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SOME RESULTS IN THE α-NORM FOR SOME NONLINEAR SECOND ORDER DIFFERENTIAL EQUATION WITH FINITE DELAY IN BANACH SPACE

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Abstract. This paper investigates the existence, regularity and compactness property in the α-norm for some second order nonlinear differential equations with finite delay in Banach spaces. The theory of the cosine family, the contraction principle, and Schauder's fixed point theorem are used to establish global existence, continuous dependence on initial data, blowing up of solutions, local existence, and compactness of the flow. Furthermore, some sufficient conditions are given to ensure the regularity of the solutions. Finally, an example is given to illustrate the theoretical results.

Keywords: cosine family; finite delay; mild and strict solutions; α-norm; second order functional differential equations.

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1. Introduction

This work consists to study the existence, regularity and compactness property in the α-norm for the following nonlinear second order differential equation

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$$
(1.1) \qquad \begin{cases} u''(t) = Au(t) + f(t, u_t, u'_t) \text{ for } t \geq 0, \\ u_0 = \varphi \in \mathscr{C}_\alpha, \\ u'_0 = \varphi' \in \mathscr{C}_\alpha, \end{cases}
$$

where A is the (possibly unbounded) infinitesimal generator of strongly continuous cosine family of linear operators in X. $\mathscr{C}_\alpha = C^1([-r,0], D((-A)^\alpha))$, $0 < \alpha < 1$, denotes the space of continuous differentiable functions from $[-r,0]$ into $D((-A)^\alpha)$, $(-A)^\alpha$ is the fractional α-power of A. This operator $((-A)^\alpha, D((-A)^\alpha))$ will be describe later. C_α is endowed with the following norm $\|h\|_{\mathscr{C}_\alpha} = \|h\|_\alpha + \|h'\|_\alpha$ for all $h \in \mathscr{C}_\alpha = C^1([-r,0], X_\alpha)$, where $\|h\|_\alpha = \sup_{-r \leq 0 \leq 0} |h(\theta)|_\alpha$. The norm $|.|_\alpha$ will be specified later. For $u \in C^1([-r,b], D((-A)^\alpha))$ $t \geq 0$, $b > 0$, and $t \in [0,b]$ u_t denotes the history function of $\mathscr{C}_\alpha$ defined by

$$
u_t(\theta) = u(t+\theta) \text{ for } \theta \in [-r,0],
$$

$f : \mathbb{R}^+ \times \mathscr{C}_\alpha \times \mathscr{C}_\alpha \to X$ and $g : \mathbb{R}^+ \times \mathscr{C}_\alpha \to X_\alpha$ are given functions.

In [14] the authors study firstly the abstract semi-linear second order initial value problem and secondly they unify and simplify some ideas from strongly continuous cosine families of linear operators in Banach spaces.

Second-order differential equations occur in many areas of science and engineering. One aspect of the study of second-order systems is the equivalent formulation of first-order equations, but this transformation may lack important information about the original evolutionary systems. In many cases, it's better to work directly with abstract second-order differential equations than to convert them into first-order systems. For this reason, Fitzgibbon [15] used the abstract second-order differential equations to establish the boundedness of solutions to the equation governing the transverse motion of an extensible beam. A useful approach to the study of abstract second-order differential equations is the theory of strongly continuous cosine families. Note that the cosine families of operators were first studied extensively by G. Da Prato [4] and M. Sova [10, 11]. Some recent contributions have made use of cosine families in studies of the quantitative theory of differential equations and control theory. We can refer the reader to [6, 5, 12] and the references they contain.

In [3], by using the theory of cosine families of linear operators in Banach space, the authors use the Schaefer theorem to study the existence of solutions of the following second-order partial neutral functional differential equation

$$
(1.2) \qquad \begin{cases} \dfrac{d}{dt}[u'(t)-g(t,u_t)] = Au(t)+f(t,u_t,u'(t)),\ t\in J=[0,T] \\ \\ u_0=\varphi\in\mathscr{B},\ u'(0)=z\in X. \end{cases}
$$

Recently in [16], the authors studied the existence and regularity of solutions for some second-order nonlinear differential equations in Banach spaces.

More recently, in [8], D. Mbainadji et al. studied the regularity and existence of solutions in α-norm for some second order partial neutral functional differential equations with finite delay in Banach spaces. They used cosine family theory and Schauder's fixed point theorem to establish the existence of solutions and then gave some sufficient conditions ensuring the regularity of the solutions. The present work is motivated by the papers of I. Zabsonre and al. [16, 17], Travis and Webb [14] and the generalization of B. Diao et al. [2].

The organization of this work is as follows, in Section 2 we recall some preliminary results on cosine families and fractional α-power and cosine family theory, in Section 3 we prove global existence, uniqueness and continuous dependence with respect to the initial data. Section 4 is devoted to local existence, blowing up phenomena and compactness of the flow. In Section 5, we study the regularity of the solutions under sufficient conditions. Finally, in Section 6, we illustrate our results by examining an example.

2. Preliminary Results

Let $(X,\|.\|)$ be a Banach space, let α be a constant such that $0<\alpha<1$ and let A be the infinitesimal generator of strongly continuous $(C(t))_{t\geq 0}$ on X. We assume without loss of generality that $0\in\rho(-A)$. Note that if the assumption $0\in\rho(-A)$ is not satisfied, one can substitute the operator $-A$ by the operator $(-A-\sigma I)$ with σ large enough such that $0\in\rho(-A-\sigma I)$. This allows us to define the fractional power $(-A)^\alpha$ for $0<\alpha<1$, as a closed linear invertible operator with domain $D((-A)^\alpha)$ dense in X. The closeness of $(-A)^\alpha$ implies that $D((-A)^\alpha)$, endowed with the graph norm of $(-A)^\alpha$, $|x|=\|x\|+\|(-A)^\alpha x\|$, is a Banach space. Since $(-A)^\alpha$

is invertible, its graph norm $|.|$ is equivalent to the norm $|x|_\alpha = \|(-A)^\alpha x\|$. Thus, $D((-A)^\alpha)$ equipped with the norm $|.|_\alpha$, is a Banach space, which we denote by X_α.

Definition 2.1. [14] *A one parameter family $\{C(t), t \in \mathbb{R}\}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if*
i) $C(s+t)+C(s-t) = 2C(s)C(t)$ for all $s,t \in \mathbb{R}$;
ii) $C(0) = I$;
iii) $C(t)x$ is continuous on $\mathbb{R}$ for each fixed $x \in X$.

The strongly continuous sine family $\{S(t), t \in \mathbb{R}\}$ associated to the given strongly continuous cosine family $\{C(t), t \in \mathbb{R}\}$ by

$$S(t)x = \int_0^t C(s)xds, \text{ for } x \in X, t \in \mathbb{R} \tag{2.1}$$

Definition 2.2. *The infinitesimal generator of strongly continuous cosine family $\{C(t), t \in \mathbb{R}\}$ is the operator $A : X \longrightarrow X$ define by*

$$Ax = \frac{d^2C(t)x}{dt^2}\Big|_{t=0}.$$

$D(A) = \{x \in X :$ *C(t)x is a twice continuously differentiable function of t*$\}$.

We shall also make use of the set

$$E = \{x : C(t)x \text{ is a once continuously differentiable function of t}\}$$

Lemma 2.3. *Let $C(t), \in \mathbb{R}$ be a strongly continuous cosine family in X with infinitesimal generator A. The following are true.*
i) $D(A)$ is dense in X and A is closed operator in X;
ii) if $x \in X$ and $s,r \in \mathbb{R}$ then $z = \int_s^r = S(u)xdu \in D(A)$ and $Az = C(s)x - C(r)x$;
iii) if $x \in X$, $s,r \in \mathbb{R}$ then $z = \int_0^s \int_0^r C(u)C(v)xdudv \in D(A)$ and
$Az = \frac{1}{2}(C(s+r)x - C(s-r)x)$;
iv) if $x \in X$, $S(t)x \in E$;
v) if $\in X$, the $S(t)x \in D(A)$ and $\frac{dC(t)}{dt} = AS(t)x$:
vi) if $x \in D(A)$, then $C(t)x \in D(A)$ and $\frac{d^2C(t)}{dt^2} = AC(t)x = C(t)Ax$;

vii) if $x \in E$, then $\lim_{t\to 0} AS(t) = 0$;

viii) if $x \in E$, then $S(t)x \in D(A)$ and $\dfrac{d^2S(t)}{dt^2} = AS(t)x$;

ix) if $x \in D(A)$, then $S(t)x \in D(A)$ and $AS(t)x = S(t)Ax$;

x) $C(t+s) + C(t-s) = 2AS(t)S(s)$ for all $s,t \in \mathbb{R}$.

In [14], for $0 < \alpha < 1$ the fractional powers $(-A)^{\alpha}$ exist as closed linear operators in X, $D((-A)^{\alpha}) \subset D((-A)^{\beta})$ for $0 \le \beta \le \alpha \le 1$ and $(-A)^{\alpha}(-A)^{\beta} = (-A)^{\alpha+\beta}$ for $0 \le \alpha+\beta \le 1$. For our objective we assume that

$(\mathbf{H}_0)$ A is the infinitesimal generator of a strongly continuous cosine family of linear operators on a Banach space X.

Lemma 2.3, $(\mathbf{H}_{0)})$ implies that the operator A is densely defined in X,
i.e $\overline{D(A)} = X$. We have the following result:

Lemma 2.4. [14] *Assume that (H_0) hols. Then there are constants $M \ge 1$ and $\sigma \ge 0$ such that $\|C(t)\| \le Me^{\sigma|t|}$ and $\|S(t_1) - S(t_2)\| \le M\left|\int_{t_1}^{t_2} e^{\sigma|s|}ds\right|$, for all $t_1, t_2 \in \mathbb{R}$.*

From previous inequality, since $S(0) = 0$ we can deduce that

$$\|S(t)\| \le \frac{M}{\sigma}e^{\sigma t} \text{ for } t \in \mathbb{R}^+.$$

In the sequel, let us pose $M_1 = \max\left(M, \dfrac{M}{\sigma}\right)$.

Theorem 2.5. [14] *If $k : \mathbb{R}^+ \to X$ is continous, $h : \mathbb{R}^+ \to X$ is continuous and u is a solution of equation* (1.1)*, the u is a solution of integral equation*

$$u(t) = C(t)x + S(t)y + \int_0^t C(t-s)k(s)ds + \int_0^t S(t-s)h(s)ds.$$

$(\mathbf{A}_1)$: For $0 < \alpha < 1$, $(-A)^{\alpha}$ maps onto X and $1-1$, so that $D((-A)^{\alpha})$ endowed with the norm $|x|_{\alpha} = \|(-A)^{\alpha}x\|$ is a Banach space. We denote by X_{α} this space. In addition we assume that A^{-1} is compact. To establish our results, we need the following Lemmas.

Lemma 2.6. [13] *Assume that $(\boldsymbol{H}_0)$ holds. The following are true*

(i) For $0 < \alpha < 1$, $(-A)^{-\alpha}$ is compact if and only if A^{-1} is compact.

(ii) For $0 < \alpha < 1$, and $t \in \mathbb{R}$ $(-A)^{-\alpha}C(t) = C(t)(-A)^{-\alpha}$ and $(-A)^{-\alpha}S(t) = S(t)(-A)^{-\alpha}$

Recall from [5], $(-A)^{-\alpha}$ is given by the following formula

$$(-A)^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^{+\infty} t^{-\alpha}(tI - A)^{-1} dt.$$

Lemma 2.7. [13] *Assume that* $(\boldsymbol{H}_0)$ *holds. Let* $v : \mathbb{R} \longrightarrow x$ *such that* v *is continuously differentiable and let* $q(t) = \int_0^t S(t-s)v(s)ds$. *Then*

(i) *q is twice continuously differentiable and for* $t \in \mathbb{R}$, $q(t) \in D(A)$,

$$q'(t) = \int_0^t C(t-s)v(s)ds$$

and

$$q''(t) = \int_0^t C(t-s)v'(s)ds + C(t)v(0) = Aq(t) + v(t).$$

(ii) *For* $0 < \alpha < 1$ *and* $t \in \mathbb{R}$, $(-A)^{\alpha-1}q'(t) \in E$.

3. Global Existence, Uniqueness and Continuous Dependence With Respect to the the Initial Data

In this section we first establish global existence, secondly continuous dependence on the initial data, and finally the results giving the blowing up of the mild solution in finite time.

Definition 3.1. *A continuous function* $u :]-r, +\infty[\to X_\alpha$, *for* $b > 0$ *is said to a mild solution of equation* (1.1) *if*

$$\begin{cases} u(t) = C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)f(s, u_s, u'_s)ds \text{ for } t \in [0, b], \\ u_0 = \varphi(0), \\ u'_0 = \varphi'(0). \end{cases}$$

In the following, we give a global existence of mild solution of equation(1.1) using the principle contraction.

For this purpose, we make this following assumptions:

$(\mathbf{H}_1)$ There exists a constant $L_F > 0$ such that

$$\|f(t, \varphi_1, \varphi_1') - f(t, \varphi_2, \varphi_2')\| \leq L_F \|\varphi_1 - \varphi_2\|_{\mathscr{C}_\alpha}, \text{ for } t \geq 0 \text{ and } \varphi_1, \varphi_2 \in \mathscr{C}_\alpha;$$

$(\mathbf{H}_2)$ A^{-1} is compact on X;

$(\mathbf{H}_3)$ The map $t \to AC(t)$ is locally bounded.

Theorem 3.2. *Assume that* $(\boldsymbol{H}_0)$-$(\boldsymbol{H}_3)$ *hold. Let* $\varphi \in \mathscr{C}_\alpha$ *such that* $\varphi(0) \in D(A)$, $\varphi'(0) \in E$ *Then equation* (1.1) *has unique mild solution which is defined for all* $t \geq 0$.

Proof. Let $b > 0$. For $\varphi \in \mathscr{C}_\alpha$, we define the set $M_b(\varphi)$ by

$$M_b(\varphi) = \big\{u \in C([0,b];X_\alpha) : u(0) = \varphi(0)\big\}.$$

We claim $M_b(\varphi)$ is a closed set of $C([0,b];X_\alpha)$, where $C([0,b];X_\alpha)$ is the set of continuous function define from $[0,b]$ to X_α endowed with the uniform norm topology. Indeed, let a sequence $(u_n)_{n\geq 0}$ of $M_b(\varphi)$ such that $\lim_{n\to+\infty} u_n = u$. By uniform convergence, u is continuous and $u(0) = \varphi(0)$. For $u \in M_b(\varphi)$, we define it extension on $[-r,b]$ by

$$\widetilde{u}(t) = \begin{cases} u(t) \text{ for } t \in [0,b], \\ \\ \varphi(t) \text{ for } t \in [-r,0]. \end{cases}$$

Let us define the operator $\mathscr{K}$ on $M_b(\varphi)$ by

$$\mathscr{K}(u)(t) = C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s)f(s,\widetilde{u}_s,\widetilde{u}'_s)ds \text{ for } t \in [0,b].$$

We claim $\mathscr{K}(M_b(\varphi)) \subset M_b(\varphi)$. In fact for $u \in M_b(\varphi)$, we have $\mathscr{K}(u)(0) = \varphi(0)$ and by the continuity of f, $t \to C(t)x$ and $t \to S(t)x$ for $x \in X$ we deduce that $\mathscr{K}(u) \in M_b(\varphi)$. Now we prove that $\mathscr{K}$ is strict contraction on $M_b(\varphi)$. To sow this, let $u,v \in M_b(\varphi)$. Then we have

$$\big(\mathscr{K}(u) - \mathscr{K}(v)\big)(t) = \int_0^t S(t-s)[f(s,\widetilde{u}_s,\widetilde{u}'_s) - f(s,\widetilde{v}_s,\widetilde{v}'_s)]ds.$$

Let μ_0 be a positive real number such that $\|AC(\xi)\| \leq \mu_0$ for all $t \in [0,b]$ and using α-norm we have

$$\begin{aligned}
\big|\big(\mathscr{K}(u) - \mathscr{K}(v)\big)(t)\big|_\alpha &= \Big\|\int_0^t (-A)^\alpha S(t-s)[f(s,\widetilde{u}_s,\widetilde{u}'_s) - f(s,\widetilde{v}_s,\widetilde{v}'_s)]ds\Big\| \\
&= \Big\|-(-A)^{\alpha-1}\int_0^t AS(t-s)[f(s,\widetilde{u}_s,\widetilde{u}'_s) - f(s,\widetilde{v}_s,\widetilde{v}'_s)]ds\Big\| \\
&= \Big\|-(-A)^{\alpha-1}\int_0^t \Big(\int_0^{t-s} AC(\xi)[f(s,\widetilde{u}_s,\widetilde{u}'_s) - f(s,\widetilde{v}_s,\widetilde{v}'_s)]d\xi\Big)ds\Big\| \\
&\leq \|(-A)^{\alpha-1}\|\mu_0 b\int_0^t \|f(s,\widetilde{u}_s,\widetilde{u}'_s) - f(s,\widetilde{v}_s,\widetilde{v}'_s)\| \\
&\leq \|(-A)^{\alpha-1}\|\mu_0 b^2 L_F\|u-v\|_{\mathscr{C}_\alpha}.
\end{aligned}$$

Using the same reasoning like previously, we have

$$\begin{aligned}\left|\left(\mathscr{K}(u)'-\mathscr{K}(v)'\right)(t)\right|_{\alpha} &= \left\|\int_0^t(-A)^{\alpha}C(t-s)[f(s,\widetilde{u}_s,\widetilde{u}'_s)-f(s,\widetilde{v}_s,\widetilde{v}'_s)]ds\right\| \\ &\leq \|(-A)^{\alpha-1}\|\mu_0 bL_F\|u-v\|_{\mathscr{C}_\alpha}.\end{aligned}$$

Adding the two previous equations

$$\left\|\left(\mathscr{K}(u)-\mathscr{K}(v)\right)(t)\right\|_{\mathscr{C}_\alpha} \leq \|(-A)^{\alpha-1}\|(1+b)b\mu_0 L_F\|u-v\|_{\mathscr{C}_\alpha}.$$

Now we choose b and μ_0 such that

$$\|(-A)^{\alpha-1}\|(1+b)b\mu_0 L_F<1.$$

Then $\mathscr{K}$ is a strict contraction on $M_b(\varphi)$, and it has a unique fixed point y which is the unique mild solution of equation (1.1) on $[0,b]$. To extend the solution of equation (1.1) in $[b,2b]$, we show that the following equation has a unique mild solution:

$$\begin{cases} z''(t)=Au(t)+f(t,z_t,z'_t) \text{ for } t\in[b,2b],\\ z_b=u_b\in C([-r,b],X_\alpha),\\ z'_b=u'_b\in C([-r,b],X_\alpha).\end{cases} \tag{3.1}$$

Notice that the solution of equation (3.1) is given by

$$z(t)=C(t-b)z(b)+S(t-b)z'(b)+\int_b^t S(t-s)f(s,z_s,z'_s)ds.$$

Let $\widetilde{z}$ the function define by

$$\widetilde{z}(t)=\begin{cases} z(t) \text{ for } t\in[b,2b],\\ u(t) \text{ for } t\in[-r,b].\end{cases}$$

Consider the set

$$M_{2b}(\varphi)=\left\{z\in C([0,b];X_\alpha): z(b)=u(b)\right\},$$

provided with the induced topological norm. Now we define the operator $\mathscr{K}_b$ on $M_{2b}(\varphi)$ by

$$(\mathscr{K}_b z)(t)=C(t-b)u(b)+S(t-b)u'(b)+\int_b^t S(t-s)f(s,\widetilde{z}_s,\widetilde{z}'_s)ds \text{ for } t\in[b,2b].$$

We have $(\mathscr{K}_b z)(b) = u(b)$ and $(\mathscr{K}_b z)$ is continuous. Then it follows that $(\mathscr{K}_b z)(M_{2b}(\varphi)) \subset M_{2b}(\varphi)$. Moreover, for $u, v \in M_{2b}(\varphi)$, one has

$$\left|\big(\mathscr{K}_b(u) - \mathscr{K}_b(v)\big)(t)\right|_\alpha = \|(-A)^{\alpha-1}\|\mu_0 b L_F \int_b^t \|\widetilde{u} - \widetilde{v}\|_{\mathscr{C}_\alpha}.$$

Since $\widetilde{u} = \widetilde{v} = \varphi$ in $[-r, 0]$, we deduce that

$$\left|\big(\mathscr{K}_b(u) - \mathscr{K}_b(v)\big)(t)\right|_\alpha = \|(-A)^{\alpha-1}\|\mu_0 b L_F \|u - v\|_{\mathscr{C}_\alpha}.$$

By similar reasoning, we have

$$\left|\big((\mathscr{K}_b(u))' - (\mathscr{K}_b(v))'\big)(t)\right|_\alpha = \|(-A)^{\alpha-1}\|\mu_0 b L_F \|u - v\|_{\mathscr{C}_\alpha}.$$

Adding the two previous equations

$$\left\|\big(\mathscr{K}_b(u) - \mathscr{K}_b(v)\big)(t)\right\|_{\mathscr{C}_\alpha} \leq \|(-A)^{\alpha-1}\|(1+b) b \mu_0 L_F \|u - v\|_{\mathscr{C}_\alpha}.$$

Then $\mathscr{K}_b$ has a unique fixed point in $M_{2b}(\varphi)$ which extends the solution u in $[b,\ 2b]$. Proceeding inductively, u is uniquely and continuously extended to $[nb,\ (n+1)b]$ for all $n \geq 1$ and this ends the proof. □

In the following we will show the continuous dependence of the mild solutions with respect to the initial data.

Theorem 3.3. *Assume that* $(\boldsymbol{H}_0)$-$(\boldsymbol{H}_3)$ *hold. The mild solution* $u(.,\varphi)$ *of equation* (1.1) *defines a continuous Lipschitz operator* $U(t)\varphi = u_t(.,\varphi)$. *Moreover there exist a real number* δ *and a scalar function* β *such that for* $t \geq 0$ *and* $\varphi_1, \varphi_2 \in \mathscr{C}_\alpha$, *we have*

$$\|U(t)\varphi_1 - U(t)\varphi_2\|_{\mathscr{C}_\alpha} \leq \beta(\delta) e^{\delta t} \|\varphi_1 - \varphi_2\|_{\mathscr{C}_\alpha}. \tag{3.2}$$

Proof. The continuity is obvious on what the map $t \to u_t(.,\varphi)$ is continuous. Now let $\varphi_1, \varphi_2 \in \mathscr{C}_\alpha$. If we pose $w(t) = u(t,\varphi_1) - u(t,\varphi_2)$, then we have

$$\begin{aligned} w(t) &= C(t)(\varphi_1(0) - \varphi_2(0)) + S(t)(\varphi_1'(0) - \varphi_2'(0)) \\ &\quad + \int_0^t S(t-s)[f(s, u_s(.,\varphi_1), u_s'(.,\varphi_1)) - f(s, u_s(.,\varphi_2), u_s'(.,\varphi_2))] ds \end{aligned}$$

On one hand taking the α-norm, we have

$$
\begin{aligned}
|w(t)|_\alpha &\leq \|C(t)(-A)^\alpha(\varphi_1(0)-\varphi_2(0))| + \|S(t)(-A)^\alpha(\varphi_1'(0)-\varphi_2'(0))\| \\
&\quad + \Big\| -(-A)^{\alpha-1}\int_0^t S(t-s)A\big[f(s,u_s(.,\varphi_1),u_s'(.,\varphi_1)) - f(s,u_s(.,\varphi_2),u_s'(.,\varphi_2))\big]ds\Big\| \\
&\leq Me^{\sigma t}\|\varphi_1+\varphi_2\|_\alpha + \frac{M}{\sigma}e^{\sigma t}\|\varphi_1'+\varphi_2'\|_\alpha \\
&\quad + \Big\|(-A)^{\alpha-1}\int_0^t \Big(\int_0^{t-s} AC(\xi)\big[f(s,u_s(.,\varphi_1),u_s'(.,\varphi_1)) - f(s,u_s(.,\varphi_2),u_s'(.,\varphi_2))\big]\Big)ds\Big\|,
\end{aligned}
$$

then we have

$$
|w(t)|_\alpha \leq M_1 e^{\sigma t}\|\varphi_1-\varphi_2\|_{\mathscr{C}_\alpha} + \|(-A)^{\alpha-1}\|\mu_0 b L_F \int_0^t \|w_s\|_{\mathscr{C}_\alpha}.
$$

On the other hand

$$
\begin{aligned}
w'(t) &= AS(t)(\varphi_1(0)-\varphi_2(0)) + C(t)(\varphi_1'(0)-\varphi_2'(0)) \\
&+ \int_0^t C(t-s)[f(s,u_s(.,\varphi_1),u_s'(.,\varphi_1)) - f(s,u_s(.,\varphi_2),u_s'(.,\varphi_2))]ds,
\end{aligned}
$$

then it follows that

$$
|w'(t)|_\alpha \leq \big(\mu_0 b + M_1 e^{\sigma t}\big)\|\varphi_1-\varphi_2\|_{\mathscr{C}_\alpha} + \|(-A)^{\alpha-1}\|\mu_0 L_F \int_0^t \|w_s\|_{\mathscr{C}_\alpha} ds.
$$

Adding the two previous inequalities, we have

$$
\|w(t)\|_{\mathscr{C}_\alpha} \leq \big(\mu_0 b + 2M_1 e^{\sigma t}\big)\|\varphi_1-\varphi_2\|_{\mathscr{C}_\alpha} + \|(-A)^{\alpha-1}\|\mu_0(1+b)L_F \int_0^t \|w_s\|_{\mathscr{C}_\alpha} ds.
$$

Let δ a real number such that $\sigma-\delta<0$. For $-r\leq\tau\leq 0$, we have

$$
e^{-\delta\tau}\|w_\tau\|_{\mathscr{C}_\alpha} \leq M_2 L, \text{ where } M_2=\max\{e^{\delta r},1\} \text{ and } L=\|\varphi_1-\varphi_2\|_{\mathscr{C}_\alpha} \tag{3.3}
$$

Now for $0\leq\tau\leq\bar{t}$ we have from (3.3)

$$
e^{-\delta\tau}\|w_\tau\|_{\mathscr{C}_\alpha} \leq \big(\mu_0 b e^{-\delta\tau} + 2M_1 e^{(\sigma-\delta)\tau}\big)L + \|(-A)^{\alpha-1}\|\mu_0(1+b)L_F \int_0^t e^{(\sigma-\delta)\tau}e^{-\sigma\tau}\|w_s\|_{\mathscr{C}_\alpha} ds.
$$

Since $\sigma-\delta\leq 0$, we have

$$
e^{-\delta\tau}\|w_\tau\|_{\mathscr{C}_\alpha} \leq \big(\mu_0 b e^{-\delta\tau} + 2M_1\big)L + \|(-A)^{\alpha-1}\|\mu_0(1+b)L_F \int_0^t e^{(\sigma-\delta)\tau}e^{-\sigma\tau}\|w_s\|_{\mathscr{C}_\alpha} ds.
$$

$$
\sup_{-r\leq\tau\leq\bar{t}} e^{-\delta\tau}\|w_\tau\|_{\mathscr{C}_\alpha} \leq KL + \|(-A)^{\alpha-1}\|\mu_0(1+b)L_F W\big(\sigma-\delta\big)^{-1}\big(e^{(\sigma-\delta)\bar{t}}-1\big), \tag{3.4}
$$

where $W = \sup_{0\leq\tau\leq\bar{t}} e^{-\delta\tau}\|w_\tau\|_{\mathscr{C}_\alpha}$ and $K = \big(\mu_0 bM_2 + 2M_1M_2\big)$. For $0 \leq \tau \leq \bar{t}$, it follows that

$$
\begin{aligned}
e^{-\delta t}\|w_t\|_{\mathscr{C}_\alpha} &= \sup_{-r\leq\theta\leq 0} e^{\delta\theta} e^{-\delta(t+\theta)}\|w(t+\theta)\|_{\mathscr{C}_\alpha} \\
&\leq M_3 \sup_{-r\leq\theta\leq 0} e^{-\delta(t+\theta)}\|w(t+\theta)\|_{\mathscr{C}_\alpha} \\
&\leq M_3 \sup_{-r\leq\tau\leq\bar{t}} e^{-\delta\tau}\|w(\tau)\|_{\mathscr{C}_\alpha}, \qquad (3.5)
\end{aligned}
$$

where $M_3 = \max\{e^{-\delta r}, 1\}$. Therefore from equations (3.4) and (3.5), we obtain

$$W \leq KLM_3 + \|(-A)^{\alpha-1}\|\mu_0(1+b)L_F M_3 W\big(\sigma-\delta\big)^{-1}\big(e^{(\sigma-\delta)\bar{t}}-1\big).$$

We deduce that

$$\|U(t)\varphi_1 - U(t)\varphi_2\|_{\mathscr{C}_\alpha} \leq \beta(\delta)e^{\delta t}\|\varphi_1-\varphi_2\|_{\mathscr{C}_\alpha},$$

$$\beta(\delta) = KM_3\Big(1 - \|(-A)^{\alpha-1}\|\mu_0(1+b)L_F M_3 W\big(\sigma-\delta\big)^{-1}\big(e^{(\sigma-\delta)\bar{t}}-1\big)\Big)^{-1}.$$

Then the proof is complete. □

4. Local Existence, Blowing Up Phenomena and Compactness of the Flow

The result of the local existence is given by the following theorem.

Theorem 4.1. *Assume that* $(\boldsymbol{H}_0)$-$(\boldsymbol{H}_3)$ *hold. Moreover, assume that* f *defined from* $I\times\Omega$ *in to* X *is continuous, where* $I\times\Omega$ *an open set of* $\mathbb{R}_+\times\mathscr{C}_\alpha$. *Then for each* $\varphi\in\Omega$, *equation* (1.1) *has at least one mild solution which is defined on some interval* $[0,b]$.

Proof. Let $\varphi\in\Omega$. For any real $\beta, p > 0$, we define the following sets:

$$I_\beta = \big\{t : 0 \leq t \leq \beta\big\} \text{ and } \beta_p = \big\{\psi\in\mathscr{C}_\alpha, \|\psi\|_{\mathscr{C}_\alpha} \leq p\big\}.$$

For $\psi\in\beta_p$ we choose such that $(t,\psi)\in I_\beta\times\beta_p$ and $\beta_p\subset\Omega$. By the continuity of f there exists $N\geq 0$ such that $\|f(t,\psi+\varphi,\psi'+\varphi')\|\leq N$ for $(t,\psi)\in I_\beta\times\beta_p$.

Let us consider $\overline{\varphi}\in C([-r,b];X_\alpha)$ as a function defined by $\overline{\varphi}(t) = C(t)\varphi(0)+S(t)\varphi'(0)$ and $\overline{\varphi}_0 = \varphi$. Suppose that $\overline{p} < p$ and choose $0 < b < \beta$ such that

$$\|(-A)^{\alpha-1}\|\mu_0 aN\int_0^a ds < \overline{p} \text{ and } \|\overline{\varphi}-\varphi\|_{\mathscr{C}_\alpha} < p-\overline{p} \text{ for } t\in I_\beta.$$

Let $K_0 = \left\{\eta \in C([-r,b];X_\alpha) : \eta_0 = 0 \text{ and } \|\eta_t\|_{\mathscr{C}_\alpha} \leq \overline{p} \text{ for } 0 \leq t \leq b\right\}$.

Then we have

$\|f(t,\overline{\varphi}_t+\eta_t,\overline{\varphi}'_t+\eta'_t)\| \leq N$ for $0 \leq t \leq a$ and $\eta \in K_0$, since $\|\overline{\varphi}_t+\eta_t-\varphi\|_{\mathscr{C}_\alpha} \leq p$. Consider the mapping $\mathscr{S}$ from K_0 to $C([-r,b];X_\alpha)$ defined by

$$\begin{cases} (\mathscr{S}\eta)(t) = \displaystyle\int_0^t S(t-s)f(s,\overline{\varphi}_s+\eta_s,\overline{\varphi}'_s+\eta'_s)ds \ \text{ for } \ 0 \leq t \leq b \\ (\mathscr{S}\eta)_0 = 0. \end{cases}$$

Notice that finding a fixed point of $\mathscr{S}$ in K_0 is equivalent is to finding of equation (1.1) in to K_0. Moreover, $\mathscr{S}$ is a mapping from K_0 to K_0. In fact let $\eta \in K_0$, then we have $(\mathscr{S}\eta)_0 = 0$ and

$$\begin{aligned} |\mathscr{S}\eta(t)|_\alpha &= \left\| -(-A)^{\alpha-1}\int_0^t S(t-s)Af(s,\overline{\varphi}_s+\eta_s,\overline{\varphi}'_s+\eta'_s)ds\right\| \\ &= \left\| -(-A)^{\alpha-1}\int_0^t \Big(\int_0^{t-s} AC(\xi)f(s,\overline{\varphi}_s+\eta_s,\overline{\varphi}'_s+\eta'_s)d\xi\Big)ds\right\| \\ &\leq \|(-A)^{\alpha-1}\|\mu_0 b\int_0^t \|f(s,\overline{\varphi}_s+\eta_s,\overline{\varphi}'_s+\eta'_s)\|ds \\ &\leq \|(-A)^{\alpha-1}\|\mu_0 bN\int_0^b ds < \overline{p}. \end{aligned}$$

Thus we have $\mathscr{S}(K_0) \subset K_0$. We suppose that $\left\{(\mathscr{S}\eta)(t) : \eta \in K_0\right\}$ is compact in X_α for fixed $t \in C([-r,b],X_\alpha)$. In fact let $0 < \alpha \leq \gamma < 1$. The above estimate show that $\left\{(-A)^\gamma(\mathscr{S}\eta)(t) : \eta \in K_0\right\}$ is bounded on X. Since the operator $(-A)^{\alpha-\gamma}$ is compact, we deduce that $\left\{(-A)^{\alpha-\gamma}(-A)^\gamma(\mathscr{S}\eta)(t) : \eta \in K_0\right\}$ is compact in X, hence $\left\{(\mathscr{S}\eta)(t) : \eta \in K_0\right\}$ is compact in X_α.

Our next objective is to show that $\left\{(\mathscr{S}\eta)(t) : \eta \in K_0\right\}$ is equicontinous. Notice that the equicontnuity of $\left\{(\mathscr{S}\eta)(t) : \eta \in K_0\right\}$ at $t = 0$ follows from the above estimation of $(\mathscr{S}\eta)(t)$. Now let $0 < t_1 < t_2 \leq b$, with t_1 be fixed. Then we have

$$\begin{aligned} |(\mathscr{S}\eta)(t_2)-(\mathscr{S}\eta)(t_1)|_\alpha &\leq \left|\int_0^{t_1}\big(S(t_2-s)-S(t_1-s)\big)f(s,\overline{\varphi}_s+\eta_s,\overline{\varphi}'_s+\eta'_s)ds\right|_\alpha \\ &+ \left|\int_{t_1}^{t_2} S(t_2-s)f(s,\overline{\varphi}_s+\eta_s,\overline{\varphi}'_s+\eta'_s)ds\right|_\alpha \\ &\leq \left\|-(-A)^{\alpha-1}\int_0^{t_1} A\big(S(t_2-s)-S(t_1-s)\big)f(s,\overline{\varphi}_s+\eta_s,\overline{\varphi}'_s+\eta'_s)ds\right\|_\alpha \\ &+ \left\|-(-A)^{\alpha-1}\int_{t_1}^{t_2} AS(t_2-s)f(s,\overline{\varphi}_s+\eta_s,\overline{\varphi}'_s+\eta'_s)ds\right\|_\alpha \end{aligned}$$

$$\begin{aligned}
&\leq \left\|(-A)^{\alpha-1}\Big[\int_0^{t_1}\frac{d}{ds}\Big(\big(C(t_2-s)-C(t_1-s)\big)f(s,\overline{\varphi}_s+\eta_s,\overline{\varphi}'_s+\eta'_s)\Big)ds\right.\\
&\quad \left.-\int_0^{t_1}\big(C(t_2-s)-C(t_1-s)\big)\frac{d}{ds}\Big()f(s,\overline{\varphi}_s+\eta_s,\overline{\varphi}'_s+\eta'_s)\Big)ds\Big]\right\|\\
&+ \left\|(-A)^{\alpha-1}\int_{t_1}^{t_2}\frac{d}{ds}\Big(C(t_2-s)f(s,\overline{\varphi}_s+\eta_s,\overline{\varphi}'_s+\eta'_s)\Big)ds\right\|\\
&+ \left\|(-A)^{\alpha-1}\int_{t_1}^{t_2}C(t_2-s)\frac{d}{ds}\Big(f(s,\overline{\varphi}_s+\eta_s,\overline{\varphi}'_s+\eta'_s)\Big)ds\right\|.
\end{aligned}$$

Then, we have

$$\begin{aligned}
|(\mathscr{S}\eta)(t_2)-(\mathscr{S}\eta)(t_1)|_\alpha &\leq \left\|(-A)^{\alpha-1}\right\|\Big[\|\big(C(t_2-t_1)-I\big)f(t_1,\overline{\varphi}_{t_1}+\eta_{t_1},\overline{\varphi}'_{t_1}+\eta'_{t_1})\|\\
&+ \|\big(C(t_2)-C(t_1)\big)f(0,\overline{\varphi}_0+\eta_0,\overline{\varphi}'_0+\eta'_0)\|\\
&+ \|f(t_2,\overline{\varphi}_{t_2}+\eta_{t_2},\overline{\varphi}'_{t_2}+\eta'_{t_2})-C(t_2-t_1)f(t_1,\overline{\varphi}_{t_1}+\eta_{t_1},\overline{\varphi}'_{t_1}+\eta'_{t_1})\|\\
&+ M_1e^{\sigma b}\|f(t_2,\overline{\varphi}_{t_2}+\eta_{t_2},\overline{\varphi}'_{t_2}+\eta'_{t_2})-f(t_1,\overline{\varphi}_{t_1}+\eta_{t_1},\overline{\varphi}'_{t_1}+\eta'_{t_1})\|\Big].
\end{aligned}$$

Since $(-A)^{\alpha-1}$ is compact from X to X and $(C(t)_{t\in\mathbb{R}})$ is uniformly continuous on compact subset of X moreover the set $\big\{(\mathscr{S}\eta)(t_1):\eta\in K_0\big\}$ is compact in X_α. We have

$$\lim_{t_2\longrightarrow t_1^+}|(\mathscr{S}\eta)(t_2)-(\mathscr{S}\eta)(t_1)|_\alpha=0 \text{ uniformly in } \eta\in K_0.$$

We obtain the same results by taking t_1 be fixed $0<t_1<t_2\leq b$. Then we we claim that

$$\lim_{t_2\longrightarrow t_1}|(\mathscr{S}\eta)(t_2)-(\mathscr{S}\eta)(t_1)|_\alpha=0 \text{ uniformly in } \eta\in K_0,$$

uniformly in $\eta\in K_0$ which means that $\big\{(\mathscr{S}\eta)(t):\eta\in K_0\big\}$ is equicontinuous. We deduce by the Ascoli-Arzela's theorem that $\big\{(\mathscr{S}\eta):\eta\in K_0\big\}$. is relatively compact in K_0. Finally, we prove that $\mathscr{S}$ is continuous.

Since f is continuous, given $\varepsilon>0$, there exists $\delta>0$ such that $\|\eta(s)-\hat{\eta}(s)\|_{\mathscr{C}_\alpha}<\delta$ implies that

$$\|f(s,\overline{\varphi}_s+\eta_s,\overline{\varphi}'_s+\eta'_s)-f(s,\overline{\varphi}_s+\hat{\eta}_s,\overline{\varphi}'_s+\hat{\eta}'_s)\|<\varepsilon.$$

Then for $0 \leq t \leq b$, we have

$$
\begin{aligned}
|(\mathscr{S}\eta)(t_2)-(\mathscr{S}\hat{\eta})(t_1)|_\alpha &= \left\|(-A)^{\alpha-1}\int_0^t S(t-s)\Big(f(s,\overline{\varphi}_s+\eta_s,\overline{\varphi}'_s+\eta'_s)-f(s,\overline{\varphi}_s+\hat{\eta}_s,\overline{\varphi}'_s+\hat{\eta}'_s)\Big)ds\right\| \\
&\leq \|(-A)^{\alpha-1}\|\mu_0 b\int_0^t \varepsilon ds.
\end{aligned}
$$

This gives the continuity of $\mathscr{S}$, and using Schauder's fixed point theorem, we deduce that $\mathscr{S}$ has a fixed point. Then The theorem's proof is complete.

□

The following result gives the blowing up phenomena of the mild solution in finite times.

Theorem 4.2. *Assume that* $(\boldsymbol{H}_0)$-$(\boldsymbol{H}_3)$ *hold. Let* $\varphi \in \mathscr{C}_\alpha$ *such that* $\varphi(0) \in D(A)$, $\varphi'(0) \in E$. *Moreover assume that f is bounded continuous. Then equation* (1.1) *has a mild solution* $u(.,\varphi)$ *on maximal interval of existence* $[-r, b_\varphi[$. *Moreover if* $b_\varphi < \infty$, *then*

$$
\overline{\lim_{t\to b_\varphi^-}}\|u(t,\varphi)\|_{\mathscr{C}_\alpha} = +\infty.
$$

Proof. Let $u(.,\varphi)$ be the mild solution of equation (1.1) defined on $[0,b]$. Similar arguments used in the local existence result can be used to prove that $b_1 > b$ exists and that a function $u(.,\varphi)$ is defined from $[b,b_1]$ to X_α which satisfies

$$
u(t,u_b(.,\varphi)) = C(t)u(b,\varphi)) + S(t)u'(b,\varphi)) + \int_b^t S(t-s)f(s,u_s,u'_s)ds.
$$

On can show that the mild solution u can be extended to a maximal interval of existence $[0,b_\varphi[$ by a similar procedure. Assume that $b_\varphi < +\infty$ and $\overline{\lim_{t\to}}\|u(t,\varphi)\|_{\mathscr{C}_\alpha} < \infty$. Since f is bounded continuous,there exists $N > 0$ such that $\|f(s,u_s,u'_s)\| \leq N$. We suppose that $u(.,\varphi)$ is uniformly continuous. In fact, let $0 < h \leq t \leq t+h < b_\varphi$. On the one hand, we have

$$
\begin{aligned}
u(t+h)-u(t) &= (C(t+h)-C(t))\varphi(0)+(S(t+h)-S(t))\varphi'(0) \\
&\quad + \int_0^t S(t-s)[f(s+h,u_{s+h},u'_{s+h})-f(s,u_s+u'_s)]ds \\
&\quad + \int_{-h}^0 S(t-s)f(s+h,u_{s+h},u'_{s+h})-f(s,u_s+u'_s)ds.
\end{aligned}
$$

Taking account the α-norm, then we have

$$\begin{aligned}
|u(t+h)-u(t)|_\alpha &\leq |(C(t+h)-C(t))\varphi(0)|_\alpha + |(S(t+h)-S(t))\varphi'(0)|_\alpha \\
&\quad + \Big|\int_0^t S(t-s)[f(s+h,u_{s+h},u'_{s+h})-f(s,u_s+u'_s)]ds\Big| \\
&\quad + \Big|\int_{-h}^0 S(t-s)f(s+h,u_{s+h},u'_{s+h})ds\Big|_\alpha \\
&\leq |(C(t+h)-C(t))\varphi(0)|_\alpha + |(S(t+h)-S(t))\varphi'(0)|_\alpha \\
&\quad + \Big\| -(-A)^{\alpha-1}\int_0^t \Big(\int_0^{t-s} AC(\sigma)[f(s+h,u_{s+h},u'_{s+h})-f(s,u_s+u'_s)]d\sigma\Big)ds\Big\| \\
&\quad + \Big\| -(-A)^{\alpha-1}\int_{-h}^0 \Big(\int_0^{t-s} AC(\sigma)f(s+h,u_{s+h},u'_{s+h})d\sigma\Big)ds\Big\|_\alpha .
\end{aligned}$$

It follows that

$$\begin{aligned}
|u(t+h)-u(t)|_\alpha &\leq |(C(t+h)-C(t))\varphi(0)|_\alpha + |(S(t+h)-S(t))\varphi'(0)|_\alpha \\
&\quad + \|(-A)^{\alpha-1}\|\mu_0 b_\varphi \int_{-h}^0 \|f(s+h,u_{s+h}+u'_{s+h})\|ds \\
&\quad + \|(-A)^{\alpha-1}\|\mu_0 b_\varphi \int_0^t \|f(s+h,u_{s+h}+u'_{s+h})-f(s,u_s,u'_s)\|ds.
\end{aligned}$$

Since the map $t\to C(t)\varphi(0)+S(t)\varphi'(0)$ is uniformly continuous, consequently for $t,t+h\in[0,b_\varphi[$ and $h\in]0,h_0[$, we have

$$\begin{aligned}
&|u(t+h)-u(t)|\alpha \\
&\leq \delta_1(h)+\|(-A)^{\alpha-1}\|\mu_0 b_\varphi Nh + \|(-A)^{\alpha-1}\|\mu_0 b_\varphi L_F\int_0^t \|u_{s+h}-u_s\|_{\mathscr{C}_\alpha}ds, \qquad (4.1)
\end{aligned}$$

where

$$\delta_1(h)=\sup_{t+h\in[0,b_\varphi[}\Big(|(C(t+h)-C(t))\varphi(0)|_\alpha + |(S(t+h)-S(t))\varphi'(0)|_\alpha\Big).$$

From [14] (in Prosition 2.4), $t\to C(t)\varphi(0)+S(t)\varphi'(0)$ belongs to $C^2([0,b_\varphi])$. By a similar reasoning, we have

$$\begin{aligned}
&|u(t+h)-u(t)|_\alpha \\
&\leq \delta_2(h)+\|(-A)^{\alpha-1}\|\mu_0 Nh + \|(-A)^{\alpha-1}\|\mu_0 L_F\int_0^t \|u_{s+h}-u_s\|_{\mathscr{C}_\alpha}ds, \qquad (4.2)
\end{aligned}$$

where

$$\delta_2(h) = \sup_{t+h\in[0,b_\varphi[} \Big(|(S(t+h)-S(t))A\varphi(0)|_\alpha + |(C(t+h)-C(t))\varphi'(0)|_\alpha \Big).$$

Adding above inequalities (4.1) and (4.2) we have

$$\|u(t+h)-u(t)\|_{\mathscr{C}_\alpha} \leq \gamma(h) + \|(-A)^{\alpha-1}\|(1+b_\varphi)\mu_0 L_F \int_0^t \|u_{s+h}-u_s\|_{\mathscr{C}_\alpha} ds,$$

where

$$\gamma(h) = \delta_1(h) + \delta_2(h) + + \|(-A)^{\alpha-1}\|(1+b_\varphi)\mu_0 N h.$$

One can see that $\gamma(0)=0$. By Gronwall's lemma, we deduce that

$$\|u(t+h)-u(t)\|_{\mathscr{C}_\alpha} \leq \gamma(h)\exp\Big(\big(\|(-A)^{\alpha-1}\|(1+b_\varphi)\mu_0 L_F\big)t \Big).$$

Then $\lim_{h\to 0} \|u(t+h)-u(t)\|_{\mathscr{C}_\alpha} = 0$ uniformly in $[0,b_\varphi[$. This means that u and u' are uniformly continuous and u can be extended over $[0,b_\varphi+\eta]$, which contradicts the maximality of $[0,b_\varphi[$. Using the same reasoning, one can show a similar result for $h<0$. □

The next result gives the global existence of the mild solutions.

Theorem 4.3. *Assume that* $(\boldsymbol{H}_0)$-$(\boldsymbol{H}_3)$ *hold. Let* $\varphi \in \mathscr{C}_\alpha$ *such that* $\varphi(0)\in D(A)$, $\varphi'(0)\in E$. *Let* k_1 *be a continuous function* $\mathbb{R}^+$ *and and* $k_2 \in L^1_{loc}(\mathbb{R}^+,\mathbb{R}^+)$ *be such that*

$$\|f(t,\varphi,\varphi')\| \leq k_1(t)\|\varphi\|_{\mathscr{C}_\alpha} + k_2(t) \textit{ for } t\geq 0 \textit{ and } \varphi,\varphi' \in \mathscr{C}_\alpha.$$

Then equation (1.1) *has a unique mild solution which is define for all* $t\geq 0$.

Proof. Let $[-r,b_\varphi[$ denote the maximal interval of existence of the mild solution $u(t,\varphi)$ of the equation (1.1), then

$$b_\varphi = +\infty \text{ or } \overline{\lim_{t\to b_\varphi^-}} \|u(t,\varphi)\|_{\mathscr{C}_\alpha} = +\infty.$$

If $b_\varphi < +\infty$, then $\overline{\lim_{t\to b_\varphi^-}} \|u(t,\varphi)\|_{\mathscr{C}_\alpha} = +\infty$. Recall that the solution of equation (1.1) is given by $u_0=\varphi$ and

$$u(t,\varphi) = C(t)\varphi(0) + S(t)\varphi'(0) + \int_0^t S(t-s) f(s,u_s(.,\varphi),u_s'(.,\varphi))ds$$

Using α-norm, we obtain

$$\begin{aligned}|u(t,\varphi)|_\alpha &\leq |C(t)\varphi(0)|+|S(t)\varphi'(0)|_\alpha+\Big|\int_0^t S(t-s)f(s,u_s(.,\varphi),u'_s(.,\varphi))ds\Big|_\alpha \\ &\leq M_1e^{\omega b_\varphi}\big(|\varphi(0)|_\alpha+|\varphi'(0)|_\alpha+\Big\|(-A)^{\alpha-1}\int_0^t\Big(\int_0^{t-s}AC(\sigma))f(s,u_s(.,\varphi),u'_s(.,\varphi))d\sigma\Big)ds \\ &\leq k_0+\|(-A)^{\alpha-1}\|\mu_0 b_\varphi\int_0^{b_\varphi}k_1\|u_s(.,\varphi)\|_{\mathscr{C}_\alpha}ds,\end{aligned}$$

where

$$k_0=M_1e^{\omega b_\varphi}\big(|\varphi(0)|_\alpha+|\varphi'(0)|_\alpha+\|(-A)^{\alpha-1}\|\mu_0 b_\varphi\int_0^{b_\varphi}k_2(s)ds.$$

On the other hand, we have

$$|u(t,\varphi)|_\alpha\leq\delta_0+\|(-A)^{\alpha-1}\|\mu_0\int_0^{b_\varphi}k_1\|u_s(.,\varphi)\|_{\mathscr{C}_\alpha}ds,$$

$$\delta_0=\mu_0 b_\alpha|\varphi(0)|_\alpha+M_1e^{\omega b_\varphi}|\varphi'(0)|_\alpha+\|(-A)^{\alpha-1}\|\mu_0\int_0^{b_\varphi}k_2(s)ds.$$

It follows that by adding above inequality

$$\|u(t,\varphi)\|_{\mathscr{C}_\alpha}\leq\gamma_0+\|(-A)^{\alpha-1}\|(1+b_\alpha)\mu_0\int_0^{b_\varphi}k_1(s)\|u_s(.,\varphi)\|_{\mathscr{C}_\alpha}ds.$$

Then by Gronwall's lemma, we deduce that

$$\|u(t,\varphi)\|_{\mathscr{C}_\alpha}\leq\gamma_0\exp\Big[\|(-A)^{\alpha-1}\|(1+b_\alpha)\mu_0\int_0^{b_\varphi}k_1(s)ds]<+\infty \text{ for } t\in[0,b_\varphi[.$$

Then we have

$$\overline{\lim_{t\to b_\varphi^-}}\|u(t,\varphi)\|_{\mathscr{C}_\alpha}<+\infty,$$

which contradicts our hypothesis. Then the mild solution is global. □

($\mathbf{H}_4$) $t\mapsto C(t)$ is compact on an interval of positive length.

Lemma 4.4. [18] *Assume that ($\boldsymbol{H}_4$) holds. Then $C(t)$ is compact for every $t\in\mathbb{R}$ and in particular the identity is compact and X is necessarily finite dimensional.*

Theorem 4.5. *Assume that ($\boldsymbol{H}_0$), ($\boldsymbol{H}_1$), ($\boldsymbol{H}_2$) and ($\boldsymbol{H}_4$) hold. Then the flow $U(t)$ defined from $\mathscr{C}_\alpha$ to $\mathscr{C}_\alpha$ by $U(t)\varphi=u_t(.,\varphi)$ is compact for $t>r$, where $u_t(.,\varphi)$ denotes the mild solution starting from φ.*

Proof. We need to use Ascoli-Arzela's theorem. To do this, let us pose $E=\{\varphi_\gamma:\gamma\in\Gamma\}$ be a bounded subset of $\mathscr{C}_\alpha$. Let $t>r$ be fixed, but arbitrary.
One can see that from $(\mathbf{H}_1)$ and equation (3.2), there exists a positive constant $N_1>0$ such that

$$\|f(t,u_t(\varphi_\gamma),u_t'(\varphi_\gamma))\|\le N\|u_t(\varphi_\gamma)\|+\|f(t,0,0)\|=N_1 \text{ for } \gamma\in\Gamma.$$

For each $\gamma\in\Gamma$, we define $F_\gamma\in\mathscr{C}_\alpha$ by $F_\gamma=u_t(.,\varphi_\gamma)$. We need to show that for $\theta\in[-r,0]$, the set $\{F_\gamma(\theta):\gamma\in\Gamma\}$ is precompact in X_α. For any $\gamma\in\Gamma$, we have

$$F_\gamma(\theta)=C(t+\theta)\varphi_\gamma(0)+S(t+\theta)\varphi_\gamma'(0)+\int_0^{t+\theta}S(t+\theta-s)f(s,u_s(.,\varphi_\gamma),u_s'(.,\varphi_\gamma))ds,$$

for $-r\le\theta\le 0$. We will prove in two steps.

Step 1: The family $\{F_\gamma:\gamma\in\Gamma\}$ is equicontinuous. Let $\gamma\in\Gamma$, let $0<\varepsilon<t-r$ and let $-r\le\hat\theta\le\theta\le 0$ with $\hat\theta$ be fixed and $h=\theta-\hat\theta$. Then we have

$$\begin{aligned}
&\|(-A)^\alpha\big(F_\gamma(\theta+h)-F_\gamma(\theta)\big)\|\\
\le\ & |\big(C(t+\hat\theta+h)-C(t+\hat\theta)\big)\varphi_\gamma(0)|_\alpha+|\big(S(t+\hat\theta+h)-S(t+\hat\theta)\big)\varphi_\gamma'(0)|_\alpha\\
+\ & \Big|\int_0^{t+\hat\theta-\varepsilon}\big(S(t+\hat\theta+h-s)-S(t+\hat\theta-s)\big)f(s,u_s(.,\varphi_\gamma),u_s'(.,\varphi_\gamma))ds\Big|_\alpha\\
+\ & \Big|\int_{t+\hat\theta-\varepsilon}^{t+\hat\theta}\big(S(t+\hat\theta+h-s)-S(t+\hat\theta-s)\big)f(s,u_s(.,\varphi_\gamma),u_s'(.,\varphi_\gamma))ds\Big|_\alpha\\
+\ & \Big|\int_{t+\hat\theta}^{t+\hat\theta+h}S(t+\hat\theta+h-s)f(s,u_s(.,\varphi_\gamma),u_s'(.,\varphi_\gamma))ds\Big|_\alpha\\
\le\ & |\big(C(t+\hat\theta+h)-C(t+\hat\theta)\big)\varphi_\gamma(0)|_\alpha+|\big(S(t+\hat\theta+h)-S(t+\hat\theta)\big)\varphi_\gamma'(0)|_\alpha\\
+\ & \Big\|(-A)^{\alpha-1}\Big[\int_0^{t+\hat\theta-\varepsilon}\frac{d}{ds}\big((C(t+\hat\theta+h-s)-C(t+\hat\theta-s))f(s,u_s(.,\varphi_\gamma),u_s'(.,\varphi_\gamma))\big)ds\\
-\ & \int_0^{t+\hat\theta-\varepsilon}(C(t+\hat\theta+h-s)-C(t+\hat\theta-s))\frac{d}{ds}\big(f(s,u_s(.,\varphi_\gamma),u_s'(.,\varphi_\gamma))\big)ds\Big]\Big\|\\
+\ & \Big\|(-A)^{\alpha-1}\Big[\int_{t+\hat\theta-\varepsilon}^{t+\hat\theta}\frac{d}{ds}\big((C(t+\hat\theta+h-s)-C(t+\hat\theta-s))f(s,u_s(.,\varphi_\gamma),u_s'(.,\varphi_\gamma))\big)ds\\
-\ & \int_{t+\hat\theta-\varepsilon}^{t+\hat\theta}(C(t+\hat\theta+h-s)-C(t+\hat\theta-s))\frac{d}{ds}\big(f(s,u_s(.,\varphi_\gamma),u_s'(.,\varphi_\gamma))\big)ds\Big]\Big\|\\
+\ & \Big\|(-A)^{\alpha-1}\Big[\int_{t+\hat\theta}^{t+\hat\theta+h}\frac{d}{ds}\big(C(t+\hat\theta+h-s)f(s,u_s(.,\varphi_\gamma),u_s'(.,\varphi_\gamma))ds\big)\\
-\ & \int_{t+\hat\theta}^{t+\hat\theta+h}(C(t+\hat\theta+h-s)\frac{d}{ds}\big(f(s,u_s(.,\varphi_\gamma),u_s'(.,\varphi_\gamma))\big)ds\Big]\Big\|.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
& \|(-A)^{\alpha}\big(F_{\gamma}(\theta+h)-F_{\gamma}(\theta)\big)\| \\
\leq\ & \|(-A)^{\alpha-1}\|\Big[\|(C(h+\varepsilon)-C(\varepsilon))f(t+\hat{\theta}-\varepsilon,u_{t+\hat{\theta}-\varepsilon}(.,\varphi_{\gamma}),u'_{t+\hat{\theta}-\varepsilon}(.,\varphi_{\gamma}))\| \\
+\ & \|\big(C(h+\hat{\theta}+h)-C(t+\hat{\theta})\big)f(0,u_0(.,\varphi_{\gamma}),u'_0(.,\varphi_{\gamma}))\| \\
+\ & \|C(t+\hat{\theta}+h)-C(t+\hat{\theta})\| \\
& \times\|f(t+\hat{\theta}-\varepsilon,u_{t+\hat{\theta}-\varepsilon}(.,\varphi_{\gamma}),u'_{t+\hat{\theta}-\varepsilon}(.,\varphi_{\gamma}))-f(0,u_0(.,\varphi_{\gamma}),u'_0(.,\varphi_{\gamma}))\| \\
+\ & \|\big(C(h+\varepsilon)-I\big)f(t+\hat{\theta},u_{t+\hat{\theta}}(.,\varphi_{\gamma}),u'_{t+\hat{\theta}}(.,\varphi_{\gamma}))\| \\
+\ & \|\big(C(h+\varepsilon)-C(\varepsilon)\big)f(t+\hat{\theta}-\varepsilon,u_{t+\hat{\theta}-\varepsilon}(.,\varphi_{\gamma}),u'_{t+\hat{\theta}-\varepsilon}(.,\varphi_{\gamma}))\| \\
+\ & \|C(t+h+\hat{\theta})-C(t+\hat{\theta})\|\|f(t+\hat{\theta},u_{t+\hat{\theta}}(.,\varphi_{\gamma}),u'_{t+\hat{\theta}}(.,\varphi_{\gamma})) \\
-\ & f(t+\hat{\theta}-\varepsilon,u_{t+\hat{\theta}-\varepsilon}(.,\varphi_{\gamma}),u'_{t+\hat{\theta}-\varepsilon}(.,\varphi_{\gamma}))\| \\
+\ & \|f(t+\hat{\theta}+h,u_{t+\hat{\theta}+h}(.,\varphi_{\gamma}),u'_{t+\hat{\theta}+h}(.,\varphi_{\gamma}))-C(h)f(t+\hat{\theta},u_{t+\hat{\theta}}(.,\varphi_{\gamma}),u'_{t+\hat{\theta}}(.,\varphi_{\gamma}))\| \\
+\ & M_1e^{\omega b}\|f(t+\hat{\theta}+h,u_{t+\hat{\theta}+h}(.,\varphi_{\gamma}),u'_{t+\hat{\theta}+h}(.,\varphi_{\gamma}))-f(t+\hat{\theta},u_{t+\hat{\theta}}(.,\varphi_{\gamma}),u'_{t+\hat{\theta}}(.,\varphi_{\gamma}))\|\Big].
\end{aligned}$$

Since $(-A)^{\alpha-1}$ is compact from X to X, $C(t)$ is compact and moreover $(C(t)_{t\in\mathbb{R}})$ is uniformly continuous on compact subset of X. Then we conclude that $\{F_{\gamma}:\gamma\in\Gamma\}$ is equicontinuous.

Step 2: For $\theta\in[-r,0]$ the set $\{F_{\gamma}(\theta):\gamma\in\Gamma\}$ is precompact in X_{α}.
For fixed $\theta\in[-r,0]$, we choose $0<\beta$ such that $\alpha<\beta<1$ and we have

$$\begin{aligned}
\|(-A)^{\beta}F_{\gamma}(\theta)\| \leq\ & \|-(-A)^{\beta-1}C(t+\theta)A\varphi_{\gamma}(0)\|+\|-(-A)^{\beta-1}S(t+\theta)A\varphi'_{\gamma}(0)\| \\
+\ & \left\|-(-A)^{\beta-1}\int_0^{t+\theta}AS(t+\theta-s)f(s,u_s(.,\varphi_{\gamma}),u'_s(.,\varphi_{\gamma}))ds\right\| \\
\leq\ & \|-(-A)^{\beta-1}C(t+\theta)A\varphi_{\gamma}(0)\|+\|-(-A)^{\beta-1}S(t+\theta)A\varphi'_{\gamma}(0)\| \\
+\ & \left\|-(-A)^{\beta-1}\int_0^{t+\theta}\Big(\int_0^{t+\theta-s}AC(\sigma)f(s,u_s(.,\varphi_{\gamma}),u'_s(.,\varphi_{\gamma}))d\sigma\Big)ds\right\| \\
\leq\ & \|(-A)^{\beta-1}\|\Big[M_1e^{\omega b}\big(\|A\varphi_{\gamma}(0)\|+\|A\varphi'\gamma(0)\|\big)+\mu_0bN_1\int_0^{t+\theta}ds\Big]<\infty.
\end{aligned}$$

Thus the family $\{(-A)^{\beta}F_{\gamma}(\theta):\gamma\in\Gamma\}$ is bounded in X. By $(\mathbf{H}_2)$, we deduce that $(-A)^{-\beta}:X\to X_{\alpha}$ is compact. It follows that the set $\{F_{\gamma}(\theta):\gamma\in\Gamma\}$ is relatively compact for each $\theta\in[-r,0]$ in X_{α}. So from **Step 1** to **Step 2** and by Ascoli-Arzela's theorem we can conclude that $U(t)$ is Compact for $t>r$. □

5. Regularity of the Mild Solution

In this section we will prove, under some conditions t that the mild solution obtained in Section 3 is the strict solution. We need the following results.

Definition 5.1. *A continuous function $u :]-r,+\infty[\to X_\alpha$ is called a strict solution of equation* (1.1) *if the following conditions hold*

(i) $u \in C^1([0,+\infty[;X_\alpha)\cap C^2([0,\infty[;X_\alpha)$;

(ii) u satisfies equation (1.1) *on* $[0,+\infty[$;

(iii) $u(\theta) = \varphi(\theta)$ for $-r \le \theta \le 0$.

Proposition 5.2. *Assume that* $(\boldsymbol{H}_{0)})$ *holds. If u is a strict solution of equation* (1.1)*, then*

$$u(t) = C(t)\phi(0) + S(t)\phi'(0) + \int_0^t S(t-s)f(s,u_s,u'_s)ds. \tag{5.1}$$

Proof. It is just the consequence of Theorem 2.5. In fact, let us pose $k(t) = g(t,u_t)$ and $h(t) = f(t,u_t,u'_t)$ for $t \ge 0$. The we get the desired results. □

Remark 5.3. *The converse is not true. In fact if u satisfies equation* (5.1)*, u may be not twice continuously differentiable, that is why we distinguish between mild and strict solutions.*

Now we make the following hypothesis:

($\mathbf{H}_4$): $f : [0,b]\times\mathscr{C}_\alpha\times\mathscr{C}_\alpha \to X$ is continuously differentiable and the partial derivatives $D_1 f$ and $D_2 f$ are are locally Lipschitz in the classical sense with respect to the second argument.

Theorem 5.4. *Assume that* $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_2)$, $(\boldsymbol{H}_3)$ *and* $(\boldsymbol{H}_4)$ *hold. Let* $\varphi \in C^3([-r,0],D((-A)^\alpha))$ *with* $\varphi(0),\varphi''(0) \in D(A)$, $\varphi'(0),\ \varphi^{(3)}(0) \in E$ *such that*

$$\varphi''(0) = A\varphi(0) + f(\varphi,\varphi') \text{ and } \varphi^{(3)} = A\varphi'(0).$$

Then the mild solution of equation (1.1) *is a strict solution.*

Proof. Let $\varphi \in C^3([-r,0],D((-A)^\alpha))$ such that $\varphi(0),\ \varphi''(0) \in D(A)$, $\varphi'(0),\ \varphi^{(3)}(0) \in E$ and $\varphi''(0) = A\varphi(0) + f(\varphi,\varphi')$ ans $\varphi^{(3)} = A\varphi'(0)$.

Let u be the corresponding mild solution of equation (1.1) which is defined on which is defined on some maximal interval $[0, b_\varphi[$ and let $b < b_\varphi$. Then by using the strict contraction principle, we can show that there exists a unique continuous function v such that

$$\begin{cases} v(t) = C(t)\Big[A\varphi(0) + f(\varphi, \varphi')\Big] + S(t)A\varphi'(0) + \int_0^t AC(t-s)g(s, u_s)ds \\ \qquad + \int_0^t C(t-s)[D_1 f(u_s, u_s')u_s' + D_2 f(u_s, u_s')v_s]ds, \\ v_0 = \varphi''. \end{cases}$$

Now, we define w by

$$\begin{cases} w(t) = \varphi'(0) + \int_0^t v(s)ds \text{ if } t \in [0, b], \\ w(t) = \varphi'(t) \text{ if } -r \le t \le 0, \\ w'(t) = \varphi''(t) \text{ if } -r \le t \le 0. \end{cases} \tag{5.2}$$

Then we can see that $w_t = \varphi' + \int_0^t v_s ds$ for $t \in [0, b]$.

Consequently the map $t \mapsto w_t$ and $t \mapsto \int_0^t C(t-s)f(u_s, w_s)ds$ are continuously differentiable. Then we have

$$\begin{aligned} \frac{d}{dt}\int_0^t C(t-s)f(u_s, w_s)dss &= \frac{d}{dt}\int_0^t C(s)f(u_{t-s}, w_{t-s})ds \\ &= C(t)f(u_0, w_0) + \int_0^t C(t-s)\Big[D_1 fs(u_s, u_s')u_s' + D_2 f(u_s, w_s)v_s\Big]ds \\ &= C(t)f(\varphi, \varphi') + \int_0^t C(t-s)\Big[D_1 fs(u_s, u_s')u_s' + D_2 f(u_s, w_s)v_s\Big]ds, \end{aligned}$$

it follows that

$$\int_0^t C(s)f(\varphi, \varphi')ds = \int_0^t C(s)f(u_s, u_s')ds - \int_0^t \int_0^s C(s-\tau)\Big[D_1 f(u_\tau, w_\tau)u_\tau' + D_2 f(u_\tau, w_\tau)v_\tau\Big]d\tau ds.$$

Consequently we have

$$\begin{aligned} w(t) &= \varphi'(0) + \int_0^t S(s)A\varphi'(0)ds + \int_0^t C(s)A\varphi(0)ds + \int_0^t C(t-s)f(u_s, w_s)ds \\ &\quad - \int_0^t \int_0^s C(s-\tau)\Big[D_1 f(u_\tau, w_\tau')u_s' + D_2 f(u_\tau, w_\tau')v_\tau\Big]d\tau ds \\ &\quad + \int_0^t \int_0^s C(s-\tau)\Big[D_1 f(u_\tau, u_\tau)u_\tau' + D_2 f(u_\tau, u_\tau)v_\tau\Big]d\tau ds. \end{aligned}$$

Moreover by Lemma 2.3, we have

$$\int_0^t C(s)A\varphi(0)ds = S(t)A\varphi(0)$$

$$\int_0^t S(s)A\varphi'(0)ds = C(t)\varphi'(0) - \varphi'(0).$$

It follows that

$$\begin{aligned} w(t) &= C(t)\varphi'(0) + S(t)A\varphi(0) + \int_0^t C(t-s)f(u_s,w_s)ds \\ &+ \int_0^t C(t-s)f(u_s,w_s)ds + \int_0^t\int_0^s C(s-\tau)\Big[D_1f(u_\tau,u'_\tau)u'_s + D_2f(u_\tau,u'_\tau)v_\tau\Big]d\tau ds \\ &- \int_0^t\int_0^s C(s-\tau)\Big[D_1f(u_\tau,w_\tau)u'_\tau + D_2f(u_\tau,w_\tau)v_\tau\Big]d\tau ds. \end{aligned}$$

Furthermore for $t \geq 0$, we know that

$$u'(t) = AS(t)\varphi(0) + C(t)\varphi'(0) + \int_0^t C(t-s)f(u_s,u'_s)ds,$$

then for $t \in [0,b]$, we have

$$\begin{aligned} u'(t) - w(t) &= \int_0^t C(t-s)[f(u_s,u'_s) - f(u_s,w_s)]ds + \int_0^t\int_0^s C(s-\tau)\Big[(D_1f(u_\tau,u'_\tau) - D_1f(u_\tau,u'_\tau))u'_\tau \\ &+(D_2f(u_\tau,u'_\tau) - D_2f(u_\tau,w_\tau))v_\tau d\tau\Big]ds. \end{aligned}$$

$$\begin{aligned} |u'(t) - w(t)|_\alpha &\leq \int_0^t |C(t-s)[f(s,u_s,u'_s) - f(s,u_s,w_s)]|_\alpha ds \\ &+ \int_0^t\int_0^s |C(s-\tau)(D_1f(u_\tau,u'_\tau) - D_1f(u_\tau,u'_\tau))u'_\tau|_\alpha d\tau ds \\ &+ \int_0^t\int_0^s |C(s-\tau)(D_2f(u_\tau,u'_\tau) - D_2f(u_\tau,w_\tau))v_\tau|_\alpha d\tau ds. \end{aligned}$$

(5.3)

Let us choose $F = \Big\{u'_s, w_s : s \in [0,b]\Big\}$. Then F is compact set. It follows that D_1f and D_2f are globally lipschitz on F. Let $L_1 > 0$ be such that for $t \in [0,b]$ and $x, y \in H$, then we have

$$\|f(x,x') - f(y,y')\| \leq L_1\|x-y\|_\alpha;$$

$$\|D_1f(x,x') - D_1f(y,y')\| \leq L_1\|x-y\|_\alpha;$$

$$\|D_2 f(x,x') - D_2 f(y,y')\| \le L_1 \|x-y\|_\alpha .$$

Consequently, using equation (5.3), we one can find a positive constance $\lambda(b)$ such that by Gronwall's lemma,

$$\|u'_t - w_t\|_\alpha \le \lambda(b) \int_0^t \|u'_s - w_s\|_\alpha ds,$$

then we deduce that $u' = w$. Consequently, we deduce that the mild solution is twice continuous differentiable from $[0,b]$ to X_α. Then functions $t \to g(t,u_t)$ and $t \to f(t,u_t,u'_t)$ are continuously differentiable on $[0,b]$. According to the Theorem 2.5, we conclude that u is a strict solution of equation (1.1) on $[0,b]$. This holds for any $b < b_\varphi$. □

6. Application

For our illustration, we propose to study the existence of solutions for the following model

$$(6.1)\quad \begin{cases} \dfrac{\partial^2}{\partial t^2} z(t,x) = \dfrac{\partial^2}{\partial x^2} z(t,x) + \displaystyle\int_{-r}^{0} h(t, \frac{\partial}{\partial x} z(t+\theta,x), \frac{\partial}{\partial x} z'(t+\theta,x)) d\theta \text{ for } t \ge 0 \text{ and } x \in [0,\pi], \\ z(t,0) = z(t,\pi) = 0 \text{ for } t \ge 0, \\ z(\theta,x) = \varphi_0(\theta)(x) \text{ for } \theta \in [-r,0] \text{ and } x \in [0,\pi], \end{cases}$$

where $h : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exists a positive constant L such that for $x, y, x_1, y_1 \in \mathbb{R}$,

$$|h(t,x,y) - h(t,x_1,y_2)| \le L\Big(|x-x_1| + |y-y_1|\Big).$$

We can choose for exemple

$$h(\theta,x,y) = e^{-\theta^2}[\sin(\frac{x}{2}) + \sin(\frac{y}{2})] \text{ for } (\theta,x,y) \in \mathbb{R}^- \times \mathbb{R} \times \mathbb{R}.$$

We can observe that

$$|h(t,x_1,y_1) - h(t,x_2,y_2)| \le \frac{1}{2}\Big(|x_1-x_2| + |y_1-y_2|\Big).$$

In the oder to rewrite equation (6.1) in the abstract form, we introduce the space $X = L^2([0,\pi];\mathbb{R})$ vanishing at 0 and π, equipped with the L^2 norm that is to say for all $x \in X$,

$$\|x\|_{L^2} = \Big(\int_0^\pi |x(s)|^2 ds\Big)^{\frac{1}{2}}.$$

Let $e_n(x) = \sqrt{\dfrac{2}{\pi}}\sin(nx)$, $x \in [0,\pi]$, $n \geq 1$, then $(e_n)_{n\geq 1}$ is an orthogonal base for X.
Let $A : X \to X$ be defined by

$$\begin{cases} Ay = y'' \\ D(A) = \big\{ y \in X : y, y' \text{ are absolutely continuous, } y'' \in X,\ y(0) = y(\pi) = 0 \big\}. \end{cases}$$

Then the operator is computed by

$$Ay = -\sum_{n=1}^{+\infty} n^2 (y, e_n) e_n,\ y \in D(A),$$

where

$$(u,v) = \int_0^{\pi} u(s)v(s)ds \ \text{ for } u, v \in X.$$

It is well known that A is the infinitesimal generator of strongly continuous cosine family $C(t), \in \mathbb{R}$ in X which is given by

$$C(t)y = \sum_{n=1}^{+\infty} \cos nt (y, e_n) e_n,\ y \in X.$$

and that the associated sine family is given by

$$S(t)y = \sum_{n=1}^{+\infty} \frac{1}{n} \sin nt (y, e_n) e_n,\ y \in X.$$

If we choose $\alpha = \dfrac{1}{2}$. then $(\mathbf{H}_0)$ and $(\mathbf{A}_1)$ are satisfied since

$$(-A)^{\frac{1}{2}} y = \sum_{n=1}^{+\infty} n (y, e_n) e_n,\ y \in D((-A)^{\frac{1}{2}}).$$

and

$$(-A)^{-\frac{1}{2}} y = \sum_{n=1}^{+\infty} \frac{1}{n} (y, e_n) e_n,\ y \in X.$$

From [13], the compactness of A^{-1} follows from Lemma 2.6 and the fact that the eigenvalues of $(-A)^{-\frac{1}{2}}$ are $\lambda_n = \dfrac{1}{n}$, $n = 1, 2 \ldots$, the $(\mathbf{H}_3)$ is satisfied.

We define the space

$$\mathscr{C}_{\frac{1}{2}} = C^1([-r,0], X_{\frac{1}{2}}),$$

where $C^1([-r,0],X_{\frac{1}{2}})$ is the space of bounded uniformly continuous differentiable from $[-r,0]$ into $X_{\frac{1}{2}}$, where $X_{\frac{1}{2}}$ is endowed with the norm

$$|\varphi|_{\frac{1}{2}} = \sup_{-r\leq\theta\leq 0} |\varphi(\theta)|.$$

Let $f : \mathbb{R}\times\mathscr{C}_{\frac{1}{2}}\times\mathscr{C}_{\frac{1}{2}} \longrightarrow X$ define by

$$f(t,\varphi,\varphi')(x) = \int_{-r}^{0} h(t,\frac{\partial}{\partial x}\varphi(\theta)(x),\frac{\partial}{\partial x}\varphi'(\theta)(x))d\theta \text{ for } x\in[0,\pi], t\geq 0,\ \varphi,\in\mathscr{C}_{\frac{1}{2}},$$

where $\varphi,\in\mathscr{C}_{\frac{1}{2}}$ define by

$$\varphi(\theta)(x) = \varphi_0(\theta,x),$$

and the norm in $\mathscr{C}_{\frac{1}{2}}$ is given by

$$\|\varphi\|_{\mathscr{C}_{\frac{1}{2}}} = \sup_{\theta\in[-r,0]}\Big(\int_0^\pi \big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]\big|^2 dx\Big)^{\frac{1}{2}} + \sup_{\theta\in[-r,0]}\Big(\int_0^\pi \big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]\big|^2 dx\Big)^{\frac{1}{2}}.$$

Let us pose $v(t) = z(t,x)$. Then equation (6.1) takes the following abstract form

$$\text{(6.2)}\qquad \begin{cases} v''(t) = Av(t) + f(t,v_t,v'_t) \text{ for } t\geq 0,\\ v_0 = \varphi\in\mathscr{C}_{\frac{1}{2}},\\ v'_0 = \varphi'\in\mathscr{C}_{\frac{1}{2}}. \end{cases}$$

From [13], for all $y\in X_{\frac{1}{2}}$, y is absolutely continuous and $|y|_{\frac{1}{2}} = |y|_{L^2}$ Let $\varphi,\psi\in C^1([-r,0],X_{\frac{1}{2}})$, since
$|h(t,x_1,y_1)-h(t,x_2,y_2)| \leq \frac{1}{2}\Big(|x_1-x_2|+\|y_1-y_2|\Big)$

$$\begin{aligned} |f(t,\varphi,\varphi')-f(t,\psi,\psi')|_{L^2} &\leq \Big(\int_0^\pi\Big(\int_{-r}^{0} h(t,\frac{\partial}{\partial x}[\varphi(\theta)(x)],\frac{\partial}{\partial x}[\varphi'(\theta)(x)]d\theta\Big) \\ &\quad +\Big(\int_0^\pi\Big(\int_{-r}^{0} h(t,\frac{\partial}{\partial x}[\psi(\theta)(x)],\frac{\partial}{\partial x}[\psi'(\theta)(x)])d\theta\Big)^2 dx\Big)^{\frac{1}{2}} \\ &\leq \frac{1}{2}rL_h\Big[\Big(\int_0^\pi\Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]-\frac{\partial}{\partial x}[\psi(\theta)(x)]\Big|^2 dx\Big)^{\frac{1}{2}} \\ &\quad +\Big(\int_0^\pi\Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]-\frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|^2 dx\Big)^{\frac{1}{2}}\Big]. \end{aligned}$$

By Minkowski Lemma, we have

$$
\begin{aligned}
|f(t,\varphi,\varphi')-f(t,\psi,\psi')|_{L^2} &\leq \frac{1}{2}rL_h\Big[\Big(\int_0^\pi\Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]-\frac{\partial}{\partial x}[\psi(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}} \\
&\quad+\Big(\int_0^\pi\Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]-\frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\Big] \\
&\leq \frac{1}{2}rL_h\Big[\sup_{\theta\in[-r,0]}\Big(\int_0^\pi\Big|\frac{\partial}{\partial x}[\varphi(\theta)(x)]-\frac{\partial}{\partial x}[\psi(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}} \\
&\quad+\sup_{\theta\in[-r,0]}\Big(\int_0^\pi\Big|\frac{\partial}{\partial x}[\varphi'(\theta)(x)]-\frac{\partial}{\partial x}[\psi'(\theta)(x)]\Big|^2dx\Big)^{\frac{1}{2}}\Big],
\end{aligned}
$$

which implies that

$$
|f(t,\varphi,\varphi')-f(t,\psi,\psi')|_{L^2}\leq\frac{1}{2}rL_h\|\varphi-\psi\|_{\mathscr{C}_{\frac{1}{2}}}.
$$

this implies that f is is a Lipschitz function with respect to the second argument. Then $(\mathbf{H}_4)$ is true. For the regularity, we make the following assumptions.
$(\mathbf{H}_5)$ $h\in C^1(\mathbb{R}\times\mathbb{R}\times\mathbb{R};\mathbb{R})$ such that $\frac{\partial h}{\partial t}$, $\frac{\partial h}{\partial x}$ and $\frac{\partial h}{\partial y}$ are locally Lipschitz continuous.
$(\mathbf{H}_6)$

$$
\begin{cases}
\varphi\in C^3([-r,0]\times[0,\pi])\text{ such that }\varphi(0),\varphi''(0)\in D(A),\ \varphi'(0),\ \varphi^{(3)}\in E,\\
\dfrac{\partial^2}{\partial\theta^2}\varphi(0,x)=\dfrac{\partial^2}{\partial x^2}\varphi(0,x)+\displaystyle\int_{-r}^0 h(0,\varphi(\theta,x))d\theta\text{ for }x\in[0,\pi],\\
\dfrac{\partial^3}{\partial\theta^3}\varphi(0)(x)=\dfrac{\partial^2}{\partial\theta^2}\varphi'(0,x)\text{ for }x\in[0,\pi].
\end{cases}
$$

We obtain the following important result.

Proposition 6.1. *Under the above assumptions, equation* (6.2) *has a unique strict solution v and the solution defined $u(t,x,x')=v(t)(x,x')$ for $t\geq0$ and $x\in[0,\pi]$ becomes a solution of* (6.2).

7. Conclusion

In this paper we have shown the existence, regularity and compactness in the α-norm for nonlinear second order differential with finite delay using the cosine family theory, such as global existence, continuous dependence on the initial data, blowing up of solutions and regularity of solutions. It is well known that impulsive differential equations, which are differential equations with impulse effect, appear as a natural description of observed evolution phenomena of several real-world problems. To this end, our next challenge is to study in α-norm some impulsive nonlinear second-order differential equations with infinite delay.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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