Available online at http://scik.org

J. Semigroup Theory Appl. 2013, 2013:4

ISSN 2051-2937

A NOTE ON TOPOLOGICAL SEMIGROUP-GROUPOID

MOHAMMAD QASIM MANN'A

Department of Mathematics, College of Science, Al-Qassim University

P.O. Box: 6644-Buraidah: 51452, Saudi Arabia

Abstract: In this paper we prove that the set of homotopy classes of paths in topological semigroup is a semigroup-groupoid. Further, we define the category TSGCov/X of topological semigroup coverings of X and prove that its equivalent to the category SGpGpdCov/ $\pi_1 X$ of covering groupoids of the semigroup-groupoid $\pi_1 X$ **.** We also prove that the topological semigroup structure of a topological semigroup-groupoid lifts to a universal topological covering groupoid.

Keywords: Fundamental groupoid, covering space, topological semigroup, groupoid, topological groupoid.

2000 AMS Subject Classification: 14H30, 57M10, 22A15, 20L05, 22A22.

1. Introduction

A *groupoid* ($[1],[2]$)G is small category consists of two sets G and O_G , called respectively the set of elements (or arrows) and the set of objects (or vertices) of the groupoid, together with, two maps $\alpha, \beta : G \to O_G$, called respectively the source and target maps, the map $\varepsilon : O_G \to G$, written as $\varepsilon(x) = 1_x$, where 1_x is called the identity element at x in O_G , and ε is called the object map and the partial

Received February 24, 2013

multiplication map,

$$\gamma : G \underset{\alpha=\beta}{\times} G \to G \text{ written } \gamma(g,h) = g \circ h \text{, on the set}$$

$$G \underset{\alpha=\beta}{\times} G = \{(g,h) \in G \times G : \alpha(g) = \beta(h)\}$$

These terms must satisfy the followings axioms:

$$\alpha(g \circ h) = \alpha(h) \; and \; \beta(g \circ h) = \beta(g),$$

$$(g \circ h) \circ k = g \circ (h \circ k),$$

$$\alpha(1_x) = \beta(1_x) = x,$$

$$g \circ 1_{\alpha(g)} = g \; and \; 1_{\beta(g)} \circ g = g,$$

$$For\ all\ g, h, k \in G \; and \; x \in O_G.$$

For a groupoid *G*, we will denote to the inverse map by:

$$\sigma : G \to G \text{, such that } g \to g^{-1},$$

Further, for we write $G(x,y)$ as the set of all morphisms $g \in G$ such that $\alpha(g) = x$ and $\beta(g) = x$. each $x, y \in O_G$ We will write $st_G(x) = \alpha^{-1}(x)$ and $\cos t_G(x) = \alpha^{-1}(x)$. The vertex group at x is $G(x) = G(x,x) = st_G(x) \cap \cos t_G(x)$. We say G is transitive (resp. 1-transitive, simply transitive) if for each $x, y \in O_G$, $G(x,y)$ is non-empty (resp. a singleton, has no more than one element). A *morphism* of groupoids *H* and *G* is a functor, that is, it consists of a pair of functions $f : H \to G$ and $O_f : O_H \to O_G$ such that:

$f(g \circ h) = f(g) \circ f(h)$, $f(g^{-1}) = f(g)^{-1}$, $\beta_G.f = O_f.\beta_H$, $\alpha_G.f = O_f.\alpha_H$ and $f.\varepsilon_H = \varepsilon_G.O_f$. Where $g \circ h$ is defined.

A *topological groupoid* is a groupoid *G* together with topologies on *G* and O_G such that all the structure maps of *G* are continuous, that is: the source map α, the target map β, the object map ε, the inversion σ and the partial multiplication map

γ are all continuous. A *morphism* of topological groupoids is a pair of maps $f: G \to H$ and $O_f: O_G \to O_H$ such that f and O_f are continuous. Covering morphism of groupoids are defined in $[2]$ as: A morphism $f: G \to H$ of groupoids is called a *covering morphism* if each $x \in O_H$ then the restriction of mapping $f_x: st_G(x) \to st_H(O_f(x))$ is bijective. Let $f: G \to H$ be a morphism of groupoids. Then for an object $x \in O_G$ the subgroup $f[G(x)]$ of $H(O_f(x))$ is called the characteristic group of f at x. So if f is the covering morphism then f maps $G(x)$ isomorphically to $f[G(x)]$. We say that a covering morphism $f: G \to H$ of transitive groupoids is a universal covering morphism if G is 1-transitive.Let G be a topological space then $\pi_1(G)$ is a groupoid which is the set of all relative to end points homotopy classes of paths in topological space.

2. Preliminaries

We recall first from $[5]$.A *semigroup-groupoid* G is a groupoid endowed with a semigroup structure such that the semigroup multiplication $m: G \times G \to G$, $(g,h) \mapsto gh,$ is a morphism of groupoids. For $g,h \in G,$ the groupoid composition is denoted by $g \circ h$, where $\alpha(g) = \beta(h)$.

A *morphism* $f: G \to H$ of semigroup-groupoids is a morphism of underlying groupoids preserving the semigroup structure i.e. $f(gh) = f(g)f(h),$ for all $g,h \in G.$

In semigroup-groupoid. The *source*, *target* and *object maps are* morphisms (homomorphisms) of semigroup and the *inversion map* σ is an isomorphism of semigroups. Also, the interchange law holds, i.e. $(b \circ a)(g \circ h) = (bg) \circ (ah)$. A *topological semigroup-groupoid* G is semigroup-groupoid with the following two properties:

Tsg1. G is a topological groupoid,

Tsg2. the morphism of groupoids $m: G\times G \to G$ is continuous.

A *morphism* of topological semigroup-groupoids $f: G \to H$ is a morphism of the underlying topological groupoids preserving the topological semigroup structure. In topological semigroup-groupoid. The set of arrows G and the set of objects O_G are topological semigroups.

3. Main Results

Its well known from $[2]$ that, if X is topological space then $\pi_1(X)$ is a groupoid by using this fact we have the following result.

Proposition 3.1 *Let X be a topological semigroup. Then the fundamental groupoid* $\pi_1(X)$ *is semigroup-groupoid.*

Proof. Since X is topological semigroup, then the group multiplication $m: X\times X \to X$, defined by $(g,h) \mapsto gh$, is continuous. And since $\pi_1(X\times X)$ isomorphic to $\pi_1(X)\times \pi_1(X)$ then we have induced map,

$$\pi_1(m): \pi_1(X)\times \pi_1(X) \longrightarrow \pi_1(X), \qquad ([g],[h]) \mapsto [gh])$$

From $[2]$ we have $\pi_1(X)$ is a groupoid. So, to prove that $\pi_1(X)$ is semigroup-groupoid, we have to show associative law as follows:

$$[g]([h][r]) = [g][hr] = [g(hr)] = [(gh)r] = ([gh])[r] = ([g][h])[r].$$

We recall from $[3]$ a subset U of the space X liftable if it is open, path connected and the inclusion $U \to X$ maps each fundamental group $\pi_1(U,x)$, $x \in X$, to the trivial subgroup of $\pi_1(X,x)$. Further, if X has a universal covering (path connected, locally path connected and semilocally simply connected space) then each point $x \in X$ has a liftable subset of X.

Proposition 3.1 *Let X be a topological semigroup whose underlying space has a universal covering. Then the fundamental groupoid* $\pi_1(X)$ *becomes a topological semigroup-groupoid.*

Proof. Since $\pi_1(X)$ is a topological groupoid and $m: X \times X \to X$ is a continuous map then its enough to show that the induced map,

$$\pi_1(m): \pi_1(X) \times \pi_1(X) \to \pi_1(X), \qquad ([g],[h]) \mapsto [gh]$$

Is continuous. By assuming that X has a universal covering, each $x \in X$ has a liftable neighbourhood. Let U consist of such sets. Then $\pi_1(X)$ has a lifted topology[2] . So the set $\widetilde{W}$, consisting of all liftings of the sets in U, forms a basis for the topology on $\pi_1(X)$. Let $\widetilde{U}$ be an open neighbourhood of $[gh]$ and a lifting of U in U. Since the multiplication $m: X \times X \to X$ is continuous, there is a neighborhood $V \times H$ of (g, h) such that $m(V \times H) \subseteq U$. Using the condition on X and choosing $V \times H$ small enough we can assume that $V \times H$ has a liftable neighbourhood. Let $\widetilde{V} \times \widetilde{H}$ be the lifting of $V \times H$.Then we have $\pi_1 m(\widetilde{V} \times \widetilde{H}) \subseteq \widetilde{U}$. So, the map $\pi_1 m$ is continuous which implies that $\pi_1(X)$ is a topological semigroup-groupoid.

Proposition3.2 *Let $f: X \to Y$ be continuous morphism of topological semigroup, then $\pi_1(f)$ is a morphism of semigroup-groupoid.*

Proof. Its Known that from[2] $\pi_1(f)$ is morphism of groupoids. It remain that to show that $\pi_1(f)$ is a morphism of semigroup: $\pi_1(f)([g][h]) = \pi_1(f)([gh]) = [f(gh)]$ $= [f(g)f(h)] = [f(g)][f(h)] = \pi_1(f)([g])\pi_1(f)([h])$.

4. Covering

Let X be a topological space. Then we have a category denoted by TCov/X whose objects are covering maps $p:\widetilde{X}\to X$ and a morphism from $p:\widetilde{X}\to X$ to $q:\widetilde{Y}\to X$ is a map $f:\widetilde{X}\to\widetilde{Y}$ (f is a covering map) such that $p=qf$. Further for X we have a groupoid called a *fundamental groupoid* [2] and have a category denoted by GpdCov/$\pi_1 X$ whose objects are the groupoid coverings $p:\widetilde{G}\to\pi_1 X$ of $\pi_1 X$ and a morphism from $p:\widetilde{G}\to\pi_1 X$ to $q:\widetilde{H}\to\pi_1 X$ is a morphism of $f:\widetilde{G}\to\widetilde{H}$ groupoids (f is a covering morphism)such that $p=qf$. We recall the following result from Brown [2].

Proposition 4.1 *Let X be a space which has a universal covering. Then the category* TCov*/X of topological covering of X is equivalent to the category of* GpdCov/$\pi_1 X$ *of the fundamental groupoid of* $\pi_1 X$.

Let X and $\widetilde{X}$ are topological semigroups. A map $p:\widetilde{X}\to X$ is called a covering morphism of topological semigroups if *p* is a morphism of semigroups and *p* is a covering map on the underlying spaces. For a topological semigroup *X*, we have the category denoted by TSGCov/*X* whose objects are topological semigroup coverings $p:\widetilde{X}\to X$ and a morphism from $p:\widetilde{X}\to X$ to $q:\widetilde{Y}\to X$ is a map $f:\widetilde{X}\to\widetilde{Y}$ such that $p=qf$. For a topological semigroup X , the fundamental groupoid $\pi_1 X$ is a semigroup-groupoid and so we have a category denoted by SGpGpdCov/$\pi_1 X$ whose objects are semigroup-groupoid coverings $p:\widetilde{G}\to\pi_1 X$ of $\pi_1 X$ and a morphism from $p:\widetilde{G}\to\pi_1 X$ to $q:\widetilde{H}\to\pi_1 X$ is a morphism of $f:\widetilde{G}\to\widetilde{H}$ semigroup-groupoids such that $p=qf$.

Proposition 4.2 *Let X be a topological semigroup whose underlying space has a universal covering. Then the category TSGCov/X of topological semigroup coverings of X is equivalent to the category SGpGpdCov/*$\pi_1 X$ *of covering groupoids of the semigroup-groupoid* $\pi_1 X$ **.**

Proof. Define a functor $\pi_1 : TSGpCov/X \to SGpCov/\pi_1 X$ as follows:

Let $p:\widetilde{X} \to X$ be a covering morphism of topological semigroups. Then the induced morphism $\pi_1 p : \pi_1\widetilde{X} \to \pi_1 X$ is a covering morphism of groupoids [2]. Moreover, the morphism $\pi_1 p$ preserves the semigroup structures as:

$$\pi_1 p([\tilde{g}\tilde{h}]) = [p(\tilde{g}\tilde{h})] = [p(\tilde{g})p(\tilde{h})] = [p(\tilde{g})][p(\tilde{h})] = \pi_1 p([\tilde{g}])\pi_1 p([\tilde{h}]).$$

So, $\pi_1 p$ becomes a covering morphism of semigroup-groupoids. We define a functor $\Psi : SGpCov/\pi_1 X \to TSGpCov/X$ as follows: If $q:\widetilde{G} \to \pi_1 X$ is a covering morphism of semigroup-groupoid, then we have a covering map $p:\widetilde{X} \to X$, where $p = O_q$ and $\widetilde{X} = O_G$. Further, we shall show that the semigroup multiplication $\tilde{m}:\widetilde{X}\times\widetilde{X} \to \widetilde{X}, (\tilde{g},\tilde{h}) \mapsto \tilde{g}\tilde{h}$ is continuous. Since *X* has a universal covering, we can choose a cover U of liftable subsets of *X*. Since the topology on $\widetilde{X}$ is lifted topology [2] the set consisting of all liftings of the sets in U forms a basis for the topology on $\widetilde{X}$. Let $\widetilde{U}$ be an open set of $\tilde{g}\tilde{h}$ and a lifting of U in U. Since the multiplication,

$$m: X\times X \to X, \quad (g,h) \mapsto gh$$

Is continuous, there is a basic open set $H\times T$ of (g,h) in $X\times X$ such that $m(H\times T) \subseteq U$. Assume that H and T are liftable subsets. Let $\widetilde{H}$ and $\widetilde{T}$ be a lifting of H and T respectively. Then $p\tilde{m}(\widetilde{H}\times\widetilde{T}) = m(H\times T) \subseteq U$ and so we have $\tilde{m}(\widetilde{H}\times\widetilde{T}) \subseteq \widetilde{U}$. So, $\tilde{m}$ is continuous. Further, since p is continuous, and $m(p\times p) = p\tilde{m}$ then p is a morphism of topological semigroups. Since by Proposition (4.1) the category of topological space covering is equivalent to the category of groupoid coverings, the proof is completed by the following digram:

$$\begin{array}{ccc} TSGCov/X & \xrightarrow{\pi_1} & SGGpd/\pi_1 X \\ \downarrow & & \downarrow \\ TCov/X & \xrightarrow{\pi_1} & Gpd/\pi_1 X. \end{array}$$

Definition 4.3 Let be $p:\widetilde{G}\to G$ a covering morphism of groupoids and $q:H\to G$ a morphism of groupoids. If there exists a unique morphism $\widetilde{q}:H\to\widetilde{G}$ such that $q=p\widetilde{q}$ we say q lifts to $\widetilde{q}$ by p $[2]$.

We recall the following theorem from $[2]$ which is an important result to have the lifting maps on covering groupoids.

Theorem 4.4 *Let $p:\widetilde{G}\to G$ be a covering morphism of groupoids, $x\in O_G$. and $\widetilde{x}\in\widetilde{O}_G$. such that $O_p(\widetilde{x})=x$. Let $q:H\to G$ be a morphism of groupoids such that H is transitive and such that $O_q(\widetilde{y})=x$. Then the morphism $q:H\to G$ uniquely lifts to a morphism $\widetilde{q}:H\to\widetilde{G}$ such that $O_{\widetilde{q}}(\widetilde{y})=\widetilde{x}$ if and only if $q[H(\widetilde{y})]$ $\subseteq p[\widetilde{G}(\widetilde{x})]$, where $H(\widetilde{y})$ and $G(\widetilde{x})$ are the object groups.*

Let G be a semigroup-groupoid, $x\in O_G$, and let $\widetilde{G}$ be just a groupoid, $\widetilde{x}\in\widetilde{O}_G$. Let $p:\widetilde{G}\to G$ be a covering morphism of groupoids such that $p(\widetilde{x})=x$. We say the semigroup structure of G lifts to $\widetilde{G}$ if there exists a semigroup structure on $\widetilde{G}$ such that $\widetilde{G}$ is a groupoid and $p:\widetilde{G}\to G$ is a morphism of semigroup-groupoids. Further, from $[4]$ an element g of G is *idempotent* if $gg=g$.

Theorem 4.5 *Let G be a semigroup-groupoid and $\widetilde{G}$ be a groupoid. Let $p:\widetilde{G}\to G$ be a universal covering on the underlying groupoids such that G and $\widetilde{G}$ are transitive groupoids. Let $x\in O_G$ and idempotent, $\widetilde{x}\in O_{\widetilde{G}}$ such that $O_P(\widetilde{x})=x$. Then the semigroup structure of G lifts to $\widetilde{G}$ and $\widetilde{x}$ is an idempotent .*

Proof. Since G is a semigroup-groupoid then it has the following map,

$$m : G \times G \to G, \quad m(g,h) = gh.$$

Since $\widetilde{G}$ is universal covering, the object group $\widetilde{G}(\widetilde{x})$ has one element at most. So, by Theorem(4.4) the map m lift to the map

$$\widetilde{m} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}, \quad \widetilde{m}(\widetilde{g}, \widetilde{h}) = \widetilde{g}\widetilde{h}$$

by $p : \widetilde{G} \to G$ such that $p(\widetilde{g}\widetilde{h}) = p(\widetilde{g})p(\widetilde{h})$. Since the multiplication m is associative then we have $m(m \times I) = m(I \times m)$, where I denotes to the identity map. Then by (4.4) these maps $m(m \times I)$ and $m(I \times m)$ respectively lift to $\widetilde{m}(\widetilde{m} \times I)$ and $\widetilde{m}(I \times \widetilde{m})$ which coincide on $(\widetilde{x}, \widetilde{x}, \widetilde{x})$. So, by the uniqueness of the lifting we have $\widetilde{m}(\widetilde{m} \times I) = \widetilde{m}(I \times \widetilde{m})$ this means that $\widetilde{m}$ is associative. Further, since $O_{\widetilde{m}}(\widetilde{x}, \widetilde{x}) = \widetilde{x}$ then $\widetilde{x}\widetilde{x} = \widetilde{x}$ which means that $\widetilde{x}$ is an idempotent.

Result 4.6 *Let G be a topological semigroup-groupoid and $\widetilde{G}$ be a topological groupoid. Let $p : \widetilde{G} \to G$ be a universal covering on the underlying groupoids such that G and $\widetilde{G}$ are transitive groupoids. Let $x \in O_G$ and idempotent, $\widetilde{x} \in \widetilde{O}_G$ such that $O_P(\widetilde{x}) = x$. Then the semigroup structure of G lifts to $\widetilde{G}$.*

Proof. Since G is semigroup-groupoid and since $\widetilde{G}$ is universal covering then by Theorem(4.5) the map m left to the map $\widetilde{m}$ by $p : \widetilde{G} \to G$, $p(\widetilde{g}\widetilde{h}) = p(\widetilde{g})p(\widetilde{h})$ and $\widetilde{m}$ is associative ($\widetilde{m}(\widetilde{m} \times I) = \widetilde{m}(I \times \widetilde{m})$). Further, by Theorem (4.4), we have $\widetilde{m}$ is continuous. So, $\widetilde{G}$ becomes a topological semigroup-groupoid.

REFERENCES

[1] M.E.-S.A.-F. Aof, Topological Aspects of Holonomy Groupoids. Ph.D. thesis, University of Wales, Bangor, 1988.

[2] R. Brown, Topology and groupoid. BookSurge LLC, U.K 2006.

[3] Fatih Ozcan , I. Icen and M. Habil Gursoy , Topological Ring Groupoids and liftings , Iranian journals of science and technology , transaction A, 30 (2006) ,355-362

[4] J. M. Howie, Fundamentals of semigroup theory. Oxford :Oxford University Press,1995

[5] M. Manna and M A. Abdulkadir, Topological semigroup-groupoid. Second Conference on Research and Education in Mathematics (ICREM 2). Institute for Mathematical research (INSPEM). 25-27 May 2005.