. 27 (2014), 69-76.

[4] A. O. Isere, Natural rhotrix, Cogent Math. 3 (2016), 1246074.

[5] B. Sani, An alternative method for multiplication of rhotrices, Int. J. Math. Educ. Sci. Technol. 35 (2004), 777-781.

[6] B. Sani, The row-column multiplication for higher dimensional rhotrices, Int. J. Math. Educ. Sci. Technol. 38 (2009), 657-662.

[7] J.B. Fountain, Adequate semigroups, Proc. Edinburgh Math. Soc. 22 (1979), 113-125.

[8] J.M. Howie, Fundamentals of semigroup theory, Oxford University Press, Inc, USA, (1995).

[9] K.T. Atanassov, A.G. Shannon, Matrix-tertions and matrix-noitrets: exercise for mathematical enrichment, Int. J. Math. Educ. Sci. Technol. 29 (1998), 898-903.

[10] M. Petrich, Inverse semigroups, Wiley and Sons, New York, (1984).

[11] M.P. Chinedu, Row-wise representation of arbitrary rhotrix, Notes Number Theory, 18 (2012), 1-27.

[12] R.U. Ndubuisi, R.B. Abubakar, N.N. Araka, et al. The concept of rhotrix type A semigroups, J. Math. Comput. Sci. 12 (2022), 1-16.

[13] U.R. Ndubuisi, C.C. Ugochukwu, M.C. Obi, et al. Representation of rhotrix type A semigroups, J. Math. Comput. Sci. 12 (2022), 163.

[14] U. Asibong-Ibe, Representations of type A monoids, Bull. Aust. Math. Soc. 44 (1991), 131-138.