Available online at http://scik.org
J. Semigroup Theory Appl. 2023, 2023:2
https://doi.org/10.28919/jsta/8046
ISSN: 2051-2937

ORDERED MATRIX SEMIGROUPS

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Abstract. A semigroup together with compatible partial order is called an ordered semigroup. In this paper we discuss various partial orders on matrix semigroups with respect to which the matrix semigroups are ordered.

Keywords: ordered semigroup; partial orders; matrix semigroups.

2020 AMS Subject Classification: 06A06.

1. Introduction and Preliminaries

A semigroup $(S,\cdot)$ is a non empty set together with an assosiative binary operation $\cdot$. Obviously every group is a semigroup, however there are semigroups which fails to be a group such as integers under usual multiplicaton, natural numbers under adition and the like. More familiar examples of semigroups are

(1) Full transformation semigroup $\mathscr{T}(X)$ of all transformations on a set X with composition of transformations as binary operation.

(2) All square matrices over a field F [ring R] $M_n(F)$ $[M_n(R)]$ with the usual multiplication of matrices as binary operation.

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Received June 08, 2023

As a mathematical structure semigroups attracted much attention and has grown to an extend that it pocesses a vivd and deep structure theory. Regular semigroups and inverse semigroups include the class of semigroups which are well developed. An element a in a semigroup S is said to be von Neumann regular if there exists an alement $x \in S$ such that $axa = a$. A semigroup S is said to be regular if every element $a \in S$ are regular. A regular semigroup in which the idempotent elements (those elements e in a semigroup S such that $e \cdot e = e$) commutes are called inverse semigroups.

Definition 1. *Let S be a semigroup and $a \in S$.*

(1) A particular solution to $axa = a$ is called the inner inverse of a and is denoted by a^-.
(2) A solution of the equation $xax = x$ is called the outer inverse of a and is denoted by $a^=$
(3) An inner inverse of a that is also an outer inverse is called a reflexive inverse and is denoted by a^+.

The set of all inner (resp. outer, reflexive) inverses of a is denoted by $a\{1\}$ (resp. $a\{12\}$, $a\{123\}$) inverses.

Definition 2. *A semigroup S is said to be weakly separative if, for any $a, b \in S$,*

$$asa = asb = bsa = bsb \Rightarrow a = b.$$

Lemma 1. *Every regular semigroup (in the sense of von Neumann) is weakly separative.*

Proof. Let S be regular. For $a, b \in S$, we have $axa = a$ and $byb = b$ for some $x, y \in S$. Assume that $asa = asb = bsa = bsb$ for all $s \in S$. In particular, $a = axa = ax$ and $ayb = byb = b$ so that $a = axb = ax(byb) = (axb)yb = ayb = b$. $\square$

Example 1.
(1) The set of all transformations on a set X with composition of maps is a regular semigroup called the full transformation semigroup.
(2) All $n \times n$ matrices M_n over a field F or a ring R are regualr semigroups with respect to usual mutliplication of matrices.

However, here it should be noted that the usual matrix product is not the only matrix product with respect to which M_n is a semigroup. In fact there are other interseting matrix products such as

(1) Kronecker product (Tensor product)
(2) Hadamard product (Schur product)

with respect to which M_n is a semigroup.

Consider $A = [a_{ij}], B = [b_{ij}] \in M_n$, then their Kronecker product $A \otimes B$ is defined to be the $n^2 \times n^2$ matrix partitioned into n^2 blocks with the $(i,j)^{th}$ block as the $n \times n$ matrix $a_{ij}B$, i.e.,

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ . & . & \cdots & . \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix}.$$

Clearly $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ for all $A,B,C \in M_n$, thus $(M, \otimes)$ is a semigroup.

The Hadamard product or Schur product of $A,B \in M_n$ is defined by the entrywise product $A \circ B = (a_{ij}b_{ij})$ that is,

$$\mathbf{A} \circ \mathbf{B} = \begin{pmatrix} a_{11}b_{11} & a_{12}b_{12} & \cdots & a_{1n}b_{1n} \\ a_{21}b_{21} & a_{22}b_{22} & \cdots & a_{2n}b_{2n} \\ . & . & \cdots & . \\ a_{n1}b_{n1} & a_{n2}b_{n2} & \cdots & a_{nn}b_{nn} \end{pmatrix}$$

Since $A \circ (B \circ C) = (A \circ B) \circ C$ for all $A,B,C \in M_n(F)$, we have $(M_n, \circ)$ is a semigroup with respect to the multiplication $\circ$.

A Partial order on a semigroup S is a binary relation $\leq$ on S which is reflexive, antisymmetric and transitive.

Definition 3. *An ordered semigroup $(S,\cdot,\leq)$ is a semigroup $(S,\cdot)$ together with a compatible partial order " $\leq$ " on S such that for any $x,y,z \in S$,*

$$x \leq y \Rightarrow xz \leq xz \quad \textit{and} \quad zx \leq zy.$$

Example 2. *The set of all relations on a set X denoted by $B(X)$ is an ordered semigroup with the composition of relations as the binary operation and inclusion as the compatible partial order on $B(X)$.*

Generally, any semigroup$(S,\cdot)$ can be considered as an ordered semigroup with respect to the order $\leq$ as the identity relation on S, that is

$$x \leq y \Longleftrightarrow x = y.$$

If P is any partial order on a semigroup S then the relation

$$P_1 = \{(x,y) \,|\, (axb, ayb) \in P \,\text{for all}\, (a,b) \in S\}$$

is the largest compatible partial order contained in P. Thus, every semigroup can be endowed with a partial order so that S becomes an ordered semigroup.

Lemma 2. *Let $(S,\cdot,\leq)$ be an ordered semigroup. For every $s \in S$ define a binary composition ρ on S by*

$$x\rho y = xsy \quad \textit{for all} \quad x,y \in S$$

Then, (S,ρ) is a semigroup such that $(S,\rho,\leq)$ is an ordered semigroup and is called the s-invariant of S.

Definition 4. *Let $(S,\cdot,\leq)$ be an ordered semigroup. A non empty subset $A \in S$ is a subsemigroup of S if for any $a,b \in A$ implies $ab \in A$.*

2. Ordered Matrix Semigroups

In the following we proceed to describe ordered semigroup $n \times n$ matrices. At the outset it is seen that the semigroup $M_n(\mathbb{R})$ square matrices over real nuumbers with order defined by, for $A,B \in M_n(\mathbb{R})$ with $A = [a_{ij}], B = [b_{ij}]$, $1 \leq i \leq n$, $1 \leq j \leq n$, then

$$A \leq B \Rightarrow a_{ij} \leq b_{ij} \,\text{for all} \quad i \quad \text{and}\, j$$

fails to be an ordered semigroup. However for $M_n(\mathbb{Z}_+)$ of all $n \times n$ matrices with non negative integer entries we have the following lemma.

Lemma 3. *Consider the semigroup $M_n((\mathbb{Z}_+))$ of all $n \times n$ matrices with non negative integer entries. For $A, B \in M_n(\mathbb{Z}_+)$ with $A = [a_{ij}], B = [b_{ij}]$, $1 \leq i \leq n$, $1 \leq j \leq n$, define the relation $\leq$ by*

$$A \leq B \Rightarrow a_{ij} \leq b_{ij} \textit{for all} \quad i \quad \textit{and } j$$

is a partial order with respect to which $M_n(\mathbb{Z}_+)$ is an ordered semigroup.

Proof. Clearly the order $\leq$ is reflexive as, $a_{ij} \leq a_{ij}$ for all i and j. Suppose $A \leq B$ and $B \leq A$, ie., $a_{ij} \leq b_{ij}$ and $b_{ij} \leq a_{ij}$ for all i and j.This gives $a_{ij} = b_{ij}$ and hence anti symmetric. If $A \leq B$ and$B \leq C$, then $a_{ij} \leq b_{ij}$ and $b_{ij} \leq c_{ij}$. This implies $a_{ij} \leq c_{ij}$,ie, $A \leq C$, hence transitive.Thus $\leq$ is a partial order.

To prove the compatibility, suppose $A \leq B$. ie,$a_{ij} \leq b_{ij}$. For $C = [c_{ij}] \in M$,

$$(CA)_{ij} = \sum c_{ik} a_{kj} \leq \sum c_{ik} b_{kj} = (CB)_{ij} \, \text{for} \, 1 \leq k \leq n$$

thus $CA \leq CB$. Similarly it is seen that $AC \leq BC$, ie., $\leq$ is compatible. □

2.1. Conrad order on Regular matrix semigroups. In (cf. [2]) Abian described a partial order on a semiprime ring R as follows

$$a \leq b \Longleftrightarrow ab = a^2 \quad \text{for all} \quad a, b \in R$$

Cornad modified this partial order on the semiprime ring R itself as

$$a \leq b \Longleftrightarrow arb = ara \quad \text{for all} \quad r \in R.$$

Further he extended this partial order to define a relation ρ on a semigroup S by

$$a\rho b \Leftrightarrow asa = asb = bsa \quad \text{for all} \quad s \in S$$

which turned out to be a partial order for 'weakly separative' semigroups.

Ler R a ring. Consider matrix semigroup $M_n(R)$. Now we proceed to describe the relation ρ on $M_n(R)$ as follows, for $A, B \in M_n(R)$.,

$$A\rho B \Leftrightarrow ASA = ASB = BSA \quad \forall \, S \in \; M_n(R)$$

and this relation ρ is called the Conrad relation on $M_n(R)$

Lemma 4. *The Conrad relation on $M_n(R)$ is a partial order with respect to which $M_n(R)$ is an ordered semigroup.*

Proof. It is easy to see that the relation ρ on $M_n(R)$ is reflexive. For transitivity, suppose $A\rho B$ and $B\rho C$, then $ASA = ASB = BSA$ and $BSB = BSC = CSB$ for all $S \in M_n(F)$. For $P \in M_n(R)$, we obtain $(ASC)P(ASC) = (ASC)P(ASB) = (ASC)P(BSA) = (ASC)P(CSA) \Rightarrow ASC = CSA$. Similarly, $(ASC)P(ASC) = (ASC)P(ASB) = (ASC)P(BSA) = (ASC)P(CSA) \Rightarrow ASC = CSA$. This gives $ASC = CSA = ASA$, ie, $A\rho C$.

Burgess and Raphael proved that ρ is a partial order on a semigroup S if and only if it is weakly separative. Since $M_n(R)$ is a regular semigroup, obviously $M_n(R)$ is weakly seperative and hence ρ is a partial order. Suppose $A\rho B$ and $C\rho D$. Then, $ASB = BSA = BSB$ and $CSC = CSD = DSC \quad \forall S \in M_n(F)$. Then,$(AC)S(BD) = (AC)S(AC) = (BD)S(AC) \Rightarrow AC\rho BD$, hence the compatibility and $(M_n(R), \rho)$ is an ordered matrix semigroup. □

2.2. More partial orders on $M_n(R)$.

Let R be any ring. Jerzy K Baksalary and Sujit Kumar Mitra introduced the following two orderings on $M_n(R)$

For $A, B \in M_n(R)$, two orders defined by

$$A\star \leq B \Leftrightarrow A^\star A = A^\star B \text{ and } R(A) \subseteq R(B)$$

where A^* stands for the conjugate transpose of A and $R(A)$ is the range space of A. In a similar way another order is given by

$$A \leq \star B \Leftrightarrow AA^\star = BA^\star \,\text{and}\, R(A^*) \subseteq R(B^\star).$$

These orders are called the left star order and the right star order respectively.

Lemma 5. *The left star order and the right star order are partial orders on $M_n(R)$*

Proof. Clearly $A\star \leq A$. Suppose $A\star \leq B$ and $B\star \leq A$. Then $A^\star A = A^\star B$, $R(A) \subseteq R(B)$, $B^\star B = B^\star A$ and $R(B) \subseteq R(A)$. This gives $R(A) = R(B)$. Let $A^\dagger$ be the Moore-Penrose inverse of A (ie., the unique matrix satisfying $AA^\dagger A = A$, and $A^\dagger AA^\dagger = A^\dagger$). Then $AA^\dagger = (AA^\dagger)^\star$ and $A^\dagger A = (A^\dagger A)^\star$ and

$$A = AA^\dagger A = (AA^\dagger)^\star A = (A^\dagger)^\star (A^\star A = (A^\dagger)^\star A^\star B = (AA^\dagger)^\star B = AA^\dagger B = B$$

hence the relation $'\star \leq'$ is antisymmetric.

For transitivity, let $A\star \leq B$ and $B\star \leq C$. Then, $A^\star A = A^\star B$, $R(A) \subseteq R(B), B^\star B = B^\star C$ and $R(B) \subseteq R(C)$, from this we get $R(A) \subseteq R(C)$. Now, $A^\star A = A^\star B = A^\star(BB^\dagger B) = A^\star(BB^\dagger)^\star B = (A^\star(B^\dagger)^\star B^\star)B = A^\star(B^\dagger)^\star(B^\star B) = A^\star(B^\dagger)^\star(B^\star C) = (BB^\dagger A)^\star C = A^\star C$. Thus $A^\star A = A^\star C$ with $R(A) \subseteq R(C)$ implying that $A\star \leq C*$.

Similarly it can be seen that the right star order "$\leq \star$" is a partial order on $M_n(R)$. $\square$

Lemma 6. *$M_n(R)$ is an ordered semigroup under the left star ordering.*

Proof. It is already seen that the left star order is a partial order on $M_n(R)$. Consider $A,B,C,D \in M_n(F)$ such that $A\star \leq B$ and $C\star \leq D$, then $A^\star A = A^\star B$, $R(A) \subseteq R(B)$, $C^\star C = C^\star D$ and $R(C) \subseteq R(D)$.
Further

$$\begin{aligned}(AC)^\star AC = C^\star(A^\star A)C = C^\star(A^\star B)C &= (AC)^\star B(CC^\dagger C) = (AC)^\star B(CC^\dagger)^\star C \\ &= (AC)^\star B(C^\dagger)^\star({}^\star C) \\ &= (AC)^\star B(C^\dagger)^\star(C^\star D) \\ &= (AC)^\star B(CC^\dagger)^\star = (AC)^\star B(CC^\dagger D) \\ &= (AC)^\star BD.\end{aligned}$$

and since $R(C) \subseteq R(D)$, we have $R(AC) \subseteq R(BD)$. Thus $AC\star \leq BD$ and hence $(M_n(R), \cdot, \star \leq)$ is an ordered matrix semigroup.
Similarly we can prove that $(M_n(R), \cdot, \leq \star)$ is an ordered semigroup. $\square$

2.3. Positive semidefinite matrices (PSD). From here onwards we restrict to the matrices $M_n(\mathbb{C})$. A matrix $A \in M_n(\mathbb{C})$ is said to be positive semi definite (positive definite) if $v^\star Av \geq 0\,(v^\star Av > 0)$ for all $v \in \mathbb{C}_n$. We write $A \geq 0\,(A > 0)$ to mean that A is positive semi definite (positive definite).

In the following we list some properties of PSD matrice

(1) A is PSD if and only if it is Hermitian, ie., $A = A^\star$
(2) If A is a PSD, then all its principal submatrices and all principal minors of A are PSD.

(3) A is PSD if and only if $A = M^{\star}M$ for some matrix M.

(4) A is PSD if and only if $A = P^{\star}P$ for some upper triangular matrix P with non negative diagonal entries only (Cholesky decomposition of A)

(5) A and B be congruent matrices then A is PSD $\Leftrightarrow$ B is PSD.

(6) A is PSD if and only if $A = B^2$ for some PSD matrix B. Then the unique $B = A^{1/2}$ is called the positive square root of A.

(7) Schur product theorem: Let A and B be PSD matrices of size n. Then the Schur product $A \circ B$ is also PSD.(but the conventional product neednot be PSD)

(8) Let A be Hermitian and PSD. Then there exists a sequence of PSD matrices $A_1, A_2 \dots$ such that $A_k \to A$ as $k \to \infty$. We can define $A_k = A + k^{-1}I$

It is easy to observe that the set of all $n \times n$ Hermitian matrices denoted by $M_n^H \subset M_n(\mathbb{C})$. For $A, B \in M_n^H$, $A \circ B$ is always Hermitian, ie., M_n^H is a subsemigroup of $M_n(\mathbb{C})$.

2.4. Loewner partial order on M_n^H. Every partial order $\preceq$ on a real linear space S can be defined by $A \preceq B$ for all $A, B \in S$ if their difference lies in a special closed convex cone. For the Loewner order the elements of the real linear space are the $n \times n$ Hermitian matrices and elements of the closed convex cone are the positive semi-definite matrices (PSD).

Also, the collection of all PSD matrices of order n is a semigroup under '$\circ$' and is denoted by $M_n^{\geq}$. Then

$$M_n^{\geq} = \{K \in M_n \mid K = LL^* \text{ for some} L \in M_n\}.$$

For $A, B \in M_n^{\geq}$, $A \circ B \in M_n^{\geq}$ and $A \circ (B \circ C) = (A \circ B) \circ C$ for all $A, B, C \in M_n^{\geq}$ and hence $M_n^{\geq}$ is a semigroup. Now we define the Loewner order $\preceq$ on M_n^H by $A \preceq B$ if and only if f $B - A$ is Hermitian and positive semidefinite.

Lemma 7. *The Loewner partial order $\preceq$ is a compatible partial order on M_n^H.*

Proof. Clearly $0 \preceq A$ means that A is PSD. For $A \in M_n^H$, $A - A = \mathbf{0} \in M_n^{\geq}$ implying that $A \preceq A$. Suppose $A \preceq B$ and $B \preceq A$. Since $A \preceq B \Rightarrow B - A \in M_n^{\geq}$, that is., $B - A = KK^*$ for some $K \in M_n$. Similarly, $B \preceq A \Rightarrow A - B = LL^{\star}$ for some $L \in M_n$. Since, $A - B = -(B - A)$ we have $LL^{\star} + KK^{\star} = 0$ which implies $LL^{\star} = KK^{\star} = 0$, ie., $A - B = 0 = B - A \Rightarrow A = B$ proving the antisymmetry.

For transitivity of $\preceq$, consider $A,B \in M_n^{\geq}$ such that $A \preceq B$ and $B \preceq C$, then, $B - A = KK^\star$ and $C - B = PP^\star$ for some $L,P \in M_n$. Now, $C - A = (C-B)+(B-A) \in M_n^{\geq}$, being the sum of two PSD matrices. Thus $\preceq$ is a partial order.

Let $A \preceq B$ and $C \preceq D$. Then $B - A = KK^\star$ and $D - C = PP^\star$ for some $L,P \in M_n$. Now, $B \circ D - A \circ C = B \circ (D-C) + (B-A) \circ C = B \circ (LL^\star) + (KK^\star) \circ C \in M_n \geq$ as the Hadamard product and sum of two PSD matrices are PSD. This implies that $A \circ C, \preceq B \circ D$, that is., $\preceq$ is compatible under $\circ$. Thus $(M_n^H, \circ, \preceq)$ is an ordered matrix semigroup. $\square$

Ordered semigroups have many applications in the theory of computer arithmetics, formal languages, error-correcting codes and the like. This class of semigroups is currently a hot topic of research and have been studied by several authors.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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