Available online at http://scik.org
J. Semigroup Theory Appl. 2013, 2013:3
ISSN 2051-2937

PRIMITIVE LEFT AMPLE SEMIGROUPS†

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Abstract. A left adequate semigroup S is called left ample if for any $a, e^2 = e \in S$, $ae = (ae)^\dagger a$. Inverse semigroups are left ample semigroups. In this paper we study primitive left ample semigroups. The structure theorem of primitive left ample semigroups with certain conditions is established.

Keywords: left ample semigroup; PLA blocked Rees matrix semigroup; inverse semigroup.

2000 AMS Subject Classification: 20M10

1. Introduction and Preliminaries

Left ample semigroups form a special class of left abundant semigroups. Interest in the latter arose originally from the study of monoids by their associated S-sets. A left abundant semigroup is a semigroup with the property that all principal left ideals are projective. All regular semigroups are left abundant semigroups and so are many other types of semigroups including right cancellative monoids. Inverse semigroups are left ample semigroups. It is interest that any left ample semigroup can be embedded into an inverse semigroup (see [1]). There are many authors having been investigating left

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†The research is supported by the NNSF of China (grant: 10961014); the NSF of Jiangxi Province and the SF of Education Department of Jiangxi Province

Received January 16, 2013

ample semigroups. The present paper continues the probe for this kind of left abundant semigroups and is inspired by Fountain [3].

Throughout this paper, we will use the terminologies and notations of [2, 4]. We recall some known results which are used in the sequel.

Let S be a semigroup and $a, b \in S$. We call elements a and b to be *related by* $\mathcal{R}^*$ if and only if a and b are related by $\mathcal{R}$ in some oversemigroup of S. Dually, we can define the relation $\mathcal{L}^*$. The following lemma gives an alternative characterization of $\mathcal{R}^*$; the dual for the relation $\mathcal{L}^*$.

Lemma 1.1 [3] *Let S be a semigroup and $a, b \in S$. Then $a\mathcal{R}^*b$ if and only if for all $x, y \in S^1$, $xa = ya \Leftrightarrow xb = yb$.*

As an easy but useful consequence of Lemma 1.1, we have

Lemma 1.2 [3] *Let S be a semigroup and $a, e^2 = e \in S$. Then $a\mathcal{R}^*e$ if and only if $ea = a$ and for any $x, y \in S^1$, $xa = ya$ implies that $xe = ye$.*

It is well known that $\mathcal{R}^*$ is a left congruence while $\mathcal{L}^*$ is a right congruence. In general, we always have $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. It is worth to pointing out that when a and b are regular elements, $a\mathcal{R}b$ $[a\mathcal{L}b]$ if and only if $a\mathcal{R}^*b$ $[a\mathcal{L}^*b]$. In particular, for a regular semigroup, we always have $\mathcal{R} = \mathcal{R}^*$ and $\mathcal{L} = \mathcal{L}^*$.

A semigroup S is called *left abundant* if each $\mathcal{R}^*$-class of S contains at least one idempotent. Dually, *right abundant semigroup* can be defined. The semigroup S is called *abundant* if S is both left abundant and right abundant. As in [2], a (left; right) abundant semigroup is called (*left*; *right*) *adequate* if all idempotents commute. It is easy to check that if S is a left (right) adequate semigroup then each $\mathcal{R}^*$-class ($\mathcal{L}^*$-class) of S contains exactly one idempotent. We shall use $a^\dagger$ to denote the idempotents in the $\mathcal{R}^*$-class containing a while a^* to denote those in the $\mathcal{L}^*$-class containing a. By a *left* (*right*) *ample semigroup*, we mean a left (right) adequate semigroup in which $ae = (ae)^\dagger a$ $(ea = a(ea)^*)$ for any element a and any idempotent e. If a semigroup is both left ample and right ample, then we call it *ample*. (Left; right) ample semigroups are formerly known as (left; right) type-A semigroups.

If $e, f \in E(S)$, we write $e \leq f$ if and only if $e = ef = fe$. Indeed, $\leq$ is an order on the set $E(S)$ of S. A nonzero idempotent g of S is called *primitive* if for any $f \in E(S)$, $f \leq g$ implies that $f = g$ or $f = 0$ if S has zero. We call S *primitive* if all idempotents of S are primitive. In what follows, we call a primitive left ample semigroup a *PLA-semigroup*.

Lemma 1.3 *Let S be a PLA-semigroup with zero 0. If $a, b \in S, e, f \in E(S)$, then*

(1) *$a\mathcal{R}^*ab$ or $ab = 0$.*

(2) *$ae = 0$ or $ae = a$. Moreover, if $aS \neq \{0\}$, then there exists unique $a^\diamond \in E(S)$ such that $aa^\diamond = a$.*

(3) *If $e \neq f$ then $ef = 0$.*

Proof. (1) Assume $ab \neq 0$. Since $a^\dagger \cdot ab = ab$, we have $a^\dagger(ab)^\dagger = (ab)^\dagger$, hence $(ab)^\dagger a^\dagger \in E(S)$, $(ab)^\dagger a^\dagger \leq a^\dagger$ and $(ab)^\dagger \mathcal{R} (ab)^\dagger a^\dagger$. But $a^\dagger$ is primitive, so $(ab)^\dagger a^\dagger = a^\dagger$ or $(ab)^\dagger a^\dagger = 0$. If $(ab)^\dagger a^\dagger = 0$, then $(ab)^\dagger = (ab)^\dagger a^\dagger \cdot (ab)^\dagger = 0$, so that $ab = (ab)^\dagger ab = 0$ since $ab\mathcal{R}^*(ab)^\dagger$. This is contrary to $ab \neq 0$. Thus $(ab)^\dagger a^\dagger = a^\dagger$, and whence $a\mathcal{R}^* a^\dagger \mathcal{R} (ab)^\dagger \mathcal{R}^* ab$.

(2) Suppose that $ae \neq 0$. Then by (1), $(ae)^\dagger \mathcal{R}^* ae \mathcal{R}^* a \mathcal{R}^* a^\dagger$, so that $(ae)^\dagger = a^\dagger$ since S is a left ample semigroup, so we have $ae = (ae)^\dagger a = a^\dagger a = a$.

Now assume still $aS \neq \{0\}$. Then there exists $x \in S$ such that $ax \neq 0$. But $ax^\dagger \cdot x = ax \neq 0$, so $ax^\dagger \neq 0$, further by the foregoing proof, $ax^\dagger = a$. If $g \in E(S)$ such that $ag = a$, then $a = ag = ax^\dagger \cdot g$, hence $x^\dagger g \neq 0$, so that $x^\dagger g \leq x^\dagger, g$. It follows that $x^\dagger = x^\dagger g = g$ since $x^\dagger, g$ are primitive idempotents. Consequently, there exists unique $a^\diamond \in E(S)$ such that $aa^\diamond = a$.

(3) Since $ef \leq e$, we have $ef = 0$ or $ef = e$. If the second equality holds, then $e \leq f$, hence $e = 0$ or $e = f$, thus $e = 0$. However $ef = 0$. □

2. PLA blocked Rees matrix semigroups

The aim of this section is to construct a class of primitive left ample semigroups.

Let I be non-empty set and J a non-empty set. Now, we index the partition $P(I) = \{I_\gamma : \gamma \in J\}$ of I. For each pair $(\alpha, \beta) \in J \times J$, if $M_{\alpha\beta}$ is a set such that for each α, $M_{\alpha\alpha} = T_\alpha$ is a monoid with identity 1_α; and for $\alpha \neq \beta$, then either $M_{\alpha\beta} = \emptyset$ or $M_{\alpha\beta}$

is a (T_α, T_β)-bisystem (such a nonempty set having an action of T_α on the left and one of T_β on the right). We now let 0 be a symbol not in any $M_{\alpha\beta}$. By the (α, β)-*block* of an $I \times I$-matrix, we mean those (i, j) positions with $i \in I_\alpha, j \in I_\beta$. The (α, α)-blocks are called the *diagonal blocks of matrix*. We denote by S the set consisting of the zero $I \times I$ matrix together with all $I \times I$ matrices with a single non-zero entry where in the (α, β)-block is a member of $M_{\alpha\beta}$.

Following the usual convention, we use $(a)_{ij}$ to denote the $I \times I$ matrix with entry a in the (i, j) position and 0 elsewhere, and 0 the $I \times I$ matrix all of whose entries are 0. Suppose that the following conditions are satisfied:

(M): For all $\alpha, \beta, \gamma \in J$, if $M_{\alpha\beta}, M_{\beta\gamma}$ are both non-empty, then $M_{\alpha\gamma}$ is non-empty and there is a (T_α, T_β)-homomorphism $\phi_{\alpha\beta\gamma} : M_{\alpha\beta} \otimes M_{\beta\gamma} \to M_{\alpha\gamma}$ such that if $\alpha = \beta$ or $\beta = \gamma$, then $\phi_{\alpha\beta\gamma}$ is a canonical isomorphism such that the square

$$\begin{array}{ccc} M_{\alpha\beta} \otimes M_{\beta\gamma} \otimes M_{\gamma\delta} & \xrightarrow{id_{\alpha\beta} \otimes \phi_{\beta\gamma\delta}} & M_{\alpha\beta} \otimes M_{\beta\delta} \\ {\scriptstyle \phi_{\alpha\beta} \otimes id_{\beta\delta}} \downarrow & & \downarrow {\scriptstyle \phi_{\alpha\beta\delta}} \\ M_{\alpha\gamma} \otimes M_{\gamma\delta} & \xrightarrow[\phi_{\alpha\gamma\delta}]{} & M_{\alpha\delta} \end{array}$$

is commutative, where $id_{\alpha\beta}$ is the identity mapping on $M_{\alpha\beta}$.

(C): (In what follows, we simply denote $(a \otimes b)\phi_{\alpha\beta\gamma}$ by ab, for $a \in M_{\alpha\beta}, b \in M_{\beta\gamma}$). If $a_1, a_2 \in M_{\alpha\beta}$, $b \in M_{\beta\gamma}$, then $a_1 b = a_2 b$ implies $a_1 = a_2$.

(R): If $M_{\alpha\beta}, M_{\beta\alpha}$ are both non-empty where $\alpha \neq \beta$, then $aba \neq a$ for all $a \in M_{\alpha\beta}, b \in M_{\beta\alpha}$.

Now, let $A, B \in S$. If $A = 0$ or $B = 0$, then $AB = 0$. Assume that $A = (a)_{ij}$ and $B = (b)_{kl}$ are neither 0. If $j \neq k$, then $AB = 0$. Assume that $j = k$. Then $AB = (ab)_{il}$. Thus, we have a product $\circ$ defined on S by $A \circ B = AB$. We can easily check that $(S, \circ)$ is a semigroup, call the *LA blocked matrix semigroup*. For the sake of convenience, we denote this semigroup by $\mathcal{LABM}(M_{\alpha\beta}; I, J)$.

Proposition 2.1 *Let $(a)_{ij} \in \mathcal{LABM}(M_{\alpha\beta}; I, J)\backslash\{0\}$ with $a \in M_{\alpha\beta}$.*

(1) *$(a)_{ij}$ is an idempotent if and only if $i = j$ and $a = 1_\alpha$.*

(2) *$(a)_{ij}\mathcal{R}^*(1_\alpha)_{ii}$.*

(3) $(a)_{ij}$ *is regular if and only if* $\alpha = \beta$ *and* a *is a unit of* T_α.

(4) $(a)_{ij}\mathcal{LABM}(M_{\alpha\beta}; I, J) \neq \{0\}$.

Proof. (1) If $(a)_{ij}$ is an idempotent, then $0 \neq (a)_{ij} = (a)_{ij}^2$, hence $i = j$ and $a^2 = a$. The second equality implies that $\alpha = \beta$ and $a = 1_\alpha$ by Condition (C). But the converse is a routine computation.

(2) Obviously, $(a)_{ii} = (1_\alpha)_{ii}(a)_{ij}$. Let $(x)_{kl}, (y)_{pq} \in \mathcal{LABM}(M_{\alpha\beta}; I, J)$ and assume $(x)_{kl}(a)_{ij} = (y)_{pq}(a)_{ij}$. We consider the following two cases:

- If $(x)_{kl}(a)_{ij} = 0$, then $(x)_{kl} = 0$ or $l \neq i$. It follows that $(x)_{kl}(1_\alpha)_{ii} = 0$. Similarly, $(y)_{pq}(1_\alpha)_{ii} = 0$.
- If $(x)_{kl}(a)_{ij} = (y)_{pq}(a)_{ij} \neq 0$, then $k = p$, $l = i = q$ and $xa = ya$. The second equality can imply that $x = y$ by Condition (C). Thus $(x)_{kl}(1_\alpha)_{ii} = (x)_{ki} = (y)_{pi} = (y)_{pq}(1_\alpha)_{ii}$.

We have proved that if $(x)_{kl}(a)_{ij} = (y)_{pq}(a)_{ij}$ then $(x)_{kl}(1_\alpha)_{ii} = (y)_{pq}(1_\alpha)_{ii}$. Consider that $(x)_{kl}(a)_{ij} = (a)_{ij} = (1_\alpha)_{ii}(a)_{ij}$. In fact, we have proved that for all $(x)_{kl}, (y)_{pq} \in \mathcal{LABM}(M_{\alpha\beta}; I, \Gamma)^1$, $(x)_{kl}(a)_{ij} = (y)_{pq}(a)_{ij}$ implies that $(x)_{kl}(1_\alpha)_{ii} = (y)_{pq}(1_\alpha)_{ii}$. Thus $(a)_{ij}\mathcal{R}^*(1_\alpha)_{ii}$.

(3) Assume $(a)_{ij}$ is regular. Then there exists $(b)_{ji}$ such that $(a)_{ij}(b)_{ji}(a)_{ij} = (a)_{ij}$. It follows that $a = aba$. By Condition (U), this shows that $\alpha = \beta$ and $a = aba$. By Condition (C), the second equality can derive that $ab = 1_\alpha$, so that a is a unit of T_α. The reverse is trivial.

(4) Let $(a)_{ij} \in \mathcal{LABM}(M_{\alpha\beta}; I, J)$ with $a \in M_{\alpha\beta}$. Clearly, $(a)_{ij}(1_\beta)_{jj} = (a)_{ij}$, so that $(a)_{ij}\mathcal{LABM}(M_{\alpha\beta}; I, J) \neq \{0\}$. □

Theorem 2.2 $\mathcal{LABM}(M_{\alpha\beta}; I, J)$ *is a PLA-semigroup.*

Proof. By Proposition 2.1, $\mathcal{LABM}(M_{\alpha\beta}; I, J)$ is a left abundant semigroup and

$$E(\mathcal{LABM}(M_{\alpha\beta}; I, J)) = \{(1_\alpha)_{ii} : \alpha \in J\} \cup \{0\}.$$

Now, by the definition of $\mathcal{LABM}(M_{\alpha\beta}; I, J)$, for any $u, v \in E(\mathcal{LABM}(M_{\alpha\beta}; I, J))$, we easily see that $uv = 0$. It follows that any idempotent of $\mathcal{LABM}(M_{\alpha\beta}; I, J)$ is primitive and that $E(\mathcal{LABM}(M_{\alpha\beta};I,J))$ is a semilattice under the multiplication of $\mathcal{LABM}(M_{\alpha\beta};I,J)$. Thus $\mathcal{LABM}(M_{\alpha\beta};I,J)$ is a primitive left adequate semigroup.

Let $(a)_{kl}\in\mathcal{LABM}(M_{\alpha\beta};I,J)$ with $a\in M_{\beta\gamma}$. Since

$$(a)_{kl}(1_\alpha)_{ii}=\begin{cases}0=0(a)_{kl} & \text{if } k\neq i;\\ (a1_\alpha)_{ki}=(a)_{ki}=(1_\beta)_{kk}(a)_{kl} & \text{if } l=i,\end{cases}$$

we have that $\mathcal{LABM}(M_{\alpha\beta};I,J)$ is left ample. We complete the proof. □

3. The structure theorem

In this section we establish the structure theorem for primitive left ample semigroups.

Theorem 3.1 *Let S be a PLA-semigroup. If $aS\neq\{0\}$ for all $a\in S\backslash\{0\}$, then S is isomorphic to some PL blocked Rees matrix semigroup.*

Proof. Assume S is a PLA-semigroup satisfying the given conditions, and P the set of nonzero idempotents of S. Denote

$$Q=\{D_x : x\in Reg(S)\backslash\{0\}\},$$

where D_x is the $\mathcal{D}$-class of S containing x. For any $\alpha\in Q$, we use P_α to stand for the set of idempotents of the set α and further we fix an element 1_α of P_α. Clearly, $P=\cup_{\alpha\in Q}P_\alpha$. Put $N_{\alpha\beta}=(1_\alpha S1_\beta)\backslash\{0\}$.

Lemma 3.2 (1) *$N_{\alpha\alpha}$ is a right cancellative monoid.*

(2) *$N_{\alpha\beta}$ is a $(N_{\alpha\alpha},N_{\beta\beta})$-bisystem.*

Proof. (1) Obviously, $N_{\alpha\alpha}$ is a subsemigroup of S and 1_α is the identity of $N_{\alpha\alpha}$. Let $a,x,y\in N_{\alpha\alpha}$. By Lemma 1.3 (1), $1_\alpha\mathcal{R}^*a$. If $xa=ya$, then $x1_\alpha=y1_\alpha$, so that $x=y$. It follows that $N_{\alpha\alpha}$ is a right cancellative monoid.

(2) By the definition of $N_{\alpha\beta}$, it is clear that $N_{\alpha\beta}$ is $(N_{\alpha\alpha},N_{\beta\beta})$-bisystem. □

Now let $\alpha,\beta,\gamma\in Q$. Define a mapping $\phi_{\alpha\beta\gamma}$ as follows:

$$\phi_{\alpha\beta\gamma}: N_{\alpha\beta}\otimes N_{\beta\gamma}\to N_{\alpha\gamma};\ a\otimes b\mapsto ab.$$

Evidently, $\phi_{\alpha\beta\gamma}$ is a $(N_{\alpha\alpha}, N_{\beta\beta})$-homomorphism. It is not difficult to see that $\phi_{\alpha\beta\gamma}$ satisfies Condition (M).

Assume $a_1, a_2 \in N_{\alpha\beta}, b \in N_{\beta\gamma}$ and $a_1 b = a_2 b$. Since $1_\beta b = b \neq 0$, we have $b\mathcal{R}^* 1_\beta$ by Lemma 1.3 (1). From the equality $a_1 b = a_2 b$, this can show that $a_1 = a_1 1_\beta = a_2 1_\beta = a_2$, so that Condition (C) holds.

Let $c \in N_{\alpha\beta}, d \in N_{\beta\alpha}$ and $\alpha \neq \beta$. Note that $1_\alpha c = c \neq 0$ and $1_\beta d = d$. We observe that $c\mathcal{R}^* 1_\alpha, d\mathcal{R}^* 1_\beta$. We claim: $cdc \neq c$. If not, then c is regular, $cd \neq 0$ and $dc \neq 0$. By Lemma 1.3, the later two equalities imply that $cd\mathcal{R}^* c, d\mathcal{R}^* dc$. Hence $cd\mathcal{R}^* 1_\alpha,\ dc\mathcal{R}^* 1_\beta$. But $cd, dc \in E(S)$, so $cd = 1_\alpha$ and $dc = 1_\beta$ since S is a left ample semigroup. It follows that $1_\alpha \mathcal{D} 1_\beta$, so that $\alpha = \beta$, contrary to $\alpha \neq \beta$. Therefore Condition (R) holds.

For any $i \in I_\alpha$, we fix a_i such that $1_\alpha \mathcal{R} a_i \mathcal{L} i$. It follows that $1_\alpha = a_i a_i^{-1}$ and $i = a_i^{-1} a_i$, where a_i^{-1} is the inverse of a_i. Let s be an arbitrary element s of S. We have the following lemma.

Lemma 3.3 *There exists unique $s^\circ \in N_{\pi\sigma}$ such that $s = a_{s^\dagger}^{-1} s^\circ a_{s^\diamond}$, where $s^\diamond$ has the same meaning as in Lemma 1.3, and $\pi, \sigma \in Q$ such that $s^\dagger \in I_\pi, s^\diamond \in I_\sigma$.*

Proof. Because $s^\dagger = a_{s^\dagger}^{-1} a_{s^\dagger},\ 1_\pi = a_{s^\dagger} a_{s^\dagger}^{-1},\ s^\diamond = a_{s^\diamond}^{-1} a_{s^\diamond}$ and $1_\sigma = a_{s^\diamond} a_{s^\diamond}^{-1}$, we have

$$s = s^\dagger s s^\diamond = a_{s^\dagger}^{-1} \cdot a_{s^\dagger} s a_{s^\diamond}^{-1} \cdot a_{s^\diamond}.$$

If $s_1 \in N_{\pi\sigma}$ such that $s = a_{s^\dagger}^{-1} \cdot s_1 \cdot a_{s^\diamond}$, then $a_{s^\dagger}^{-1} \cdot s_1 \cdot a_{s^\diamond} = a_{s^\dagger}^{-1} \cdot a_{s^\dagger} s a_{s^\diamond}^{-1} \cdot a_{s^\diamond}$, so that

$$\begin{aligned} s_1 &= 1_\pi s_1 1_\sigma = a_{s^\dagger} \cdot a_{s^\dagger}^{-1} s_1 a_{s^\diamond} \cdot a_{s^\diamond}^{-1} = a_{s^\dagger} a_{s^\dagger}^{-1} \cdot a_{s^\dagger} s a_{s^\diamond}^{-1} \cdot a_{s^\diamond} a_{s^\diamond}^{-1} = 1_\pi \cdot a_{s^\dagger} s a_{s^\diamond}^{-1} \cdot 1_\sigma \\ &= a_{s^\dagger} s a_{s^\diamond}^{-1}. \end{aligned}$$

This proves the lemma. □

Form the PL blocked Rees matrix semigroup $\mathcal{LABM}(N_{\alpha\beta}; P, Q)$ and define

$$\theta:\ S \to \mathcal{LABM}(N_{\alpha\beta}; P, Q);\ s \mapsto (s^\circ, s^\dagger, s^\diamond),\ 0 \mapsto 0.$$

By Lemmas 1.3 (2) and 3.3, θ is well defined. Now let $s, t \in S$. If $(s^\circ, s^\dagger, s^\diamond) = (t^\circ, t^\dagger, t^\diamond)$, then $s^\circ = t^\circ, s^\dagger = t^\dagger$ and $s^\diamond = t^\diamond$. By Lemma 3.3, $s = a_{s^\dagger}^{-1} s^\circ a_{s^\diamond} = a_{t^\dagger}^{-1} t^\circ a_{t^\diamond} = t$, so that θ is injective.

Let $(x, i, j) \in \mathcal{LABM}(N_{\alpha\beta}; P, Q)$ with $i \in I_\alpha, j \in I_\beta$ and $x \in N_{\alpha\beta}$. Then $x = 1_\alpha x 1_\beta$, so that $x = a_i \cdot a_i^{-1} x a_j \cdot a_j^{-1}$ since $1_\alpha = a_i a_i^{-1}$ and $1_\beta = a_j a_j^{-1}$. It follows that $a_i^{-1} x a_j \neq 0$ since $x \neq 0$. Denote $b = a_i^{-1} x a_j$. We have $ib = i a_i^{-1} \cdot x a_j = b$, hence by Lemma 1.3 (1), $i \mathcal{R}^* b$, in other words, $b^\dagger = i$. Also, $b = bj$, and further by Lemma 1.3 (2), $b^\diamond = j$. Now by Lemma 3.3, $b^\circ = x$. Therefore $b\theta = (x, i, j)$. This shows that θ is surjective.

Compute

$$(s\theta)(t\theta) = (s^\circ, s^\dagger, s^\diamond)(t^\circ, t^\dagger, t^\diamond) = \begin{cases} 0 & \text{if } s^\diamond \neq t^\dagger \\ (s^\circ t^\circ, s^\dagger, t^\diamond) & \text{else.} \end{cases}$$

On the other hand, we consider the following two cases:

- If $st = 0$, then $st^\dagger = 0$. We claim: $s^\diamond t^\dagger = 0$; if not, we have $s^\diamond = s^\diamond t^\dagger = t^\dagger$ since $s^\diamond t^\dagger \leq s^\diamond, t^\dagger$, it follows that $s = ss^\diamond = st^\dagger \neq 0$, contrary to $st^\dagger = 0$. Thus by Lemma 1.3 (3), $s^\diamond \neq t^\dagger$, so that $(st)\theta = 0 = (s\theta)(t\theta)$.
- If $st \neq 0$, then

$$a_{(st)^\dagger}^{-1} (st)^\circ a_{(st)^\diamond} = st = a_{s^\dagger}^{-1} s^\circ a_{s^\diamond} a_{t^\dagger}^{-1} t^\circ a_{t^\diamond} = a_{s^\dagger}^{-1} s^\circ a_{s^\diamond} \cdot s^\diamond t^\dagger \cdot a_{t^\dagger}^{-1} t^\circ a_{t^\diamond}, \tag{1}$$

so $s^\diamond = t^\dagger$ by Lemma 1.3 (3), hence

$$a_{(st)^\dagger}^{-1} (st)^\circ a_{(st)^\diamond} = st = a_{s^\dagger}^{-1} s^\circ a_{s^\diamond} a_{t^\dagger}^{-1} t^\circ a_{t^\diamond} = a_{s^\dagger}^{-1} s^\circ t^\circ a_{t^\diamond} \tag{2}$$

Note that $s^\dagger \cdot st = st$ and $st \cdot (st)^\diamond = st = st \cdot t^\diamond$. By Lemma 1.3 (1) and (2), we observe that $s^\dagger \mathcal{R}^* st \mathcal{R}^* (st)^\dagger$ and $t^\diamond = (st)^\diamond$. The prior formulae implies that $s^\dagger = (st)^\dagger$ since S is a left ample semigroup. Now by Lemma 3.3 and Eq. (2), we can obtain that $(st)^\circ = s^\circ t^\circ$, so that $(st)\theta = (s^\circ t^\circ, s^\dagger, t^\diamond) = (s\theta)(t\theta)$.

However we have $(st)\theta = (s\theta)(t\theta)$. Consequently, θ is a homomorphism. We complete the proof. □

Definition 3.4 A primitive left ample semigroup S is called a *PLARI- semigroup* if for any $a \in S \backslash \{0\}$, $aS \neq \{0\}$.

Summarizing Proposition 2.1, and Theorems 2.2 and 3.1, we can obtain the structure theorem of PLARI-semigroups.

Theorem 3.5 *A semigroup S is a PLARI-semigroup if and only if S is isomorphic to some LA blocked Rees matrix semigroup.*

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