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EXISTENCE AND APPROXIMATE BOUNDARY CONTROLLABILITY OF SOME PARTIAL FUNCTIONAL INTEGRODIFFERENTIAL EQUATIONS WITH NONLOCAL INITIAL CONDITION IN BANACH SPACES

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Abstract. This work concerns the study of the existence of solutions and the approximate boundary controllability for some nonlinear partial functional integrodifferential equations with nonlocal initial condition arising in the modelling of materials with memory, in the framework of general Banach spaces. We give sufficient conditions that ensure the approximate boundary controllability of the system by supposing that its linear part is approximately controllable, admits a resolvent operator in the sense of Grimmer, and by making use of the Banach fixed-point Theorem and the continuity of the resolvent operator in the uniform norm-topology. As a result, we obtain a generalization of several important results in the literature, without assuming the compactness of the resolvent operator and the uniform boundedness of the nonlinear term. An example of applications is given for illustration.

Keywords: approximate boundary controllability; semigroup; functional integrodifferential equation; nonlocal condition; resolvent operator; Banach fixed-point theorem.

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Control theory plays a significant role in many modern applications within the physical sciences, positioned at the intersection of engineering and mathematics. A control system is a dynamic system that can be influenced through appropriate parameters, known as controls, to achieve a desired behavior or state. A key challenge in studying such systems is the controllability problem, which involves determining whether it is possible to guide the system from an initial state (or condition) to a desired final state (boundary condition) through a suitable choice of control functions. Controllability, a qualitative property of dynamic control systems, is fundamental to control theory. This problem is further divided into two key concepts: exact controllability and approximate controllability.

Several researchers have explored the concept of exact controllability for systems governed by nonlinear evolution equations, often employing fixed-point techniques to achieve their results (see, e.g., [20, 29, 30, 34, 32, 33] and the references therein). While in finite-dimensional spaces, exact and approximate controllability are equivalent, in infinite-dimensional spaces, exact controllability tends to be too stringent and thus has limited practical use (see [28] and related works). In many applications, the notion of approximate controllability proves to be more suitable and sufficient (see [28] and references therein). It is therefore crucial to investigate this weaker form of controllability for nonlinear integrodifferential systems. In many real-world applications—such as engineering, environmental sciences, and demography—nonlocal constraints (e.g., isoperimetric or energy conditions, multipoint boundary conditions, and flux boundary conditions) frequently arise and have attracted significant attention in recent decades (see [16] and [17]). The concept of nonlocal initial conditions not only generalizes the classical Cauchy initial condition but also offers practical advantages, as it can incorporate future measurements over a period following the initial time t equals 0.

The approximate controllability of nonlinear differential and integrodifferential systems with distributed controls, both with and without delays in infinite-dimensional spaces, has been extensively investigated (see, e.g., [2, 11, 12, 24, 25, 26, 27, 28] and the references therein).

Similarly, many researchers have explored the exact boundary controllability of nonlinear systems in infinite-dimensional spaces (see, e.g., [5, 6, 7, 8, 13] and related works).

However, only a limited number of studies address the approximate boundary controllability of nonlinear control systems (see, e.g., [13, 14, 15, 9]). The primary challenges in studying boundary controllability include formulating an appropriate integral equation suitable for a given fixed-point theorem, ensuring the existence of sufficiently regular solutions for the state-space system, and selecting controls from a space of sufficiently smooth functions.

In [13], the authors studied the following integrodifferential boundary control system:

$$\begin{cases} x'(t) = \sigma x(t) \ + \ f(t,x(t), \int_0^t g(t,s,x(s)))ds \quad \text{for } t \in J = [0,b] \\ \tau x(t) = B_1 u(t), \\ x(0) = x_0, \end{cases} \tag{1}$$

where under some sufficient conditions, they established the exact boundary controllability of system (1).

In [14], the author considered the following Sobolev-type stochastic differential boundary control system:

$$\begin{cases} d(Fx(t)) = (\rho x(t) \ + f(t,x(\gamma_1(t)),x(\gamma_2(t)),\cdots,x(\gamma_n(t))))dt + g(x(\gamma_1(t)),x(\gamma_2(t)),\cdots,x(\gamma_n(t)))dW(t) \\ \text{for } t \in J = [0,b] \\ \tau x(t) = B_1 u(t), \\ x(0) = x_0. \end{cases} \tag{2}$$

Using the compactness of the semigroup operator, the author obtained existence and approximate boundary controllability results for equation (2).

In [15], the authors considered the following semi-linear delay differential system:

$$\begin{cases} x'(t) = \sigma x(t) \ + \ f(t,x_t) \quad \text{for } t \in J = [0,b] \\ \tau x(t) = B_1 u(t), \\ x_0 = \xi(t), \ \text{for } t \in [-r,0]. \end{cases} \tag{3}$$

Assuming the approximate controllability of the corresponding linear system, the authors obtained existence and approximate controllability results for equation (3).

The controllability problem of nonlinear deterministic systems described by integrodifferential equations with nonlocal initial condition in infinite dimensional Banach spaces has been studied by several authors by applying the resolvent operator theory (see for example [20] and the references contained in them).

Motivated by the above works, we study in this paper, the approximate controllability of the following abstract model of partial functional integrodifferential equation with nonlocal initial conditions in a Banach space $(X, \|\cdot\|)$:

$$\begin{cases} x'(t) = \sigma x(t) + \int_0^t \gamma(t-s)x(s)ds + f(t,x(t)) \text{ for } t \in I = [0,b] \\ \tau x(t) = B_1 u(t), \\ x(0) = x_0 + g(x). \end{cases} \tag{4}$$

where $x_0 \in X,\ g : \mathscr{C}(I,X) \to X$ and $f : I \times X \to X$ are functions satisfying some conditions, $\sigma : \mathscr{D}(\sigma) \subset X \to X$ is a closed and densely defined linear operator on X; the system state $x(t)$ takes values in $D(\sigma) \subseteq X$ where X is a Banach space, the control u belongs to $L^2(I,U)$ which is a Banach space of admissible controls, where U is also a Banach space; $\tau : X \to E$ is a linear operator from X to a Banach space E; $B_1 : U \to E$ is a bounded linear operator, $(\gamma(t))_{t\in I}$ are closed linear operators on X and $\mathscr{C}(I,X)$ denotes the Banach space of continuous functions $x : I \to X$ with supremum norm $\|x\|_\infty = \sup_{t\in I} \|x(t)\|_X$.

In this work, we extend and complement the works above without a compactness assumption.

To the best of our knowledge, up to now no work has reported on approximate controllability of partial functional integrodifferential equation (4) with nonlocal initial condition in Banach spaces. It has been an untreated topic in the literature, and this fact also motivates the present work. We use the settings developped in [4] to achieve our goal.

The rest of the work is organized as follows: Section 2 is devoted to stating some preliminary results and introducing most of the notation we need. In section 3, we study the existence of mild solutions to equation (4). In section 4, we prove the approximate controllability of the control system (4), assuming the approximate controllability of the associated linear undelayed part. In section 5, we give an example to illustrate the results we obtain.

1. PRELIMINARIES

In this section we introduce some definitions and Lemmas that will be used throughout the paper.

Let $I = [0,b], \ b > 0$ and let X be a Banach space. A measurable function $x : I \to X$ is Bochner integrable if $\|x\|$ is Lebesgue integrable. We denote by $L^1(I,X)$ the Banach space of Bochner integrable functions $x : I \to X$ normed by

$$\|x\|_{L^1} = \int_0^b \|x(t)\| dt.$$

Consider the following linear homogeneous equation:

$$\begin{cases} x'(t) = Ax(t) + \displaystyle\int_0^t \gamma(t-s)x(s)ds \ \text{ for } \quad t \geq 0, \\ x(0) = x_0 \in X. \end{cases} \tag{5}$$

where A and $\gamma(t)$ are closed linear operators on a Banach space X.

In the sequel, we assume A and $\big(\gamma(t)\big)_{t\geq 0}$ satisfy the following conditions:

($\mathbf{H_1}$) A is a densely defined closed linear operator in X. Hence $\mathscr{D}(A)$ is a Banach space equipped with the graph norm defined by, $|y| = \|Ay\| + \|y\|$ which will be denoted by $(X_1, |\cdot|)$.

($\mathbf{H_2}$) $\big(\gamma(t)\big)_{t\geq 0}$ is a family of linear operators on X such that $\gamma(t)$ is continuous when regarded as a linear map from $(X_1, |\cdot|)$ into $(X, \|\cdot\|)$ for almost all $t \geq 0$ and the map $t \mapsto \gamma(t)y$ is measurable for all $y \in X_1$, and belongs to $W^{1,1}(\mathbb{R}^+, X)$. Moreover there is a locally integrable function $b : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\gamma(t)y\| \ \leq \ b(t)|y| \quad \text{ and } \quad \left\| \frac{d}{dt}\gamma(t)y \right\| \ \leq \ b(t)|y| \ .$$

Remark 1. Note that ($\mathbf{H_2}$) is satisfied in the modelling of Heat Conduction in materials with memory and viscosity. More details can be found in [18].

Let $\mathscr{L}(X)$ be the Banach space of bounded linear operators on X.

Definition 1.1. [21] A resolvent operator $\big(R(t)\big)_{t\geq 0}$ for equation (5) is a bounded operator valued function

$$R : \ [0, +\infty) \ \longrightarrow \ \mathscr{L}(X)$$

such that

(i) $R(0) = Id_X$ and $\|R(t)\| \leq Ne^{\beta t}$ for some constants N and β.

(ii) For all $x \in X$, the map $t \mapsto R(t)x$ is continuous for $t \geq 0$.

(iii) Moreover for $x \in X_1$, $R(\cdot)x \in \mathscr{C}^1(\mathbb{R}^+;X) \cap \mathscr{C}(\mathbb{R}^+;X_1)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t \gamma(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)\gamma(s)xds. \end{aligned}$$

Observe that the map defined on $\mathbb{R}^+$ by $t \mapsto R(t)x_0$ solves equation (5) for $x_0 \in \mathscr{D}(A)$.

Theorem 1.2. [31] *Assume that* $(\mathbf{H_1})$ *and* $(\mathbf{H_2})$ *hold. Then, the linear equation (5) has a unique resolvent operator* $\left(R(t)\right)_{t\geq 0}$.

Remark 2. In general, the resolvent operator $\left(R(t)\right)_{t\geq 0}$ for equation (5) does not satisfy the semigroup law, namely,

$$R(t+s) \neq R(t)R(s) \qquad \text{for some } t, s > 0.$$

We have the following lemmas that will be useful in proving the main results.

Lemma 1.3. *(see [[10], Lemma 2.3]) $AR(t)$ is continuous for $t > 0$ in the uniform operator topology of $\mathscr{B}(X)$.*

Theorem 1.4. *(see [[20], Theorem 6]) Let A be the infinitesimal generator of a c_0-semigroup $\left(T(t)\right)_{t\geq 0}$ and let $\left(\gamma(t)\right)_{t\geq 0}$ satisfy* $(\mathbf{H_2})$. *Then the resolvent operator $\left(R(t)\right)_{t\geq 0}$ for equation* (5) *is operator-norm continuous (or continuous in the uniform operator topology) for $t > 0$ if and only if $\left(T(t)\right)_{t\geq 0}$ is operator-norm continuous for $t > 0$.*

Let $A : X \to X$ be the linear operator defined by:

$$\mathscr{D}(A) = \{x \in \mathscr{D}(\sigma); \tau x = 0\} \text{ and } \quad Ax = \sigma x, \quad \text{for } x \in \mathscr{D}(A). \tag{6}$$

Throughout the paper, we shall require the following hypotheses to be satisfied:

$(\mathbf{H_3})$ $\mathscr{D}(\sigma) \subset \mathscr{D}(\tau)$ and the restriction of τ to $\mathscr{D}(\sigma)$ is continuous relative to the graph norm of $\mathscr{D}(\sigma)$.

$(\mathbf{H_4})$ The operator A as defined above is the infinitesimal generator of a C_0-semigroup $\left(S(t)\right)_{t\geq 0}$

on X and equation (5) has a resolvent operator $\big(R(t)\big)_{t\geq 0}$ that is continuous in the operator-norm topology for $t>0$.

($\mathbf{H_5}$) There exists a bounded linear operator $B: U \to X$ with $B(U) \subseteq \mathscr{D}(\sigma)$ and a positive constant K such that

$$
\sigma B \in \mathscr{L}(U,X), \quad \tau(Bu) = B_1 u, \quad \forall u \in U,
\tag{7}
$$

$$
\|Bu\|_X \leq K \|B_1 u\|_E, \ \forall u \in U.
$$

($\mathbf{H_6}$) For each $t \in (0,b]$ and $u \in U$, one has $R(t)Bu \in \mathscr{D}(A)$. Also, there exists a positive function $\delta(\cdot) \in L^2(I)$ such that

$$
\|AR(t)B\|_{\mathscr{L}(U,X)} \leq \delta(t), \quad a.e.;\ t \in (0,b].
\tag{8}
$$

($\mathbf{H_7}$) There exists a positive number L_f such that

$$
\|f(t,y) - f(t,z)\| \leq L\|y-z\|, \ \text{ for all } \ y,\ z \in \mathscr{C}([0,b],X) \text{ and } t \in I.
\tag{9}
$$

(($\mathbf{H_8}$))There exist positive numbers L_g and M_g such that

(10)

$$
\|g(y)-g(z)\| \leq L_g\|y-z\|, \ \text{ for all } \ y,\ z \in \mathscr{C}([0,b],X), \text{ and } \|g(x)\| \leq M_g, \ \text{ for all } \ x \in \mathscr{C}([0,b],X)
$$

Let $x(t)$ be a solution of equation (4), then we can define $z(t) = x(t) - Bu(t) \in \mathscr{C}([0,b],X)$. So, equation (4) becomes

$$
\begin{cases}
x'(t) = Az(t) \ + \sigma Bu(t) + \displaystyle\int_0^t \gamma(t-s)x(s)ds \ + \ f(t,x(t)) \ \text{ for } t \in I = [0,b] \\
x(t) = z(t) + Bu(t), \\
x(0) = x_0 + g(x).
\end{cases}
\tag{11}
$$

If $u \in \mathscr{C}^1([0,b])$, then, $z(t)$ can be defined as the mild solution of:

(12)

$$
\begin{cases}
z'(t) = Az(t) + \sigma Bu(t) - Bu'(t) + \displaystyle\int_0^t \gamma(t-s)z(s)ds + \int_0^t \gamma(t-s)Bu(s)ds + f(t, z(t)+Bu(t)), \ \ t \in I \\
x(t) = z(t) + Bu(t),\ t \geq 0 \\
z(0) = x(0) - Bu(0)
\end{cases}
$$

This implies that

$$z(t) = R(t)(x(0) - Bu(0)) + \int_0^t R(t-s)[\sigma Bu(s) - Bu'(s) + f(s, z(s) + Bu(s))]\,ds + \int_0^t R(t-s)\int_0^s \gamma(s-\tau)Bu(\tau)d\tau\,ds$$

and the solution of (4) is given by:

$$x(t) = R(t)(x(0) - Bu(0)) + Bu(t) + \int_0^t R(t-s)[\sigma Bu(s) - Bu'(s) + f(s, x(s))]\,ds + \int_0^t R(t-s)\int_0^s \gamma(s-\tau)Bu(\tau)d\tau\,ds. \tag{13}$$

Since the differentiability of the control u is an unrealistic and severe requirement, it is necessary to extend the concept of the solution to accommodate general inputs $u \in L^2(I,U)$. Integrating by parts yields:

$$x(t) = R(t)[x_0 + g(x)] + \int_0^t [R(t-s)\sigma + AR(t-s)]Bu(s)\,ds + \int_0^t R(t-s)f(s,x(s))\,ds + \int_0^t\int_0^s \gamma(s-\tau)R(\tau)Bu(\tau)\,d\tau\,ds + \int_0^t R(t-s)\int_0^s \gamma(s-\tau)Bu(\tau)d\tau\,ds. \tag{14}$$

Definition 1.5. Let $u \in L^2(I,U)$ and $x_0 \in X$. A function $x : [0,b] \to X$ is called a mild solution of equation (4) if $x \in \mathscr{C}([0,b];X)$ and satisfies the following integral equation

$$x(t) = \begin{cases} R(t)[x_0 + g(x)] + \int_0^t [R(t-s)\sigma + AR(t-s)]Bu(s)\,ds \\ \quad + \int_0^t R(t-s)f(s,x(s))\,ds + \int_0^t\int_0^s \gamma(s-\tau)R(\tau)Bu(\tau)\,d\tau\,ds \\ \quad + \int_0^t R(t-s)\int_0^s \gamma(s-\tau)Bu(\tau)d\tau\,ds, \quad \text{for } t \in I \end{cases} \tag{15}$$

2. Main Results

In this section, we state and prove the main results of this work which are the existence and uniqueness of solution, and the approximate boundary controllability of the control system under consideration.

2.1. Existence and Uniqueness Result.

In this section, we prove an existence and uniqueness result for the mild solution of (4).

Theorem 2.1. *Suppose hypotheses* $(\mathbf{H_3}) - (\mathbf{H_8})$ *are satisfied. Then, equation (4) has a unique mild solution.*

Proof. For any $u(\cdot)\in L^2(I,U)$, define an operator $K:\mathscr{C}([0,b];X)\to\mathscr{C}([0,b];X)$ as follows:

$$(Kx)(t)=\begin{cases} R(t)[x_0+g(x)]+\int_0^t R(t-s)[\sigma Bu(s)+f(s,x(s))]\,ds+\int_0^t AR(t-s)Bu(s)\,ds\\ +\int_0^t\int_0^s \gamma(s-\tau)R(\tau)Bu(\tau)\,d\tau\,ds+\int_0^t R(t-s)\int_0^s\gamma(s-\tau)Bu(\tau)d\tau\,ds, \quad \text{for } t\in I\end{cases} \tag{16}$$

We need to show that K is well-defined. First we show that for any $x\in\mathscr{C}([0,b];X)$, that is the integrals in $(Kx)(t)$ are finite. Indeed, we have from $(\mathbf{H_7})$ that $\|f(t,x)\|\le L_f\|x\|_X+M_1$, where $M_1=\sup_{t\in I}\|f(t,0)\|$. Let $M=\sup_{t\in I}\|R(t)\|$.

We have that

$$\begin{aligned}\left\|\int_0^t AR(t-s)Bu(s)\,ds\right\| &\le \left(\int_0^t \|AR(t-s)B\|^2\,ds\right)^{\frac12}\left(\int_0^t \|u(s)\|^2\,ds\right)^{\frac12}\\ &\le \|\delta\|_{L^2}\|u\|_{L^2}\end{aligned} \tag{17}$$

$$\begin{aligned}\left\|\int_0^t R(t-s)[\sigma Bu(s)+f(s,x(s))]\,ds\right\| &\le M\int_0^t\Big[\|\sigma Bu(s)\|+L_f\|x(s)\|_X+M_1\Big]\,ds\\ &\le M\|\sigma B\|_{\mathscr{L}(U,X)}\sqrt{b}\|u\|_{L^2}+bMM_1+ML_fb\|x\|;\end{aligned} \tag{18}$$

$$\begin{aligned}&\left\|\int_0^t R(t-s)\int_0^s\gamma(s-\tau)Bu(\tau)\,d\tau\,ds\right\|+\left\|\int_0^t\int_0^s\gamma(s-\tau)R(\tau)Bu(\tau)\,d\tau\,ds\right\|\\ &\le\int_0^t\left\|R(t-s)\int_0^s\gamma(s-\tau)Bu(\tau)\,d\tau\right\|\,ds+\int_0^t\left\|\int_0^s\gamma(s-\tau)R(\tau)Bu(\tau)\,d\tau\right\|\,ds\\ &\le\int_0^t M\int_0^s\|\gamma(s-\tau)Bu(\tau)\|\,d\tau\,ds+\int_0^t\int_0^s\|\gamma(s-\tau)R(\tau)Bu(\tau)\|\,d\tau\,ds\\ &\le 2\int_0^t M\int_0^s M_2\|B\|\|u(\tau)\|\,d\tau\,ds\\ &\le 2MM_2\|B\|b\sqrt{b}\|u\|_{L^2},\end{aligned} \tag{19}$$

where M_2 is such that $\|\gamma(t)\|\le M_2,\quad \forall t\in I$.

Combining (17), (18) and (19), we get that the integrals in $(Kx)(t)\in X$ for any $x\in\mathscr{C}([0,b];X)$ are finite.

Next, we show that K maps $\mathscr{C}([0,b];X)$ into $\mathscr{C}([0,b];X)$, in other words, $Kx \in \mathscr{C}([0,b];X)$ for any $x \in \mathscr{C}([0,b];X)$. Taking $t, t+\varepsilon \in I$ with $\varepsilon > 0$, then

$$\begin{aligned}
&\|(Kx)(t+\varepsilon)-(Kx)(t)\| \\
&= \Big\| \int_0^{t+\varepsilon} R(t+\varepsilon-s)[\sigma Bu(s)+f(s,x(s))]\,ds + \int_0^{t+\varepsilon} AR(t+\varepsilon-s)Bu(s)\,ds \\
&+ \int_0^{t+\varepsilon} R(t+\varepsilon-s)\int_0^s \gamma(s-\tau)Bu(\tau)\,d\tau\,ds + \int_0^{t+\varepsilon}\int_0^s \gamma(s-\tau)R(\tau)Bu(\tau)\,d\tau\,ds \\
&- \int_0^t R(t-s)[\sigma Bu(s)+f(s,y_s+\tilde{\varphi}_s)]\,ds - \int_0^t AR(t-s)Bu(s)\,ds \\
&- \int_0^t R(t-s)\int_0^s \gamma(s-\tau)Bu(\tau)\,d\tau\,ds - \int_0^t\int_0^s \gamma(s-\tau)R(\tau)Bu(\tau)\,d\tau\,ds \Big\| \\
&\le \Big\| \int_0^t [R(t+\varepsilon-s)-R(t-s)][\sigma Bu(s)+f(s,x(s))]\,ds \Big\| \\
&+ \Big\| \int_0^t AR(t+\varepsilon-s)Bu(s)-AR(t-s)Bu(s)\,ds \Big\| \\
&+ \Big\| \int_0^t \Big[R(t+\varepsilon-s)-R(t-s)\Big]\int_0^s \gamma(s-\tau)Bu(\tau)\,d\tau\,ds \Big\| \\
&+ \Big\| \int_t^{t+\varepsilon} R(t+\varepsilon-s)[\sigma Bu(s)+f(s,x(s))]\,ds \Big\| \\
&+ \Big\| \int_t^{t+\varepsilon} AR(t+\varepsilon-s)Bu(s)\,ds \Big\| \\
&+ \Big\| \int_t^{t+\varepsilon} R(t+\varepsilon-s)\int_0^s \gamma(s-\tau)Bu(\tau)\,d\tau\,ds \Big\| \\
&+ \Big\| \int_t^{t+\varepsilon}\int_0^s \gamma(s-\tau)R(\tau)Bu(\tau)\,d\tau\,ds \Big\| \\
&:= I_1+I_2+I_3+I_4+I_5+I_6+I_7.
\end{aligned}$$

It follows that

$$I_1 = \left\| \int_0^t [R(t+\varepsilon-s) - R(t-s)][\sigma Bu(s) + f(s,x(s))]\, ds \right\|$$

$$\leq \int_0^t \|R(t+\varepsilon-s) - R(t-s)\| \|\sigma Bu(s) + f(s,x(s))\|\, ds$$

$$\leq \int_0^t \|R(t+\varepsilon-s) - R(t-s)\|\, ds \Big(\|\sigma B\|_{\mathscr{L}(U,X)} \|u\|_{L^2} + M_1 + L_f \|x\| \Big).$$

Now, since $R(\cdot)$ is continuous in the operator norm topology, it follows by the Lebesgue dominated convergence Theorem that the left hand side of the above inequality tends to 0 as $\varepsilon \to 0$. Also, by the continuity of $AR(\cdot)$ in the operator norm topology (Lemma 1.3), and the Lebesgue dominated convergence theorem, we have

$$I_2 = \left\| \int_0^t AR(t+\varepsilon-s)Bu(s) - AR(t-s)Bu(s)\, ds \right\|$$

$$\leq \int_0^t \|AR(t+\varepsilon-s)Bu(s) - AR(t-s)Bu(s)\|\, ds \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$

Also, it follows from (19) and from the Lebesgue dominated convergence Theorem that

$$I_3 = \left\| \int_0^t \Big[R(t+\varepsilon-s) - R(t-s) \Big] \int_0^s \gamma(s-\tau)Bu(\tau)\, d\tau\, ds \right\|$$

$$\leq \int_0^t \|R(t+\varepsilon-s) - R(t-s)\| \int_0^s M_2 \|B\| \|u(\tau)\|\, d\tau\, ds \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0$$

Again, it follows from (18) that

$$I_4 = \left\| \int_t^{t+\varepsilon} R(t+\varepsilon-s)[\sigma Bu(s) + f(s,x(s))]\, ds \right\|$$

$$\leq \int_t^{t+\varepsilon} \|R(t+\varepsilon-s)[\sigma Bu(s) + f(s,x(s))]\|\, ds$$

$$\leq M \int_t^{t+\varepsilon} [\|\sigma Bu(s)\| + L_f \|x(s)\|_X + M_1]\, ds$$

$$\leq M \|\sigma B\|_{\mathscr{L}(U,X)} \sqrt{\varepsilon} \|u\|_{L^2} + \varepsilon M M_1 + \varepsilon M L_f \|x\| \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$

Also, we have from the following estimate

$$I_5 = \left\| \int_t^{t+\varepsilon} AR(t+\varepsilon-s)Bu(s)\,ds \right\|$$

$$\leq \int_t^{t+\varepsilon} \|AR(t+\varepsilon-s)Bu(s)\|\,ds$$

$$\leq \left(\int_t^{t+\varepsilon} \delta^2(s)\,ds \right)^{\frac{1}{2}} \|u\|_{L^2}.$$

that $I_5 \longrightarrow 0$ as $\varepsilon \longrightarrow 0$.

Again, it follows from (19) that

$$I_6 = \left\| \int_t^{t+\varepsilon} R(t+\varepsilon-s) \int_0^s \gamma(s-\tau)Bu(\tau)\,d\tau\,ds \right\|$$

$$\leq \int_t^{t+\varepsilon} \|R(t+\varepsilon-s)\| \int_0^s \|\gamma(s-\tau)Bu(\tau)\|\,d\tau\,ds$$

$$\leq \int_t^{t+\varepsilon} M \int_0^s M_2\|B\|\|u(\tau)\|\,d\tau\,ds$$

$$\leq \varepsilon\sqrt{b}MM_2\|B\|\|u\|_{L^2} \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$

Also, we have that:

$$I_7 = \left\| \int_t^{t+\varepsilon} \int_0^s \gamma(s-\tau)R(\tau)Bu(\tau)\,d\tau\,ds \right\|$$

$$\leq \int_t^{t+\varepsilon} \int_0^s \|\gamma(s-\tau)R(\tau)Bu(\tau)\|\,d\tau\,ds$$

$$\leq \int_t^{t+\varepsilon} M \int_0^s M_2\|B\|\|u(\tau)\|\,d\tau\,ds$$

$$\leq \varepsilon\sqrt{b}MM_2\|B\|\|u\|_{L^2} \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0.$$

Thus, we have $\|(Kx)(t+\varepsilon)-(Kx)(t)\| \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0$ and, hence, $Kx \in \mathscr{C}([0,b];X)$.

We now prove that K is a contraction mapping. In fact, let $b_1 \in [0,b]$ and $y, z \in \mathscr{C}([0,b_1];X)$. Then, we have

$$\|(Ky)(t)-(Kz)(t)\| = \left\| [R(t)g(y)-R(t)g(z)] + \int_0^t R(t-s)[f(s,y(s))-f(s,z(s))]\,ds \right\|$$
$$\leq M\|g(y)-g(z)\| + ML_f \int_0^t \|y(s)-z(s)\|_X\,ds$$
$$\leq ML_g\|y-z\| + ML_f t\|y-z\| = (ML_g+ML_f t)\|y-z\|$$

It follows that:

$$\|Ky-Kz\| \leq (ML_g+ML_f b_1)\|y-z\|$$

Now, if we choose b_1 such that $(ML_g+ML_f b_1)<1$, then K is a contraction mapping. The contraction mapping principle implies that K has a unique fixed-point in $\mathscr{C}([0,b_1];X)$, which is the unique mild solution x of equation (4). A similar argument can be used for $[b_1,2b_1],\cdots,[nb_1,(n+1)b_1]$, for all $n\geq 0$, which implies that the mild solution exists uniquely in $[0,+\infty[$, and hence on $[0,b]$. This completes the proof.

■

2.2. Approximate Controllability Results. In this section we establish the approximate controllability of system (4) whose mild solution is given by (15). The corresponding linear boundary control system to (4) is:

$$\begin{cases} x'(t) = \sigma x(t) + \int_0^t \gamma(t-s)x(s)ds \ \text{ for } t\in I=[0,b] \\ \tau x(t) = B_1 u(t), \\ x(0) = x_0 + g(x), \end{cases} \tag{20}$$

and its mild solution is given by the following corresponding linear system to (15):

$$x(t) = \begin{cases} R(t)[x_0+g(x)] + \int_0^t [R(t-s)\sigma + AR(t-s)]Bu(s)\,ds + \int_0^t \int_0^s \gamma(s-\tau)R(\tau)Bu(\tau)\,d\tau\,ds \\ + \int_0^t R(t-s)\int_0^s \gamma(s-\tau)Bu(\tau)\,d\tau\,ds, \ \text{ for } t\in I \end{cases} \tag{21}$$

Let Λ be a nonempty bounded subset of U. We denote the solution of (4) by $x(t;0,x(0),u)$ to emphasize the initial time $t=0$, the initial state $x(0)$, and the control function u. $x(t_1;0,x(0),u)$ is called the system state at time t_1 corresponding to the initial pair $(0,x(0))$ and the control function u. Introduce the set

$$\mathscr{R}(t_1;0,x(0))(N) = \{x(t_1;0,x(0),u),\, u \in L^2(I;\Lambda)\},$$

which is called the reacheable set of system (4) at time t_1 corresponding to the initial pair $(0,x(0))$. $\overline{\mathscr{R}(t_1;0,x(0))(N)}$ denotes the closure of the reachable set $\mathscr{R}(t_1;0,x(0))(N)$.
Let us now define the notion of approximate controllability which is the main topic of this paper.

Definition 2.2. Equation (4) is said to be approximately controllable on the interval $[0,t_1] \subset [0,b]$ if $\mathscr{R}(t_1;0,x(0))(N)$ is dense in X, i.e., $\overline{\mathscr{R}(t_1;0,x(0))(N)} = X$.

Definition 2.3. Equation (4) is said to be approximately null controllable on the interval $[0,t_1] \subset [0,b]$ if for any $x(0) \in X$ and $\varepsilon > 0$ there exists a control function $u \in L^2([0,t_1];\Lambda)$ such that $\|x(t_1;0,x(0),u)\| < \varepsilon$.

Similarly to the nonlinear system (4), we define the reachable set of system (20) at time t_1 corresponding to the initial pair $(0,x(0))$ as $\mathscr{R}(t_1;0,x(0))(L)$. The approximate controllability and the approximate null controllability for equation (20) can be defined in a similar way as for the case of equation (4).
For any $t_1,\, t_2 \in I$ with $t_2 > t_1$, we define the operators $L(t_1,t_2) : L^2([t_1,t_2];\Lambda) \to X$, and $N(t_1,t_2) : L^2([t_1,t_2];\Lambda) \to X$ as follows:

$$\begin{aligned} L(t_1,t_2)u &= \int_{t_1}^{t_2} [R(t_2-s)\sigma + AR(t_2-s)]Bu(s)\,ds + \int_{t_1}^{t_2}\int_{t_1}^{s} \gamma(s-\tau)R(\tau)Bu(\tau)\,d\tau\,ds \\ &+ \int_{t_1}^{t_2} R(t_2-s)\int_{t_1}^{s}\gamma(s-\tau)Bu(\tau)\,d\tau\,ds \\ N(t_1,t_2)u &= \int_{t_1}^{t_2} R(t_2-s)f(s,x(s))\,ds, \end{aligned}$$

where in the definition of $N(t_1,t_2)$, $x(t;u)$ is the solution of (11) with the initial pair $(t_1,x(t_1))$ and the control function $u \in L^2([t_1,t_2];\Lambda)$.

One observes that the mild solution of (4) with initial time t_1 is given by

$$x(t) = \begin{cases} R(t-t_1)x(t_1) + \int_{t_1}^{t} [R(t-s)\sigma + AR(t-s)]Bu(s)\,ds \\ \\ \qquad + \int_{t_1}^{t} R(t-s)f(s,x(s))\,ds + \int_{t_1}^{t}\int_{t_1}^{s} \gamma(s-\tau)R(\tau)Bu(\tau)\,d\tau\,ds \\ \\ \qquad + \int_{t_1}^{t} R(t-s)\int_{t_1}^{s} \gamma(s-\tau)Bu(\tau)d\tau\,ds, \quad \text{for } t \in I \end{cases} \tag{22}$$

We now state and prove a result that provides sufficient conditions for the approximate controllability of system (4).

Theorem 2.4. *Suppose that system (20) is approximately controllable on the interval $[a,b]$ for any $a \geq 0$, and there exists a function $q(\cdot) \in L^1(I,\mathbb{R}^+)$ such that $\|f(t,x)\| \leq q(t), \quad \forall (t,x) \in I \times \mathscr{C}(I,X)$. Then, system (4) is approximately controllable on I.*

Proof. We need to prove that the reachable set of system (4) at time b is dense in the Banach space X, in other words,

$$\overline{\mathscr{R}(b;0,x(0))(N)} = X \tag{23}$$

for any $x(0) \in X$.

To this end, given any $\varepsilon > 0$ and $x_b \in X$, since (20) is approximately controllable on $[0,b]$, there exists a control function $v_0 \in L^2([0,b];\Lambda)$ such that

$$\|R(b)x(0) + L(0,b)v_0 - x_b\| < \frac{\varepsilon}{11}.$$

Observe that since $q(\cdot) \in L^1(I)$, we can select an increasing sequence $\{t_n\} \subset I$ such that

$$\int_{t_n}^{b} q(t)\,dt \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Let $x_1 := x(t_1;0,x(0),v_0)$. Again, the approximate controllability of (20) on $[t_1,b]$ implies that there exists a control function $v_1 \in L^2([t_1,b];\Lambda)$ such that $\|x(b;t_1,x(t_1),v_1) - x_b\| < \frac{\varepsilon}{11}$. This implies that

$$\|R(b-t_1)x_1 + L(t_1,b)v_1 - x_b\| < \frac{\varepsilon}{11}.$$

Define

$$u_1(t) = \begin{cases} v_0(t), & 0 \le t \le t_1 \\ v_1(t), & t_1 \le t \le b. \end{cases} \tag{24}$$

Then, $u_1(\cdot) \in L^2([0,b];\Lambda)$. Repeating the procedure, we have three sequences $x_n,\ v_n$ and u_n such that $v_n(\cdot) \in L^2([t_n,b];\Lambda),\ \ u_n(\cdot) \in L^2([0,b];\Lambda)$, defined as follows:

$$u_n(t) = \begin{cases} u_{n-1}(t), & 0 \le t \le t_n \\ v_n(t), & t_n \le t \le b. \end{cases} \tag{25}$$

$$x_n = x(t_n;0,x(0),u_{n-1}), \quad \|R(b-t_n)x_n + L(t_n,b)v_n - x_b\| < \frac{\varepsilon}{11}.$$

We now write the mild solution of equation (4) under the sequence of control functions as follows:

$$
\begin{aligned}
& x(t;0,x(0),u_n) \\
= \;& R(t)x(0)+L(0,t_n)u_n+N(0,t_n)u_n+L(t_n,t)u_n+N(t_n,t)u_n \\
+ \;& \int_0^{t_n}\Big[\Big(R(t-s)-R(t_n-s)\Big)\sigma+A\Big(R(t-s)-R(t_n-s)\Big)\Big]Bu_{n-1}(s)\,ds \\
+ \;& \int_0^{t_n}\Big[R(t-s)-R(t_n-s)\Big]\int_0^s\gamma(s-\tau)Bu_{n-1}(\tau)\,d\tau\,ds+\int_0^{t_n}\Big[R(t-s)-R(t_n-s)\Big]f(s,x(s))ds \\
+ \;& \int_{t_n}^t R(t-s)\int_0^s\gamma(s-\tau)Bu_n(\tau)\,d\tau\,ds-\int_{t_n}^t R(t-s)\int_{t_n}^s\gamma(s-\tau)Bu_n(\tau)\,d\tau\,ds \\
+ \;& \int_{t_n}^t\int_0^s\gamma(s-\tau)R(\tau)Bu_n(\tau)\,d\tau\,ds-\int_{t_n}^t\int_{t_n}^s\gamma(s-\tau)R(\tau)Bu_n(\tau)\,d\tau\,ds \\
= \;& [R(t_n)x(0)+L(0,t_n)u_{n-1}+N(0,t_n)u_{n-1}]+[R(t)-R(t_n)]x(0)+L(t_n,t)v_n+N(t_n,t)v_n \\
+ \;& \int_0^{t_n}\Big[\Big(R(t-s)-R(t_n-s)\Big)\sigma+A\Big(R(t-s)-R(t_n-s)\Big)\Big]Bu_{n-1}(s)\,ds \\
+ \;& \int_0^{t_n}\Big[R(t-s)-R(t_n-s)\Big]\int_0^s\gamma(s-\tau)Bu_{n-1}(\tau)\,d\tau\,ds+\int_0^{t_n}\Big[R(t-s)-R(t_n-s)\Big]f(s,x(s))ds \\
+ \;& \int_{t_n}^t R(t-s)\left[\int_0^s\gamma(s-\tau)Bu_n(\tau)\,d\tau\,ds+\int_s^{t_n}\gamma(s-\tau)Bu_n(\tau)\,d\tau\,ds\right] \\
+ \;& \int_{t_n}^t\left[\int_0^s\gamma(s-\tau)R(\tau)Bu_n(\tau)\,d\tau\,ds+\int_s^{t_n}\gamma(s-\tau)R(\tau)Bu_n(\tau)\,d\tau\,ds\right] \\
= \;& x_n+L(t_n,t)u_n+N(t_n,t)u_n+[R(t)-R(t_n)]x_0+[R(t)-R(t_n)]g(x) \\
+ \;& \int_0^{t_n}\Big[\Big(R(t-s)-R(t_n-s)\Big)\sigma+A\Big(R(t-s)-R(t_n-s)\Big)\Big]Bu_{n-1}(s)\,ds \\
+ \;& \int_0^{t_n}\Big[R(t-s)-R(t_n-s)\Big]f(s,x(s))ds \\
+ \;& \int_0^{t_n}\Big[R(t-s)-R(t_n-s)\Big]\int_0^s\gamma(s-\tau)Bu_{n-1}(\tau)\,d\tau\,ds \\
+ \;& \int_{t_n}^t R(t-s)\int_0^{t_n}\gamma(s-\tau)Bu_{n-1}(\tau)\,d\tau\,ds+\int_{t_n}^t\int_0^{t_n}\gamma(s-\tau)R(\tau)Bu_{n-1}(\tau)\,d\tau\,ds \\
= \;& R(t-t_n)x_n+L(t_n,t)u_n+N(t_n,t)u_n+[R(t)-R(t_n)]x_0+[R(0)-R(t-t_n)]x_n \\
+ \;& \int_0^{t_n}\Big[\Big(R(t-s)-R(t_n-s)\Big)\sigma+A\Big(R(t-s)-R(t_n-s)\Big)\Big]Bu_{n-1}(s)\,ds \\
+ \;& \int_0^{t_n}\Big[R(t-s)-R(t_n-s)\Big]f(s,x(s))ds+[R(t)-R(t_n)]g(x). \\
+ \;& \int_0^{t_n}\Big[R(t-s)-R(t_n-s)\Big]\int_0^s\gamma(s-\tau)Bu_{n-1}(\tau)\,d\tau\,ds \\
+ \;& \int_{t_n}^t R(t-s)\int_0^{t_n}\gamma(s-\tau)Bu_{n-1}(\tau)\,d\tau\,ds+\int_{t_n}^t\int_0^{t_n}\gamma(s-\tau)R(\tau)Bu_{n-1}(\tau)\,d\tau\,ds
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& \|x(b;0,x(0),u_n)-x_b\| \\
\leq\ & \|R(b-t_n)x_n+L(t_n,b)u_n-x_b\|+\|N(t_n,b)u_n\| \\
+\ & \|[R(b)-R(t_n)]x_0\|+\|[R(0)-R(t-t_n)]x_n\| \\
+\ & \int_0^{t_n}\left\|\Big(R(b-s)-R(t_n-s)\Big)\sigma Bu_{n-1}(s)\right\|ds \\
+\ & \int_0^{t_n}\left\|A\Big(R(b-s)-R(t_n-s)\Big)Bu_{n-1}(s)\right\|ds \\
+\ & \int_0^{t_n}\left\|R(b-s)-R(t_n-s)\right\|\int_0^{s}\|\gamma(s-\tau)Bu_{n-1}(\tau)\|\,d\tau\,ds \\
+\ & \int_0^{t_n}\left\|R(b-s)-R(t_n-s)\right\|\|f(s,x(s))\|\,ds+\|[R(b)-R(t_n)]g(x)\| \\
+\ & \int_{t_n}^{b}\|R(b-s)\|\int_0^{t_n}\|\gamma(s-\tau)Bu_{n-1}(\tau)\|\,d\tau\,ds+\int_{t_n}^{b}\int_0^{t_n}\|\gamma(s-\tau)R(\tau)Bu_{n-1}(\tau)\|\,d\tau\,ds \\
<\ & \frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\int_{t_n}^{b}\|R(b-s)f(s,x(s))\|\,ds \\
+\ & \int_0^{t_n}\left\|R(b-s)-R(t_n-s)\right\|q(s)ds+M_g\|R(b)-R(t_n)\| \\
+\ & M\int_{t_n}^{b}\int_0^{t_n}\|\gamma(s-\tau)R(\tau)Bu_{n-1}(\tau)\|\,d\tau\,ds+\int_{t_n}^{b}\int_0^{t_n}\|\gamma(s-\tau)R(\tau)Bu_{n-1}(\tau)\|\,d\tau\,ds \\
<\ & \frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+M\int_{t_n}^{b}q(s)\,ds \\
+\ & M\int_{t_n}^{b}\int_0^{t_n}\|\gamma(s-\tau)R(\tau)Bu_{n-1}(\tau)\|\,d\tau\,ds+\int_{t_n}^{b}\int_0^{t_n}\|\gamma(s-\tau)R(\tau)Bu_{n-1}(\tau)\|\,d\tau\,ds \\
<\ & \varepsilon.
\end{aligned}
$$

For a sufficiently large n such that $M\int_{t_n}^{b}q(s)\,ds<\frac{\varepsilon}{11}$ and by the continuity of $R(\cdot)$ and that of $AR(\cdot)$ in the operator norm topology (Lemma 1.3), and the Lebesgue dominated convergence theorem, such that

$$
\int_0^{t_n}\left\|\Big(R(b-s)-R(t_n-s)\Big)\sigma Bu_{n-1}(s)\right\|ds+\int_0^{t_n}\left\|A\Big(R(b-s)-R(t_n-s)\Big)Bu_{n-1}(s)\right\|ds<\frac{\varepsilon}{11}+\frac{\varepsilon}{11},
$$

Similarly,

$$
\int_0^{t_n}\left\|R(b-s)-R(t_n-s)\right\|\int_0^{s}\|\gamma(s-\tau)Bu_{n-1}(\tau)\|\,d\tau\,ds<\frac{\varepsilon}{11}.
$$

Also we have for sufficiently large n that

$$
M\int_{t_n}^{b}\int_0^{t_n}\|\gamma(s-\tau)R(\tau)Bu_{n-1}(\tau)\|\,d\tau\,ds<\frac{\varepsilon}{11},
$$

$$\int_{t_n}^{b}\int_{0}^{t_n}\|\gamma(s-\tau)R(\tau)Bu_{n-1}(\tau)\|\,d\tau\,ds<\frac{\varepsilon}{11}.$$

Hence, (23) follows, and the proof is complete.

■

The next result is about the approximate null controllability of system (4).

Theorem 2.5. *Suppose that system (20) is approximately null controllable on the interval $[a,b]$ for any $a \geq 0$, and that there exists a function $q(\cdot) \in L^1(I,\mathbb{R}^+)$ such that $\|f(t,x)\| \leq q(t),\ \forall\,(t,x) \in I\times\mathscr{C}(I,X)$. Then, system (4) is approximately null controllable on I.*

Proof. Given any $\varepsilon > 0$, since (20) is approximately null controllable on $[0,b]$, there exists a control function $v_0 \in L^2([0,b];\Lambda)$ such that

$$\|R(b)x(0)+L(0,b)v_0\|<\frac{\varepsilon}{11}.$$

Select a sequence $\{t_n\}$ as in the proof of Theorem 2.4. Let $x_1 := x(t_1;0,x(0),v_0)$. Again, the approximate null controllability of (20) on $[t_1,b]$ implies that there exists a control function $v_1 \in L^2([t_1,b];\Lambda)$ such that

$$\|R(b-t_1)x_1+L(t_1,b)v_1\|<\frac{\varepsilon}{11}.$$

Similar to the proof of Theorem 2.4, we obtain three sequences $x_n,\ v_n$ and u_n such that $v_n(\cdot) \in L^2([t_n,b];\Lambda),\ \ u_n(\cdot) \in L^2([0,b];\Lambda)$, defined as follows:

$$u_n(t)=\begin{cases} u_{n-1}(t), & 0\leq t\leq t_n\\ \\ v_n(t), & t_n\leq t\leq b. \end{cases} \tag{26}$$

$x_n = x(t_n;0,x(0),u_{n-1}),\quad \|R(b-t_n)x_n+L(t_n,b)v_n\| < \frac{\varepsilon}{11}$.

It follows that,

$$\begin{aligned}
&\|x(b;0,x(0),u_n)\|\\
\leq\ &\|R(b-t_n)x_n+L(t_n,b)u_n\|+\|N(t_n,b)u_n\|\\
+\ &\|[R(b)-R(t_n)]x_0\|+\|[R(b)-R(t_n)]g(x)\|+\|[R(0)-R(t-t_n)]x_n\|\\
+\ &\int_0^{t_n}\left\|\Big(R(b-s)-R(t_n-s)\Big)\sigma Bu_{n-1}(s)\right\|ds
\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^{t_n} \left\| A\Big(R(b-s)-R(t_n-s)\Big)Bu_{n-1}(s)\right\| ds \\
&+ \int_0^{t_n} \left\| R(b-s)-R(t_n-s)\right\| \int_0^s \|\gamma(s-\tau)Bu_{n-1}(\tau)\|\, d\tau\, ds \\
&+ \int_0^{t_n} \left\| R(b-s)-R(t_n-s)\right\| \|f(s,x(s))\|\, ds \\
&+ \int_{t_n}^b \|R(b-s)\| \int_0^{t_n} \|\gamma(s-\tau)Bu_{n-1}(\tau)\|\, d\tau\, ds + \int_{t_n}^b \int_0^{t_n} \|\gamma(s-\tau)R(\tau)Bu_{n-1}(\tau)\|\, d\tau\, ds \\
&< \frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+M_g\|R(b)-R(t_n)\| + \int_{t_n}^b \|R(b-s)f(s,x_s)\|\, ds \\
&+ \int_0^{t_n} \left\| R(b-s)-R(t_n-s)\right\| q(s) ds \\
&+ M\int_{t_n}^b \int_0^{t_n} \|\gamma(s-\tau)Bu_{n-1}(\tau)\|\, d\tau\, ds + \int_{t_n}^b \int_0^{t_n} \|\gamma(s-\tau)R(\tau)Bu_{n-1}(\tau)\|\, d\tau\, ds \\
&< \frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+\frac{\varepsilon}{11}+M\int_{t_n}^b q(s)\, ds \\
&+ M\int_{t_n}^b \int_0^{t_n} \|\gamma(s-\tau)Bu_{n-1}(\tau)\|\, d\tau\, ds + \int_{t_n}^b \int_0^{t_n} \|\gamma(s-\tau)R(\tau)Bu_{n-1}(\tau)\|\, d\tau\, ds \\
&< \varepsilon.
\end{aligned}
$$

For a sufficiently large n such that $M\int_{t_n}^b q(s)\, ds < \frac{\varepsilon}{11}$ and by the continuity of $R(\cdot)$ and that of $AR(\cdot)$ in the operator norm topology (Lemma 1.3), and the Lebesgue dominated convergence theorem, such that

$$
\int_0^{t_n} \left\| \Big(R(b-s)-R(t_n-s)\Big)\sigma Bu_{n-1}(s)\right\| ds + \int_0^{t_n} \left\| A\Big(R(b-s)-R(t_n-s)\Big)Bu_{n-1}(s)\right\| ds < \frac{\varepsilon}{11}+\frac{\varepsilon}{11},
$$

Similarly,

$$
\int_0^{t_n} \left\| R(b-s)-R(t_n-s)\right\| \int_0^s \|\gamma(s-\tau)Bu_{n-1}(\tau)\|\, d\tau\, ds < \frac{\varepsilon}{11}.
$$

Also we have for sufficiently large n that

$$
M\int_{t_n}^b \int_0^{t_n} \|\gamma(s-\tau)R(\tau)Bu_{n-1}(\tau)\|\, d\tau\, ds < \frac{\varepsilon}{11},
$$

$$
\int_{t_n}^b \int_0^{t_n} \|\gamma(s-\tau)R(\tau)Bu_{n-1}(\tau)\|\, d\tau\, ds < \frac{\varepsilon}{11}.
$$

Hence, we have shown that for any $\varepsilon > 0$ and $x_0 \in X$, there exists a control function $u(\cdot) \in L^2(I;\Lambda)$ such that $\|R(b)x(0)+L(0,b)u+N(0,b)u\| < \varepsilon$, and the proof is complete.

■

We now illustrate our main result by the following example.

3. ILLUSTRATIVE EXAMPLE

Let Ω be bounded domain in $\mathbb{R}^n$ with smooth boundary Γ and consider the following nonlinear integrodifferential equation.

(27)
$$\begin{cases} \dfrac{\partial v(t,\xi)}{\partial t} = \Delta v(t,\xi) + \displaystyle\int_0^t \zeta(t-s)\Delta v(s,\xi)\,ds + f(t,v(t,\xi)) \ \text{ for } t \in I = [0,1] \text{ and } \xi \in \Omega \\ v(t,\xi) = u(t) \text{ for } t \in [0,1] \text{ and } \xi \in \Gamma \\ v(0,\xi) = v_0(\xi) + g(v)(\xi) \text{ for } \ \xi \in \Omega, \end{cases}$$

where $\zeta \in \mathscr{C}^1(\mathbb{R}^+,\mathbb{R}^+)$, and f and g satify $(\mathbf{H_7}) - (\mathbf{H_8})$.
Let $X = L^2(\Omega)$, $E = H^{-1/2}(\Gamma)$, $U = L^2(\Gamma)$, $B_1 = Id$, $\mathscr{D}(\sigma) = \{v \in L^2(\Omega) : \Delta v \in L^2(\Omega)\}$, and $\sigma = \Delta$
We define $A : \mathscr{D}(A) \subset X \to X$ by:

$$\begin{cases} \mathscr{D}(A) = H^2(\Omega) \cap H_0^1(\Omega) \\ Av = \Delta v \ \text{ for } v \in \mathscr{D}(A). \end{cases}$$

Then, one can easily convert system (27) into its abstract form like equation (4).
We have the following result.

Theorem 3.1. *(***Theorem 4.1.2, p. 79 of** [23]*) The operator A defined above is the infinitesimal generator of a C_0-semigroup $S(t)_{t\geq 0}$ of contractions on $L^2(\Omega)$. Moreover, A is self-adjoint and $(\mathscr{D}(A), \|\cdot\|_{\mathscr{D}(A)})$ is continuously included in $H_0^1(\Omega)$. If Ω is bounded with $\mathscr{C}^1$-boundary, then $(\mathscr{D}(A), \|\cdot\|_{\mathscr{D}(A)})$ is compactly imbedded in $L^2(\Omega)$.*

By Theorem 3.1 above, A generates a c_0-semigroup $\big(S(t)\big)_{t\geq 0}$ of contractions on $L^2(\Omega)$. Moreover, $\big(S(t)\big)_{t\geq 0}$ generated by A above, is compact for $t > 0$ (see Corollary 6.3.2, p.143 of [23]) and therefore is operator-norm continuous for $t > 0$. Hence by Theorem 1.4, the corresponding resolvent operator is operator-norm continuous for $t > 0$.
The operator τ is the trace operator $\gamma_0 v$ which is well defined and belongs to $H^{-1/2}(\Gamma)$ for each $v \in \mathscr{D}(\sigma)$. Clearly, assumptions $(\mathbf{H_1}) - (\mathbf{H_4})$ are satisfied. Define the linear operator

$B : L^2(\Gamma) \to L^2(\Omega)$ by $Bu = w_u$, where $w_u \in L^2(\Omega)$ is the unique solution to the Dirichlet boundary-value problem:

$$\begin{cases} \Delta w_u = 0 \quad \text{in } \Omega \\ w_u = u \;\text{ on } \Gamma. \end{cases} \tag{28}$$

From [3], we have that for every $u \in H^{-1/2}(\Gamma)$, equation (28) has a unique solution $w_u \in L^2(\Omega)$ satisfying $\|Bu\|_{L^2(\Omega)} = \|w_u\|_{L^2(\Omega)} \leq C_1\|u\|_{H^{-1/2}(\Gamma)}$, for some $C_1 > 0$. This shows that $(\mathbf{H_5})$ is satisfied. Also, in [36], it was proven that there exists a constant $C_2 > 0$ independent of u and t such that

$$\|AS(t)Bu\|_{L^2(\Omega)} \leq C_2 t^{\theta-1}\|u\|_{L^2(\Gamma)}, 0 < \theta < 1 \tag{29}$$

for all $u \in L^2(\Gamma)$ and $t > 0$.

Now consider the following system:

$$\begin{cases} x'(t) = Ax(t) + \int_0^t [\gamma_1(t-s) + \gamma_2(t-s)]\,x(s)ds \;\text{ for t} \geq 0 \\ x(0) = x_0 \in X, \end{cases} \tag{30}$$

where $\gamma_1(t)$ and $\gamma_2(t)$ are closed linear operators in X and satisfy $(\mathbf{H_2})$.
Then we have the next Lemma coming from [35].

Lemma 3.2. *(Perturbation result)([35]) Suppose A satisfies* $(\mathbf{H_1})$ *and* $\big(\gamma_1(t)\big)_{t\geq 0}$ *and* $\big(\gamma_2(t)\big)_{t\geq 0}$ *satisfy* $(\mathbf{H_2})$. *Let* $\big(R_{\gamma_1}(t)\big)_{t\geq 0}$ *be a resolvent operator of equation* (5) *and* $\big(R_{\gamma_1+\gamma_2}(t)\big)_{t\geq 0}$ *be a resolvent operator of equation* (30). *Then*

$$R_{\gamma_1+\gamma_2}(t)x - R_{\gamma_1}(t)x \;=\; \int_0^t R_{\gamma_1}(t-s)Q(s)x\,ds \tag{31}$$

where the operator Q is defined by

$$Q(t)x \;=\; \int_0^t \gamma_2'(t-s)\int_0^s R_{\gamma_1+\gamma_2}(\tau)x\,d\tau\,ds \;+\; \gamma_2(0)\int_0^t R_{\gamma_1+\gamma_2}(s)x\,ds,$$

is uniformly bounded on bounded intervals, and for each $x \in X$, $Q(\cdot)x$ *belongs to* $\mathscr{C}([0,\infty),X)$.

Corollary 3.3. *([35]) Let A be a closed, densely defined linear operator in X,* $\gamma(t) = 0$ *for all* $t \geq 0$, *and* $\big(R(t)\big)_{t\geq 0}$ *be a resolvent operator for equation* (5). *Then* $\big(R(t)\big)_{t\geq 0}$ *is a* C_0*-semigroup with infinitesimal generator A.*

By Lemma 3.2 and Corollary 3.3 (applied to equation (30)), when $\gamma_1 = 0$, $\left(R_{\gamma_1}(t)\right)_{t\geq 0}$ is a C_0-semigroup (denoted $S(t)$) with infinitesimal generator A. By replacing $R_{\gamma_1}(t)$ in equation (31) by $S(t)$, we have the following relationship between the semigroup $S(t)_{t\geq 0}$ and the resolvent operator $R(t)_{t\geq 0}$:

$$R(t)x \;=\; S(t)x + \int_0^t S(t-s)Q(s)x\,ds \tag{32}$$

for $x \in X$, and where the operators $\left(Q(t)\right)_{t\geq 0}$ are uniformly bounded for t on bounded intervals. Now using (29) and (32), we have that there exists a constant $C_3 > 0$ such that

$$\|AR(t)Bu\|_{L^2(\Omega)} \leq C_3 t^{\theta-1}\|u\|_{L^2(\Gamma)},\ 0<\theta<1$$

for all $u \in L^2(\Gamma)$ and $t > 0$. In other words, hypothesis $(\mathbf{H_6})$ holds with $\delta(t) = C_3 t^{-1/3}$, taking $\theta = \frac{2}{3}$. Hence, system (27) can be formulated in the form of equation (4).

The associated linear part to equation (27) is given by:

$$\begin{cases} \dfrac{\partial v(t,\xi)}{\partial t} = \Delta v(t,\xi) + \displaystyle\int_0^t \zeta(t-s)\Delta v(s,\xi)\,ds \ \text{ for } t \in I = [0,1] \text{ and } \xi \in \Omega \\ v(t,\xi) = u(t) \text{ for } t \in [0,1] \text{ and } \xi \in \Gamma \\ v(0,\xi) = v_0(\xi) + g(v)(\xi) \text{ for } \xi \in \Omega, \end{cases} \tag{33}$$

From Theorem 2 of [1], equation (33) is approximately boundary controllable on any interval $[b,1]$ with $b \geq 0$. Hence, it follows that system (27) is approximately controllable on I if the nonlinear perturbation function f satisfies hypothesis $(\mathbf{H_7})$ and the function g is chosen to satisfy hypothesis $(\mathbf{H_8})$.

CONCLUSION

This paper contains the approximate boundary controllability of some partial functional integrodifferential differential equation with nonlocal initial condition in Banach spaces. We use the resolvent operator theory, and fixed point theory techniques to prove the existence of mild solutions. The result shows that without assuming the compactness of the resolvent operator for the associated linear homogeneous part, one can obtain approximate controllability results under some sufficient conditions such as the approximate controllability of the associated linear homogeneous part. Moreover, the example presented in Section 5 illustrates an application of the obtained results.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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