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BENJAMIN-BONA-MAHONY TYPE EQUATION

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Abstract. The aim of this paper, is to find an atomic solution for a Benjamin-Bona-Mahony type equation, using the theory of tensor product of Banach spaces.

Keywords: Benjamin-Bona-Mahony type equation; fractional derivative; atomic solution.

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1. Introduction

Linear or non-linear differential equations, sometimes it is difficult to find a general solution. There are many different partial differential equations where the method of separation of variables can't be applied. For example

$$\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial^2 y \partial^2 x} = u$$

is linear but the method of separation of variables cannot be applied. Here comes the concept of atomic solutions, which gives a solution of that equation. In this paper, we find an atomic solution of the Benjamin-Bona-Mahony equation. Review [1], [2], [3], [4], [5] , and [6] for more on atomic solutions and fractional calculus.

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2. Preliminaries

2.1. Conformable Fractional Derivative.

Definition 1. [5] *Given a function $f : [0,\infty) \to \mathbb{R}$. Then, the conformable fractional derivative of f of order α is defined by*

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon},$$

for all $t > 0$, $\alpha \in (0,1)$. If f is α-differentiable in some $(0,a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

The conformable fractional derivative satisfies the following properties:

Let $\alpha \in (0,1]$ and f,g be α-differentiable at a point $t > 0$. Then

(1) $T_\alpha(af+bg)(t) = aT_\alpha(f) + bT_\alpha(g)$, for all $a,b \in \mathbb{R}$.
(2) $T_\alpha(\lambda) = 0$ for all constant functions $f(t) = \lambda$, $\lambda \in \mathbb{R}$.
(3) $T_\alpha(fg) = T_\alpha(f)g + fT_\alpha(g)$.
(4) $T_\alpha\left(\frac{f}{g}\right) = \frac{T_\alpha(f)g - fT_\alpha(g)}{g^2}$, where $g(t) \neq 0$ for any t.
(5) If, in addition, f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$

2.2. Atomic Solution.

Let X and Y be Banach spaces, and X^* be the dual space of X. Assume $x \in X$ and $y \in Y$. The operator $T : X^* \to Y$, defined by $T(\varphi) = \varphi(x)y$ is a bounded one rank linear operator, and we write $x \otimes y$ for T. The operator $x \otimes y$ is called an atom. Now we will consider crucial results concerning atoms.

Lemma 2.3. *Let*

$$x_1 \otimes y_1 + x_2 \otimes y_2 = x_3 \otimes y_3$$

Then either $x_1 = x_2 = x_3$ or $y_1 = y_2 = y_3$.

Lemma 2.4. *Let X and Y be Banach spaces, $a,c \in X$ and $b,d \in Y$ be nonzero vectors. If $a \otimes b = c \otimes d$ then there exists a non-zero scalar β such that $a = \beta c$ and $b = \frac{1}{\beta}d$. With no loss of generality, we can assume $\beta = 1$.*

3. Main Results

We now consider the Benjamin-Bona-Mahony fractional differential equation:

$$D_x^\alpha u + D_y^{2\alpha} u + D_{xy}^{2\alpha} u = D_{xxy}^{3\alpha} u \tag{1}$$

with conditions: $u(0,0) = 1,\ u_x(0,0) = u_y(0,0) = 1$

Clearly equation (1) is linear but the method of separation of variables does not work since the variables cannot be separated, we will try to find an atomic solution.

3.1. Procedure.

Let us look for an atomic solution of the form

$$u(x,y) = P(x) \otimes Q(y). \tag{2}$$

From (1) we see that we can assume $P(0) = P(1) = 1$, $P(0) = P^\alpha(0) = 1$, $Q(0) = Q(1) = 1$ and $Q(0) = Q^\alpha(0) = 1$. Now substitute (2) into (1) to get

$$P^\alpha \otimes Q + P \otimes Q^{2\alpha} + P^\alpha \otimes Q^\alpha = P^{2\alpha} \otimes Q^\alpha. \tag{3}$$

Using the fact that $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$, we see that (3) can be written as

$$P \otimes Q^{2\alpha} + P^\alpha \otimes (Q + Q^\alpha) = P^{2\alpha} \otimes Q^\alpha.$$

Hence by (2.3) we see that we have two cases to consider

3.1.1. *Case(1).*

$$P = P^\alpha = P^{2\alpha}.$$

We get the following situations

(1) $P = P^\alpha$
(2) $P = P^{2\alpha}$
(3) $P^\alpha = P^{2\alpha}$

Situation(1).

$$P = P^\alpha.$$

From which we know that $P^\alpha - P = 0$. Using the method described in [3] we find that $r - 1 = 0$. Thus $P(x) = ce^{\frac{x^\alpha}{\alpha}}$ and using the initial conditions we have $P(x) = e^{\frac{x^\alpha}{\alpha}}$.

Situation(2).

$$P = P^{2\alpha}.$$

Thus $P^{2\alpha} - P = 0$ which gives $r^2 - 1 = 0$, thus

$$P(x) = c_1 e^{\frac{x^\alpha}{\alpha}} + c_2 e^{\frac{-x^\alpha}{\alpha}}.$$

Using the initial conditions we have

$$c_1 + c_2 = 1$$

$$c_1 - c_2 = 1$$

Hence $c_1 = 1$ and $c_2 = 0$. Therefore, $P(x) = e^{\frac{x^\alpha}{\alpha}}$.

Situation(3).

$$P^\alpha = P^{2\alpha}.$$

Thus $P^{2\alpha} - P^\alpha = 0$ which gives $r^2 - r = 0$, then

$$P(x) = c_1 e^{\frac{x^\alpha}{\alpha}} + c_2.$$

Using the initial conditions we see that

$$c_1 + c_2 = 1$$

$$c_1 = 1$$

Therefore $c_2 = 0$ and $P(x) = e^{\frac{x^\alpha}{\alpha}}$.

Now, since all situations gave the same solution, we get substitute $P(x) = e^{\frac{x^\alpha}{\alpha}}$ in (3) to get

$$e^{\frac{x^\alpha}{\alpha}} \otimes (Q + Q^\alpha) + e^{\frac{x^\alpha}{\alpha}} \otimes Q^{2\alpha} = e^{\frac{x^\alpha}{\alpha}} \otimes Q^\alpha. \tag{4}$$

Now (4) can be written as $e^{\frac{x^\alpha}{\alpha}} \otimes (Q + Q^\alpha + Q^{2\alpha}) = e^{\frac{x^\alpha}{\alpha}} \otimes Q^\alpha$, so using (2.4) we have

$$Q + Q^\alpha + Q^{2\alpha} = Q^\alpha.$$

Which simplifies to $Q + Q^{2\alpha} = 0$, hence $r^2 + 1 = 0$ and we have that

$$Q(y) = c_1 \cos\left(\frac{y^\alpha}{\alpha}\right) + c_2 \sin\left(\frac{y^\alpha}{\alpha}\right).$$

So using the initial conditions we have $c_1 = c_2 = 1$ and $Q(y) = \cos(\frac{y^\alpha}{\alpha}) + \sin(\frac{y^\alpha}{\alpha})$. Therefore

$$u(x,y) = e^{\frac{x^\alpha}{\alpha}} \otimes \left(\cos(\frac{y^\alpha}{\alpha}) + \sin(\frac{y^\alpha}{\alpha})\right)$$

3.1.2. *Case(2).*

$$Q + Q^\alpha = Q^{2\alpha} = Q^\alpha.$$

Which gives the following situations

a) $Q + Q^\alpha = Q^\alpha$

b) $Q^{2\alpha} = Q^\alpha$

c) $Q + Q^\alpha = Q^{2\alpha}$

Situation(a).

$$Q + Q^\alpha = Q^\alpha.$$

From which we get that $Q = 0$.

Situation(b).

$$Q^{2\alpha} = Q^\alpha.$$

Which gives $Q^{2\alpha} - Q^\alpha = 0$, hence $r^2 - r = 0$ and

$$Q(y) = c_1 e^{\frac{y^\alpha}{\alpha}} + c_2.$$

Thus using the initial conditions we have that $c_1 = 1$ and $c_2 = 0$, therefore $Q(y) = e^{\frac{y^\alpha}{\alpha}}$, thus we see that situations (a) and (b) give different solutions hence we can conclude that there is no atomic solution in this case.

Conflict of Interests

The authors declare that there is no conflict of interests.

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