Available online at http://scik.org
J. Semigroup Theory Appl. 2013, 2013:5
ISSN 2051-2937

A COMPUTER APPLICATION ON ARF SEMIGROUPS AND TYPE SEQUENCES OF NUMERICAL SEMIGROUPS

SEDAT İLHAN*, CENGİZ COŞKUN AND AZİZ M. YÜCELEN

Department of Mathematics, Dicle University, Diyarbakır, Turkey

Abstract: In this paper, we give a brief description of the properties of numerical semigroup. In addition, we implement algorithms to determine the type sequence and to investigate whether any numerical semigroup is Arf using C++ programming language.

Keywords: Numerical semigroup, Apery set, gaps, symmetric, Arf.

2000 AMS Subject Classification: 20M14

1. Introduction

A numerical semigroup S is a subset of $\mathbb{N}$ (the set of nonnegative integers) closed under addition, $0 \in S$.

For a numerical semigroup S, $A = \{s_1, s_2, ..., s_p\} \subset S$ is a generating set of S provided that $S = < s_1, s_2, ..., s_p > = \{ k_1 s_1 + k_2 s_2 + ... + k_p s_p : k_i \in \mathbb{N}, 1 \le i \le p \}$. The set $A = \{s_1, s_2, ..., s_p\}$ is called minimal generating set of S if no proper subset is a generating set of S. It was observed in [1] that the set $\mathbb{N} \backslash S$ is finite if and only if $gcd\{ s_1, s_2, ..., s_p \} = 1$ (gcd stands for greatest common divisor).

---

*Corresponding author

Received February 27, 2013

Another important invariant of S is the largest integer not belonging to S , known as the Frobenius number of S and denoted by $g(S)$, that is $g(S) = \max\{ x \in \mathbb{Z} : x \notin S \}$ ([1]). We define $n(S) = \#(\{0,1,2,..., g(S)\} \cap S)$ where $\#(A)$ denotes the cardinality of A. Furthermore, the set determine of S is defined as $N(S) = \{ s \in S : s < g(S) \}$. It is also well-known that $S = \{0, s_1, s_2, ..., s_{n-1}, s_n = g(S)+1, \rightarrow ...\}$ where $\rightarrow$ means that every integer greater than $g(S)+1$ belongs to S, $n = n(S)$ and $s_i < s_{i+1}$ for $i = 1,2,...,n$.

For $m \in S \setminus \{0\}$, the Apery set of m in S is the set $Ap(S,m) = \{ s \in S : s - m \notin S \}$. It can easily be proved that $Ap(S,m)$ is formed by the smallest elements of S belonging to the different congruence classes $\mod m$. According to this, we can write $\#(Ap(S,m)) = m$ and $g(S) = \max(Ap(S,m)) - m$. Various aspects and properties of Apery sets are detailed in [4, 5]. The elements of $\mathbb{N} \backslash S$, denoted by $G(S)$, are called the gaps of S, that is $G(S) = \{ x : x \in \mathbb{N}, x \notin S \}$ ([6]) .

Given a numerical semigroup S, the set of gaps of S is defined as $G(S) = \mathbb{N} \backslash S$, and an element $x \in G(S)$ is said to be fundamental gap of S if $\{2x, 3x\} \subset S$, and the set of all fundamental gaps of S is written as $F(S) = \{ x \in G(S) : \{2x, 3x\} \subset S \}$ ([6]).

S is symmetric if for every $x \in \mathbb{Z} \backslash S$, the integer $g(S) - x \in S$. Similarly, a numerical semigroup S is Pseudo-Symmetric if $g(S)$ is even and the only integer such that $x \in \mathbb{Z} \backslash S$ and $g(S) - x \notin S$ is $x = \frac{g(S)}{2}$ ([7]). Especially, $S = < s_1, s_2 >$ which is generated by two elements is symmetric numerical semigroup and the Frobenius number of S is $g(S) = s_1 s_2 - s_1 - s_2$ ([3]).

The holes of S is defined by $H(S) = \ x \in \mathbb{Z} : x \notin S \ , \ g(S) - x \notin S$. Thus, it can be written that S is symmetric if and only if $H(S) = \phi$ ([4]).

Now, we give follows definition from ([2]);

Let $S = \{0, s_1, s_2, ..., s_{n-1}, s_n = g(S) + 1, \rightarrow ...\}$ is numerical semigroup $S_i = \ x \in S : x \geq s_i$ and $S(i) = \ x \in \mathbb{N} : \ x + S_i \subset S$, and for $i = 0, 1, 2, ..., n(S) = k$. Thus, we write that

$S_k \subset S_{k-1} \subset ... \subset S_1 \subset S_0 = S = S(0) \subset S(1) \subset ... \subset S(k-1) \subset S(k) = \mathbb{N}$.

In this case, the number $t = t_1(S) = \# \ \ S \ 1 / S$ is defined as the type of S . So, we obtain the set $t_1, t_2, ..., t_{n(S)}$ which is defined as the type sequence of S , where

$t_i = t_i(S) = \# \ \ (S(i) / S(i-1))$, for $i = 0, 1, 2, ..., n(S) = k$.

S is called Arf numerical semigroup if $x + y - z \in S$, for all $x, y, z \in S$ where $x \geq y \geq z$ ([8]). Alternatively, S can be defined as Arf numerical semigroup using the type sequence of S : S is Arf numerical semigroup if $t_i = s_i - s_{i-1} - 1$, for every $1 \leq i \leq n(S)$ ([2]).

2. Preliminaries

Study of numerical semigroups necessitiates use of computer programs in the sense that it deals with numbers and sets which involves very large amount of operations that can be time consuming, error prone and bothersome to an extent.

Manually computing the type sequence of a given numerical semigroup S and determining whether S is an Arf Numerical Semigroup is a very time consuming and error prone task to accomplish. On the other hand using a computer program to generate the elements of the numerical semigroup from its generators and determining the type sequence – a series of processes which includes the constructing of S(i) sets and counting the number of elements of set differences - can be achieved easily and accurately.

Likewise, given any numerical semigroup S, checking whether S is an Arf Numerical Semigroup requires similar hand work which can easily be achieved by the help of comupter programs.

3. Main results

3.1. Computer Program Menu

Our program, implemented with C++ programming language, has a main menu that incorporates many operations on numerical semigroups. Some of these operations had been added previously by İlhan et al. in their studies ([9]).

In this study we have added the modules to determine the type sequence of a given numerical semigroup (H) and another two modules that determine whether the given numerical semigroup is an Arf numerical semigroup (I, J). The newly added menu items (I) and (J) use two different methods to achieve the same goal.

Main menu of our program includes the following menu items;

A - Enter generators and compute the numerical semigroup, S.

B - Compute H(S), the holes of S.

C - Compute G(S),the set of gaps of S

D - Compute F(S),the set of fundamental gaps of S

E - Compute N(S),the set determinant of S

F – Apery set

G - Display the elements of S.

H - Type Sequence.

I - Is the Semigroup an Arf? (Algorithm 1)

J - Is the Semigroup an Arf? (Algorithm 2)

X - Leave this world.

3.2. Contribution to Computer Program

Initially this program is a product of Grace and Craven. They designed this program in their study of their master thesis. Then, it was extended to include some more functionalities by İlhan et al ([9]). In this study, we have included three more functionalities to determine the type sequence of a numerical semigroup and to determine whether the given numerical semigroup is an Arf Numerical Semigroup using C++ programming language.

The first algorithm to determine the Arf'ness of a numerical semigroup depends on the type sequence and the second algorithm depends on the requirement of the existense of every value of $x+y-z$ in the semigroup itself for $z=\min(x,y,z)$. Menu items H, I and J are the interfaces added for the contribution to this program.

3.3. Methods

Determining the type sequence of any numerical semigroup defined as the cardinality of the set $(S(i)/S(i-1))$ where $S_i = \; x \in S : x \geq s_i$ and $S(i) = \; x \in \mathbb{N} : \; x + S_i \subset S$ for every $1 \leq i \leq n(S)$ is implemented, thus reducing the burden and the error propagation risks of manual calculations. Constructing the sets S(i) and Si , and then calculating the cardinality of the difference of the sets $S(i), S(i-1)$ for every value of $1 \leq i \leq n(S)$, manually requires a hard work and is open to errors. Our program eliminates the cost of the burden and reduces the error risks.

In this study Arf'ness of a numerical semigroup is determined by two different algorithms. First algorithm uses the type sequence of the numerical semigroup to check whether $\mathrm{t}_i = \mathrm{s}_i - \mathrm{s}_{i-1} - 1$. This algorithm obviously requires the calculation of ti values for every $1 \leq i \leq n(S)$. The second algorithm uses the restriction that for all $x, y, z \in S$ where $z = \min(x,y,z)$, the statement $x+y-z \in S$ holds. Using our computer program it becomes easier and error free.

Thus, our program can be used in scientific studies in numerical semigroups to improve effectiveness and accuracy. Our product can contribute to:

- New product implementations,
- Scientific studies on related sciences and multidisciplinary studies,
- Similar and new approaches in numerical semigroups.

REFERENCES

[1] V. Barucci, D.E. Dobbs and M. Fontana, Maximality properties in numerical semigroups and applications to one –dimensional analytically irreducible local domains, Mem. Amer. Math. Soc. 125/598 (1997), 13-25.

[2] Marco D'Anna, Type sequences of numerical semigroups, Semigroup Forum 56/1 (1998), 1-31.

[3] R. Froberg, C. Gottlieb and R. Haggkvist, On Numerical Semigroups, Semigroup Forum 35/1 (1987), 63-83.

[4] M. Madero, Apery Sets of Numerical semigroups, Master Thesis, (2003).

[5] J.C. Rosales, Numerical Semigroups with Apery sets of unique expression, J. Algebra 226/1 (2000), 497-487.

[6] J.C. Rosales, Fundamental gaps of numerical semigroups generated by two elements, Linear Algebra Appl. 405 (2008), 200-208.

[7] J.C. Rosales, One half of a pseudo-symmetric numerical semigroup, Bull. London Math. Soc. 40/2 (2008), 347-352.

[8] J.C. Rosales, P.A. Garcia-Sanchez, J.I. Garcia-Garcia and M.B. Branco, Arf Numerical Semigroups, Journal of Algebra 276 (2004), 3-12.

[9] S. İlhan, A. Baykal and A. Kaya, Some Applications of Computational Semigroups, International Journal of Information Science and Computer Mathematics, Vol. 1, Number 1 (2010), 49-57.