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ON EPIMORPHISMS AND SEMINORMAL IDENTITIES

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Abstract. Khan and Shah associated two natural numbers with a seminormal identity. Using these natural numbers, we further enlarge the class of heterotypical identities of which both sides contain repeated variables which are preserved under epis in conjunction with a seminormal permutation identity.

Keywords: Zigzag, epimorphism, dominion, preserved under epis, heterotypical identity, seminormal identity.

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1. Introduction

Let U and S be any semigroups with U a subsemigroup of S. Following Isbell [5], we say that U dominates an element d of S if for every semigroup T and for all homomorphisms $\alpha, \beta : S \to T$, $u\alpha = u\beta$ for all $u \in U$ implies $d\alpha = d\beta$. The set of all elements of S dominated by U is called the *dominion* of U in S, and we denote it by $Dom(U, S)$. It may easily be seen that $Dom(U, S)$ is a subsemigroup of S containing U. A semigroup U is said to be *saturated* if $Dom(U, S) \neq S$ for every properly containing semigroup S, and *epimorphically embedded* or *dense* in S if $Dom(U, S) = S$.

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A morphism $\alpha : S \to T$ in the category of all semigroups is called an *epimorphism* (*epi* for short) if for all morphisms β, γ, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$. Every onto morphism is epi, but the converse is not true in general. It may easily be checked that $\alpha : S \to T$ is epi if and only if the inclusion map $i : S\alpha \to T$ is epi and the inclusion map $i : U \to S$ is epi if and only if $Dom(U, S) = S$. A variety $\mathcal{V}$ of semigroups is said to be *saturated* if all its members are saturated and *epimorphically closed* or *closed under epis* if whenever $S \in \mathcal{V}$ and $\varphi : S \to T$ is epi in the category of all semigroups, then $T \in \mathcal{V}$ or equivalently whenever $U \in \mathcal{V}$ and $Dom(U, S) = S$, then $S \in \mathcal{V}$.

An identity μ is said to be preserved under epis in conjunction with an identity τ if whenever S satisfies τ and μ, and $\varphi : S \to T$ is an epimorphism in the category of all semigroups, then T also satisfies τ and μ; or equivalently, whenever U satisfies τ and μ and $Dom(U, S) = S$, then S also satisfies τ and μ.

An identity of the form

$$x_1x_2 \cdots x_n = x_{i_1}x_{i_2} \cdots x_{i_n} \ (n \geq 2) \tag{1}$$

is called a permutation identity, where i is any permutation of the set $\{1, 2, 3, \ldots, n\}$ and i_k $(1 \leq k \leq n)$ is the image of k under the permutation i. A permutation identity of the form (1) is said to be nontrivial if the permutation i is different from the identity permutation. Further, a nontrivial permutation identity $x_1x_2 \cdots x_n = x_{i_1}x_{i_2} \cdots x_{i_n}$ is called *seminormal* if $i_1 = 1$ and $i_n = n$. A semigroup S satisfying a nontrivial permutation identity is said to be permutative while a variety $\mathcal{V}$ of semigroups is said to be permutative if it admits a nontrivial permutation identity. For any word u, the *content* of u (necessarily finite) is the set of all variables appearing in u and is denoted by $C(u)$. An identity $u = v$ is said to be *heterotypical* if $C(u) \neq C(v)$; otherwise *homotypical*. A variety $\mathcal{V}$ of semigroups is said to be *heterotypical* if it admits a heterotypical identity.

Khan [8], jointly with Higgins, has shown that any semigroup variety satisfying a permutation identity $x_1x_2 \cdots x_n = x_{i_1}x_{i_2} \cdots x_{i_n}$, where $i_1 \neq 1$ or $i_n \neq n$, is epimorphically closed. Higgins [3] found an example of an identity whose both sides contain repeated variables and is not preserved under epis in conjunction with the identity $xyzt = xzyt$ (a seminormal identity). Thus the problem of finding those semigroup identities whose both sides contain repeated variables and are preserved under epis in conjunction with a seminormal identity appears worthwhile.

A necessary condition for a heterotypical variety to be saturated is that it admits a heterotypical identity of which atleast one side has no repeated variable (see Higgins [2], and Khan [6] for sufficient condition). Since every saturated variety of semigroups is epimorphically closed, all heterotypical identities of which atleast one side has no repeated variable are preserved under epis in conjunction with any non trivial permutation identity. In [8], Khan found some homotypical as well as heterotypical identities containing repeated variables on both sides that are preserved under epis in conjunction with any seminormal identity. Recently in [10], Khan and Shah have found some suffecient condition on homotypical identities containing repeated variables on both sides that are preserved under epis in conjunction with a seminormal identity. It is, therefore, natural to find all those heterotypical identities whose both sides contain repeated variables and are preserved under epis in conjunction with a seminormal identity.

In the present paper, we enlarge the class of heterotypical identities whose both sides contain repeated variables and are preserved under epis in conjunction with a seminormal identity. However, a complete determination of all such heterotypical identities to be preserved under epis in conjunction with a seminormal identity remains still an open problem.

2. Preliminaries

Now, we quote some results that will be used in rest of the paper. We shall be using standard notation and refer the reader to Clifford and Preston [1] and Howie [4] for any unexplained symbols and terminology. Further, in what follows, we will often speak of a semigroup *satisfying a semigroup identity* to mean that the semigroup in question *satisfies an identity of that type.*

Result 2.1 ([7, Proposition 3.1]). *Let S be any permutative semigroup satisfying (1) with $n \geq 3$.*

(i) *For each $g \in \{2,\ 3, \ldots, n\}$ such that $x_{g-1}x_g$ is not a subword of $x_{i_1}x_{i_2}\cdots x_{i_n}$, S also satisfies the permutation identity*

$$x_1x_2\cdots x_{g-1}xyx_g\cdots x_n = x_1x_2\cdots x_{g-1}yxx_g\cdots x_n.$$

(ii) *If $x_1 \neq x_{i_1}$, then S also satisfies the permutation identity*

$$xyx_1x_2\cdots x_n = yxx_1x_2\cdots x_n.$$

In the following result and elsewhere in the paper $S^{(m)}$, for any positive integer m and semigroup S, will denote the set of all m-fold products of elements of S.

Result 2.2 ([7, Proposition 6.3]). *Let S be any semigroup satisfying (1) with $n \geq 3$. Then for each $g \in \{2, 3, \ldots, n\}$ such that $x_{g-1}x_g$ is not a subword of $x_{i_1}x_{i_2}\cdots x_{i_n}$, for all $m \geq g-1$, $p \geq n-g+1$ and for all $u \in S^{(m)}, v \in S^{(p)}$, we have*

$$ux_1x_2v = ux_2x_1v, \qquad \text{for all } x_1, x_2 \in S.$$

In particular, $S^{(k)}$ satisfies the normality identity for all $k \geq max(g-1, n-g+1)$.

Result 2.3 ([8, Theorem 3.1]). *All permutation identities are preserved under epis.*

A most useful characterization of semigroup dominions is provided by Isbell's Zigzag Theorem.

Result 2.4 ([5, Theorem 2.3] or [4, Theorem VII.2.13]). *Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U, S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows*:

$$d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m}, \qquad (2)$$

where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $y_i, t_i \in S$ $(i = 1, 2, \ldots, m)$, and

$$a_0 = y_1a_1, \qquad a_{2m-1}t_m = a_{2m},$$

$$a_{2i-1}t_i = a_{2i}t_{i+1}, \qquad y_ia_{2i} = y_{i+1}a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorization is called a zigzag in S *over* U *with value* d*, length* m *and spine* $a_0, a_1, \ldots, a_{2m}$.

In whatever follows, we refer to the equations in Result 2.4 as *the zigzag equations*.

Result 2.5 ([7, Result 3]). *Let* U *be any subsemigroup of a semigroup* S *and let* $d \in Dom(U,S) \setminus U$. *If* (2) *is a zigzag of minimal length* m *over* U *with value* d*, then* $y_j, t_j \in S \setminus U$ *for all* $j = 1, 2, \ldots, m$.

Result 2.6 ([9, Proposition 2.1]). *Let* S *be any permutative semigroup satisfying* (1) *with* $n \geq 3$. *Then for each* $g \in \{2, 3, \ldots, n\}$ *such that* $x_{g-1}x_g$ *is not a subword of* $x_{i_1}x_{i_2}\cdots x_{i_n}$, *for all* $m \geq g-1,\ p \geq n-g+1$ *and for all* $u \in S^{(m)},\ v \in S^{(p)}$, *we have*

$$ux_1x_2\cdots x_\ell v = ux_{\lambda_1}x_{\lambda_2}\cdots x_{\lambda_\ell}v$$

for all $x_1, x_2, \ldots, x_\ell \in S$ $(\ell \geq 2)$, *where* λ *is any permutation of the set* $\{1, 2, \ldots, \ell\}$.

In the following results, let U and S be any semigroups with U dense in S.

Result 2.7 ([7, Result 4]). *For any* $d \in S \setminus U$ *and* k *any positive integer, if* (2) *is a zigzag of minimal length over* U *with value* d*, then there exist* $b_1, b_2, \ldots, b_k \in U$ *and* $d_k \in S \setminus U$ *such that* $d = b_1b_2\cdots b_kd_k$.

Result 2.8 ([7, Corollary 4.2]). *If* U *be permutative, then*

$$sx_1x_2\cdots x_kt = sx_{j_1}x_{j_2}\cdots x_{j_k}t$$

for all $x_1, x_2, \ldots, x_k \in S,\ s, t \in S \setminus U$ *and any permutation* j *of the set* $\{1, 2, \ldots, k\}$.

The following corollary easily folllows by Result 2.8

Corollary 2.9 ([9, Corollary 1.8]). *For any* $d \in S$ *and positive integer* k*, if* $d = b_1b_2\cdots b_kd_k$ *for some* $b_1, b_2, \ldots, b_k \in U$ *and* $d_k \in S \setminus U$ *such that* $b_1 = {y_1}'c_1$ *for some* ${y_1}'$ *in* $S \setminus U$, $c_1 \in U$, *then* $d^p = b_1^pb_2^p\cdots b_k^pd_k^p$ *for any positive integer* p.

The symmetrical statement in the following result is in addition to the original statement.

Result 2.10 ([8, Proposition 4.6]). *Assume that U is permutative. If $d \in S \setminus U$ and (2) is a zigzag of length m over U with value d such that $y_1 \in S \setminus U$, then $d^k = a_0^k t_1^k$ for each positive integer k; in particular, the conclusion holds if (2) is of minimal length. Symmetrically, if $d \in S \setminus U$ and (2) is a zigzag of length m over U with value d such that $t_m \in S \setminus U$, then $d^k = y_m^k a_{2m}^k$ for each positive integer k; in particular, the conclusion holds if (2) is of minimal length.*

Result 2.11 ([9, Proposition 2.2]). *Let U be any semigroup satisfying (1) with $n \geq 3$. Then for each $g \in \{2, 3, \ldots, n\}$ such that $x_{g-1}x_g$ is not a subword of $x_{i_1}x_{i_2}\cdots x_{i_n}$, for all $m \geq g-1$ and for all $u \in S^{(m)}$, $v \in S \setminus U$, we have*

$$ux_1x_2\cdots x_\ell v = ux_{\lambda_1}x_{\lambda_2}\cdots x_{\lambda_\ell}v$$

for all $x_1, x_2, \ldots, x_\ell \in S$ $(\ell \geq 2)$, where λ is any permutation of the set $\{1, 2, \ldots, \ell\}$. Symmetrically, for all $p \geq h-1$ such that $x_{n-h}x_{n-(h-1)}$ is not a subword of $x_{i_1}x_{i_2}\cdots x_{i_n}$ and for all $v \in S^{(p)}, u \in S \setminus U$, we have

$$ux_1x_2\cdots x_\ell v = ux_{\lambda_1}x_{\lambda_2}\cdots x_{\lambda_\ell}v$$

for all $x_1, x_2, \ldots, x_\ell \in S$ $(\ell \geq 2)$, where λ is any permutation of the set $\{1, 2, \ldots, \ell\}$.

3. Main results

Throughout the paper, we shall assume that U is any permutative semigroup satisfying a seminormal permutation identity and is dense in the semigroup S.

To avoid introduction of new symbols, we shall treat, wherever is appropriate, $\mathbf{x_1}, \ldots, \mathbf{x_r}$, $\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_s}, \mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w}_\ell, \mathbf{z_1}, \mathbf{z_2}, \ldots, \mathbf{z_p}$ etc. as variables as well as the members of a semigroup without explicit mention of distinction. Further for any word $\mathbf{u}$ and any variable $\mathbf{x}$ of $\mathbf{u}$, $|\mathbf{x}|_\mathbf{u}$ will denote the number of occurrences of $\mathbf{x}$ in the word $\mathbf{u}$.

Lemma 3.1. *Let u and v be any words in $w_1, w_2, \ldots, w_\ell$ and $z_1, z_2, \ldots, z_p$ respectively. Let $p_1, p_2, \ldots, p_r, q_1, q_2, \ldots, q_s$ are any positive integers such that $p_1 \leq p_2 \leq \cdots \leq p_r$; $q_s \leq \cdots \leq q_2 \leq q_1 (r, s \geq 1)$. If U satisfies the semigroup identity*

$$x_1^{p_1}\cdots x_r^{p_r}u(w_1,\ldots,w_\ell)y_1^{q_1}\cdots y_s^{q_s}=x_1^{p_1}\cdots x_r^{p_r}v(z_1,\ldots,z_p)y_1^{q_1}\cdots y_s^{q_s} \tag{3}$$

then (3) *is also satisfied for all* $x_1,x_2,\ldots,x_r,y_1,y_2,\ldots,y_s\in S$ *and* $w_1,w_2,\ldots,w_\ell,\ z_1,$ $z_2,\ldots,z_p$ *in* U.

Proof. Since U satisfies a seminormal identity, by Result 2.3, S also satisfies a seminormal identity. Now we shall show that the identity (3) satisfied by U is also satisfied when $x_1,x_2,\ldots,x_r,y_1,y_2,\ldots,y_s\in S$ and $z_1,z_2,\ldots,z_p,w_1,w_2,\ldots,w_\ell\in U$.

Case (i): First, take any $x_1,x_2,\ldots,x_r\in S$ and $y_1,\ldots,y_s,w_1,\ldots,w_\ell,z_1,\ldots,z_p\in U$. If $x_1,x_2,\ldots,x_r\in U$, then (3) holds trivially. So assume without loss of generality that $x_1\in S\setminus U$. Let (2) be a zigzag of minimal length m over U with value x_1. Then

$$\begin{aligned}
&x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}\\
&= y_m^{p_1}a_{2m}^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}\\
&\quad\text{(by the zigzag equations and Result 2.10)}\\
&= y_m^{p_1}a_{2m}^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}\text{(as U satisfies (3))}\\
&= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}.\\
&\quad\text{(by the zigzag equations and Result 2.10)}
\end{aligned}$$

We, now, assume inductively that the result is true for all $x_1,\ldots,x_{k-1}\in S$ and $x_k,\ldots,x_r$ in U. We shall prove that the result is also true for all $x_1,\ldots,x_k\in S$ and $x_{k+1},\ldots,x_r\in U$. Again if $x_k\in U$, then the result follows by the inductive hypothesis. So assume that $x_k\in S\setminus U$. Let (2) be a zigzag of minimal length in S over U with value x_k. Now,

$$\begin{aligned}
&x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}\\
&= x_1^{p_1}x_2^{p_2}\cdots x_{k-1}^{p_{k-1}}y_m^{p_k}a_{2m}^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}\\
&\quad\text{(by Result 2.10 and zigzag equations)}\\
&= w{y_m^{(m)}}^{p_k}{b_1^{(m)}}^{p_k}\cdots {b_{k-1}^{(m)}}^{p_k}a_{2m}^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}
\end{aligned}$$

(by Results 2.4 and 2.5 for some $b_1^{(m)},\ldots,b_{k-1}^{(m)}\in U$ and $y_m^{(m)}\in S\setminus U$ as y_m

in $S\backslash U$ and $a_{2m} = a_{2m-1}t_m$ with $t_m \in S\backslash U$ and where $w = x_1^{p_1}x_2^{p_2}\cdots x_{k-1}^{p_{k-1}}$)

$$= wy_m^{(m)^{p_k}}v^{(m)}b_1^{(m)^{p_1}}\cdots b_{k-1}^{(m)\,p_{k-1}}a_{2m}^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

(by Result 2.5 as $y_m^{(m)}, t_m \in S\backslash U$ and where $v^{(m)} = b_1^{(m)^{p_k-p_1}}\cdots b_{k-1}^{(m)\,p_k-p_{k-1}}$)

$$= wy_m^{(m)^{p_k}}v^{(m)}b_1^{(m)^{p_1}}\cdots b_{k-1}^{(m)\,p_{k-1}}a_{2m}^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

(as U satisfies (3))

$$= wy_m^{(m)^{p_k}}b_1^{(m)^{p_k}}\cdots b_{k-1}^{(m)\,p_k}a_{2m}^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

(by Result 2.5 and the definition of $v^{(m)}$)

$$= x_1^{p_1}x_2^{p_2}\cdots x_{k-1}^{p_{k-1}}y_m^{p_k}a_{2m}^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

(as $y_m^{(m)^{p_k}}b_1^{(m)^{p_k}}\cdots b_{k-1}^{(m)\,p_k} = y_m^{p_k}$ and $w = x_1^{p_1}x_2^{p_2}\cdots x_{k-1}^{p_{k-1}}$)

$$= x_1^{p_1}x_2^{p_2}\cdots x_{k-1}^{p_{k-1}}x_k^{p_k}x_{k+1}^{p_{k+1}}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

(by Result 2.10 and zigzag equations)

as required.

Case(ii): Now we show that (3) is satisfied for all $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s \in S$ and $w_1, w_2, \ldots, w_\ell, z_1, z_2, \ldots, z_p \in U$. Again, we can assume without loss of generality that $y_1 \in S \setminus U$. Let (2) be a zigzag of minimal length m over U with value y_1, we have

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)a_0^{q_1}t_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

(by the zigzag equations and Result 2.10)

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)a_0^{q_1}c_2^{(1)^{q_1}}\cdots c_s^{(1)^{q_1}}t_1^{(1)^{q_1}}y_2^{q_2}\cdots y_s^{q_s} \tag{4}$$

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)a_0^{q_1}c_2^{(1)^{q_2}}\cdots c_s^{(1)^{q_s}}w^{(1)}t_1^{(1)^{q_1}}y_2^{q_2}\cdots y_s^{q_s} \tag{5}$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)a_0^{q_1}{c_2^{(1)}}^{q_2}\cdots {c_s^{(1)}}^{q_s}w^{(1)}{t_1^{(1)}}^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

(as U satisfies (3))

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)a_0^{q_1}{c_2^{(1)}}^{q_1}\cdots {c_s^{(1)}}^{q_1}{t_1^{(1)}}^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

(by Result 2.5 and definition of $w^{(1)}$)

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)a_0^{q_1}t_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

(by Result 2.5 as ${c_2^{(1)}}^{q_1}\cdots {c_s^{(1)}}^{q_1}{t_1^{(1)}}^{q_1} = t_1^{q_1}$)

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

(by the zigzag equations and Result 2.10)

as the equalities (4) and (5) follow by Results 2.4 and 2.5 for some $c_2^{(1)},\ldots,c_s^{(1)}$ in U and $t_1^{(1)} \in S\setminus U$ as $y_1, t_1 \in S\setminus U$ and where $w^{(1)} = {c_2^{(1)}}^{q_1-q_2}\cdots {c_s^{(1)}}^{q_1-q_s}$ respectively.

Now, we assume inductively that the result is true for all $y_1,\ldots,y_{k-1} \in S$ and $y_k,\ldots,y_s$ in U. We shall prove that the result is also true for all $y_1,\ldots,y_{k-1},y_k \in S$ and $y_{k+1},\ldots,y_s$ in U. Again if $y_k \in U$, then the result follows by the inductive hypothesis. So assume that $y_k \in S\setminus U$. Let (2) be a zigzag of minimal length m in S over U with value y_k. Now as the equalities (6) and (7) follow by Results 2.4 and 2.5 for some $c_{k+1}^{(1)},\ldots,c_s^{(1)}$ in U and $t_1^{(1)} \in S\setminus U$ as $y_1, t_1 \in S\setminus U$ and where $v = y_{k+1}^{q_{k+1}}\cdots y_s^{q_s}$, and by Result 2.5 as $a_0 = y_1a_1,\ y_1, t_1^{(1)} \in S\setminus U$ and where $w^{(1)} = {c_{k+1}^{(1)}}^{q_k-q_{k+1}}\cdots {c_s^{(1)}}^{q_k-q_s}$ respectively, we have

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}t_1^{q_k}y_{k+1}^{q_{k+1}}\cdots y_s^{q_s}$$

(by Result 2.10 and zigzag equations)

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}{c_{k+1}^{(1)}}^{q_k}\cdots {c_s^{(1)}}^{q_k}{t_1^{(1)}}^{q_k}v \tag{6}$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}{c_{k+1}^{(1)}}^{q_{k+1}}\cdots {c_s^{(1)}}^{q_s}w^{(1)}{t_1^{(1)}}^{q_k}v \tag{7}$$

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}y_2^{q_2}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}{c_{k+1}^{(1)}}^{q_{k+1}}\cdots {c_s^{(1)}}^{q_s}w^{(1)}{t_1^{(1)}}^{q_k}v$$

(by the inductive hypothesis)

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}y_2^{q_2}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}c_{k+1}^{q_k}\cdots c_s^{q_k}{t_1^{(1)}}^{q_k}v$$

(by Result 2.5 and the definition of $w^{(1)}$)

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}\cdots y_{k-1}^{q_{k-1}}a_0^{q_k}t_1^{q_k}y_{k+1}^{q_{k+1}}\cdots y_s^{q_s}$$

(by Result 2.5 as ${c_{k+1}^{(1)}}^{q_k}\cdots {c_s^{(1)}}^{q_k}{t_1^{(1)}}^{q_k} = t_1^{q_k}$ and the definition of v)

$$= \quad x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}y_2^{q_2}\cdots y_{k-1}^{q_{k-1}}y_k^{q_k}y_{k+1}^{q_{k+1}}\cdots y_s^{q_s}$$

(by Result 2.10 and zigzag equations)

as required. This completes the proof of the lemma. □

Following [10], for any seminormal identity (1), let $g_0 = \min P$, the minimum of P, where

$$P = \{2 \le g \le n-2 : x_{g-1}x_g \text{ is not a subword of } x_{i_1}x_{i_2}\cdots x_{i_n}\}.$$

Similarly, let $h_0 = \min Q$, where

$$Q = \{1 \le h \le n-g_0-1 : x_{n-h}x_{n-(h-1)} \text{ is not a subword of } x_{i_1}x_{i_2}\cdots x_{i_n}\}.$$

In whatever follows, $\mathbf{g_0}$ and $\mathbf{h_0}$ will stand as defined above. We shall also assume that $\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_r},\ \mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_s}$ be any positive integers such that

$$\mathbf{p_1} + \cdots + \mathbf{p_r} \ge \mathbf{g_0} - \mathbf{1}, \mathbf{q_1} + \cdots + \mathbf{q_s} \ge \mathbf{h_0} - \mathbf{1},$$

$\mathbf{p_1} \le \cdots \le \mathbf{p_r}$ and $\mathbf{q_s} \le \cdots \le \mathbf{q_1} (\mathbf{r}, \mathbf{s} \ge \mathbf{1})$ without further mention.

The following corollary directly follows from Result 2.11 and Lemma 3.1.

Corollary 3.2. *Let $m_1, m_2, \ldots, m_\ell, n_1, n_2, \ldots, n_p$ be any positive integers. If U satisfies the identity*

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}w_1^{m_1}\cdots w_\ell^{m_\ell}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} = x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}w_{j_1}^{m_{j_1}}\cdots w_{j_\ell}^{m_{j_\ell}}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_1^{n_1}\cdots z_p^{n_p}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} = x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}z_{\lambda_1}^{n_{\lambda_1}}\cdots z_{\lambda_p}^{n_{\lambda_p}}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} \tag{8}$$

where j and λ are any permutations of $\{1,2,\ldots,\ell\}$ and $\{1,2,\ldots,p\}$ respectively, then (8) *is satisfied for all $x_1,x_2,\ldots,x_r,y_1,y_2,\ldots,y_s \in S$ and $w_1,w_2,\ldots,w_\ell,\ z_1,z_2,\ldots,z_p \in\ U$.*

The following corollary follows easily by Result 2.11 as $p_1+p_2+\cdots+p_r \geq g_0-1$ and $q_1+q_2+\cdots+q_s \geq h_0-1$.

Corollary 3.3. *Let S be any permutative semigroup satisfying a seminormal identity. Let u be any word in the variables $z_1,z_2,\ldots,z_\ell$ and let $z_j \in C(u)$, for some $j \in \{1,2,\ldots,\ell\}$, be such that $z_j \in S$. If $z_j = xa = by$, for all $x,y,a,b \in S$, then*

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(z_1,z_2,\ldots,z_j,\ldots,z_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}(x)^{|z_j|_u}u(z_1,z_2,\ldots,z_{j-1},a,z_{j+1},\ldots,z_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
$$[x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(z_1,z_2,\ldots,z_j,\ldots,z_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(z_1,z_2,\ldots,z_{j-1},b,z_{j+1},\ldots,z_\ell)(y)^{|z_j|_u}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}].$$

Further, if $z_j = s_1cs_2$ for all $s_1,c,s_2 \in S$, then

$$x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(z_1,z_2,\ldots,z_j,\ldots,z_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}(s_{k_1})^{|z_j|_u}(s_{k_2})^{|z_j|_u}u(z_1,z_2,\ldots,z_{j-1},c,z_{j+1},\ldots,z_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
$$[x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(z_1,z_2,\ldots,z_j,\ldots,z_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(z_1,z_2,\ldots,z_{j-1},c,z_{j+1},\ldots,z_\ell)(s_{k_1})^{|z_j|_u}(s_{k_2})^{|z_j|_u}y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}],$$

where k is any permutation on the set {1,2}.

Proposition 3.4. *Let u and v be any words in $w_1,w_2,\ldots,w_\ell$ and $z_1,z_2,\ldots,z_p$ respectively and let the identity*

$$x_1^{p_1}\cdots x_r^{p_r}u(w_1,\ldots,w_\ell)y_1^{q_1}\cdots y_s^{q_s} = x_1^{p_1}\cdots x_r^{p_r}v(z_1,\ldots,z_p)y_1^{q_1}\cdots y_s^{q_s}$$

holds for all $x_1,x_2,\ldots,x_r,y_1,y_2,\ldots,y_s \in S$ and $w_1,w_2,\ldots,w_\ell,\ z_1,z_2,\ldots,z_p$ in U.
Then the identity

$$x^pu(w_1,\ldots,w_\ell)y_1^{q_1}\cdots y_s^{q_s} = x^pv(z_1,\ldots,z_p)y_1^{q_1}\cdots y_s^{q_s}$$

$$[x_1^{p_1}\cdots x_r^{p_r}u(w_1,\ldots,w_\ell)y^q = x_1^{p_1}\cdots x_r^{p_r}v(z_1,\ldots,z_p)y^q]$$

holds for all $x\in S\setminus U$, $y_1,\ldots,y_s\in S$, $w_1,\ldots,w_\ell$, $z_1,\ldots,z_p$ *in* U, *and positive integer* $p\geq p_r$ *[for all* $y\in S\setminus U$, $x_1,\ldots,x_r\in S$, $w_1,\ldots,w_\ell$, $z_1,\ldots,z_p$ *in* U, *and positive integer* $q\geq q_1$*].*

Proof. We have

$$\begin{aligned}
&x^pu(w_1,w_2,\ldots,w_\ell)y_1^{q_1}\cdots y_s^{q_s}\\
&= x^{p-p_r}x^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}\cdots y_s^{q_s}\\
&= x^{p-p_r}x^{'p_r}a_1^{p_r}\ldots a_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}\cdots y_s^{q_s}\\
&\quad\text{(by Result 2.7 and Corollary 2.9 for some } \mathrm{a}_1,\ldots,a_r\in U \text{ and}\\
&\quad x^{'}\in S\backslash U \text{ as } a_r=b_rz_r^{'} \text{ for some } z_r^{'}\in S\backslash U,\ b_r\in U)\\
&= x^{p-p_r}x^{'p_r}wa_1^{p_1}\ldots a_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}\cdots y_s^{q_s}\\
&\quad\text{(by Corollary 2.9 as } a_r=b_rz_r^{'} \text{ and where } w=a_1^{p_r-p_1}\cdots a_{r-1}^{p_r-p_{r-1}})\\
&= x^{p-p_r}x^{'p_r}wa_1^{p_1}\ldots a_r^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}\cdots y_s^{q_s}\\
&= x^{p-p_r}x^{'p_r}a_1^{p_r}\ldots a_r^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}\cdots y_s^{q_s}\text{(by defenition of } w)\\
&= x^{p-p_r}x^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}\cdots y_s^{q_s}\\
&\quad\text{(by Result 2.7 and Corollary 2.9 as } x^{p_r}=x^{'p_r}a_1^{p_r}\ldots a_r^{p_r})\\
&= x^pv(z_1,z_2,\ldots,z_p)y_1^{q_1}\cdots y_s^{q_s}
\end{aligned}$$

as required. Dual statement may be proved on the similar lines. □

Theorem 3.5. *Let* u *and* v *be any words in* $w_1,w_2,\ldots,w_\ell$ *and* $z_1,z_2,\ldots,z_p$ *respectively such that* $\forall\ i\in\{1,2,\ldots,\ell\}$ *and* $\forall\ j\in\{1,2,\ldots,p\}$, $min\{|w_i|_u,|z_j|_v\}\geq min\{p_r,q_1\}$. *Then all heterotypical identities of the form*

$$x_1^{p_1}\cdots x_r^{p_r}u(w_1,\ldots,w_\ell)y_1^{q_1}\cdots y_s^{q_s} = x_1^{p_1}\cdots x_r^{p_r}v(z_1,\ldots,z_p)y_1^{q_1}\cdots y_s^{q_s} \tag{9}$$

are preserved under epis in conjunction with a seminormal identity.

Proof. We shall prove the theorem for Case when $\min\{p_r, q_1\} = p_r$, the proof in other case follows along similar lines. As U satisfy a seminormal identity, by Result 2.3, S also satisfy a seminormal identity. We shall show that if U satisfies (9), then so does S. So let $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s, w_1, w_2, \ldots, w_\ell, z_1, z_2, \ldots, z_p$ in S. If all of $w_1, w_2, \ldots, w_\ell, z_1, z_2, \ldots, z_p$ are from U, then the result holds by Lemma 3.1. So, assume that not all of $w_1, w_2, \ldots, w_\ell, z_1, z_2, \ldots, z_p$ are from U. Now to show that the identity (9) is satisfied by S, we shall first prove that

$$x_1^{p_1}\cdots x_r^{p_r}u(w_1,\ldots,w_\ell)y_1^{q_1}\cdots y_s^{q_s} = x_1^{p_1}\cdots x_r^{p_r}v(v_1,\ldots,v_p)y_1^{q_1}\cdots y_s^{q_s} \tag{10}$$

for all $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s, w_1, w_2, \ldots, w_\ell \in S$ and $v_1, v_2, \ldots, v_p \in U$. We prove the equality (10) by induction on the number k of arguments $w_1, w_2, \ldots, w_k$ in S, by assuming that the remaining arguments $w_{k+1}, \ldots, w_\ell \in U$. So, first, assume that $w_1 \in S$ and $w_2, \ldots, w_\ell \in U$. When $w_1 \in U$, equality (10) is satisfied by Lemma 3.1. So, let $w_1 \in S\backslash U$. Let (2) be a zigzag of minimal length m over U with value w_1. Letting $x = x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}$ and $y = y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$, we have

$$\begin{aligned}
& x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s} \\
= \;& xu(y_m a_{2m}, w_2,\ldots,w_\ell)y \text{ (by the zigzag equations)} \\
= \;& x(y_m)^{|w_1|_u}u(a_{2m}, w_2,\ldots,w_\ell)y \text{ (by Corollary 3.3)} \\
= \;& x(y_m)^{|w_1|_u}v(v_1, v_2,\ldots,v_p)y \text{ (by Proposition 3.4 as } y_m \in S\setminus U \text{ and } |w_1|_u \geq p_r) \\
= \;& x(y_m)^{|w_1|_u}u(a_{2m-1}, w_2,\ldots,w_\ell)y \\
& \text{(by Proposition 3.4 as } y_m \in S\setminus U \text{ and } |w_1|_u \geq p_r) \\
= \;& xu(y_m a_{2m-1}, w_2,\ldots,w_\ell)y \text{ (by Corollary 3.3)}
\end{aligned}$$

$$\begin{aligned}
&= xu(y_{m-1}a_{2m-2}, w_2 \ldots, w_\ell)y \text{ (by the zigzag equations)}\\
&= x(y_{m-1})^{|w_1|_u}u(a_{2m-2}, w_2, \ldots, w_\ell)y \text{ (by Corollary 3.3)}\\
&\vdots\\
&= x(y_1)^{|w_1|_u}u(a_2, w_2, \ldots, w_\ell)y\\
&= x(y_1)^{|w_1|_u}v(v_1, v_2, \ldots, v_p)y \text{ (by Proposition 3.4 as } y_1 \in S \setminus U \text{ and } |w_1|_u \geq p_r)\\
&= x(y_1)^{|w_1|_u}u(a_1, w_2, \ldots, w_\ell)y \text{ (by Proposition 3.4 as } y_1 \in S \setminus U \text{ and } |w_1|_u \geq p_r)\\
&= xu(y_1a_1, w_2, \ldots, w_\ell)y \text{ (by Corollary 3.3)}\\
&= xu(a_0, w_2, \ldots, w_\ell)y \text{ (by the zigzag equations)}\\
&= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(v_1, v_2, \ldots, v_p)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}\\
&\quad \text{(by Lemma 3.1 and as } x = x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r} \text{ and } y = y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s})
\end{aligned}$$

as required.

Next, assume inductively that the equality (10) holds for all $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s$, $w_1, w_2, \ldots, w_{k-1}$ in S and $w_k, w_{k+1}, \ldots, w_\ell$ in U. From this we shall prove that the equality (10) also holds for all $x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_s, w_1, w_2, \ldots, w_{k-1}, w_k$ in S and $w_{k+1}, \ldots, w_\ell \in U$. If $w_k \in U$, then the equality (10) follows by the inductive hypothesis. So, assume that $w_k \in S \setminus U$. Let (2) be a zigzag of minimal length m over U with value w_k. Now, for any $v_1, v_2, \ldots v_p \in U$, we have

$$\begin{aligned}
&x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1, w_2, \ldots, w_{k-1}, w_k, w_{k+1}, \ldots, w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}\\
&= xu(w_1, w_2, \ldots, w_{k-1}, y_ma_{2m}, w_{k+1}, \ldots, w_\ell)y \text{ (by the zigzag equations)}\\
&= x(y_m)^{|w_k|_u}u(w_1, w_2, \ldots, w_{k-1}, a_{2m}, w_{k+1}, \ldots, w_\ell)y \text{ (by Corollary 3.3)}
\end{aligned}$$

$= x(y_m)^{|w_k|_u} v(v_1, v_2, \ldots, v_p)y$(by the inductive hypothesis and Proposition 3.4 as $y_m \in S \setminus U$ and $|w_k|_u \geq p_r$)

$= x(y_m)^{|w_k|_u} u(w_1, w_2, \ldots, w_{k-1}, a_{2m-1}, w_{k+1}, \ldots, w_\ell)$(by the inductive hypothesis and Proposition 3.4 as $y_m \in S \setminus U$ and $|w_k|_u \geq p_r$)

$= xu(w_1, w_2, \ldots, w_{k-1}, y_m a_{2m-1}, w_{k+1}, \ldots, w_\ell)y$ (by Corollary 3.3)

$= xu(w_1, w_2, \ldots, w_{k-1}, y_{m-1} a_{2m-2}, w_{k+1}, \ldots, w_\ell)$ (by the zigzag equations)

$= x(y_{m-1})^{|w_k|_u} u(w_1, w_2, \ldots, w_{k-1}, a_{2m-2}, w_{k+1}, \ldots, w_\ell)y$ (by Corollary 3.3)

$\vdots$

$= xy_1^{|w_k|_u} u(w_1, w_2, \ldots, w_{k-1}, a_2, w_{k+1}, \ldots, w_\ell)y$

$= xy_1^{|w_k|_u} v(v_1, v_2, \ldots, v_p)y$(by the inductive hypothesis and Proposition 3.4 as $y_1 \in S \setminus U$ and $|w_k|_u \geq p_r$)

$= xy_1^{|w_k|_u} u(w_1, w_2, \ldots, w_{k-1}, a_1, w_{k+1}, \ldots, w_\ell)y$
(by the inductive hypothesis as $y_1 \in S \setminus U$ and $|w_k|_u \geq p_r$)

$= xu(w_1, w_2, \ldots, w_{k-1}, y_1 a_1, w_{k+1}, \ldots, w_\ell)y$ (by Corollary 3.3)

$= xu(w_1, w_2, \ldots, w_{k-1}, a_0, w_{k+1}, \ldots, w_\ell)y$ (by the zigzag equations)

$= x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} v(v_1, v_2, \ldots, v_p) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$
(by the inductive hypothesis and as $x = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r}$ and $y = y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$)

as required.

Similarly, we may prove that

$$x_1^{p_1}\cdots x_r^{p_r}v(z_1,\ldots,z_p)y_1^{q_1}\cdots y_s^{q_s}=x_1^{p_1}\cdots x_r^{p_r}u(u_1,\ldots,u_\ell)y_1^{q_1}\cdots y_s^{q_s} \tag{11}$$

for all $x_1,x_2,\ldots,x_r,y_1,y_2,\ldots,y_s,z_1,z_2,\ldots,z_p\in S$ and $u_1,u_2,\ldots,u_\ell\in U$.

Now, using Lemma 3.1 and equations (10) and (11), we have

$$\begin{aligned}
&x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,w_2,\ldots,w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}\\
&= xv(v_1,v_2,\ldots,v_p)y\\
&= xu(u_1,u_2,\ldots,u_\ell)y\\
&= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(z_1,z_2,\ldots,z_p)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}.\\
&\quad(\text{as } x=x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r} \text{ and } y=y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s})
\end{aligned}$$

This completes the proof of Theorem. □

Proposition 3.6. *Let u and v be any words in $w_1,\ldots,w_\ell,z_1,\ldots,z_p$ $(\ell,p\geq 1)$ and $w_1,\ldots,w_\ell$ $(\ell\geq 1)$ respectively such that $\forall\ i\in\{1,2,\ldots,\ell\}$, $min\{|w_i|_u,|w_i|_v\}\geq min\{p_r,q_1\}$. If U satisfies*

$$x_1^{p_1}\cdots x_r^{p_r}u(w_1,\ldots,w_\ell,z_1,\ldots,z_p)y_1^{q_1}\cdots y_s^{q_s}=x_1^{p_1}\cdots x_r^{p_r}v(w_1,\ldots,w_\ell)y_1^{q_1}\cdots y_s^{q_s}, \tag{12}$$

then (12) is also satisfied for all $x_1,\ldots,x_r,y_1,\ldots,y_s,w_1,\ldots,w_\ell\in S$ and for all $z_1,\ldots,z_p$ in U.

Proof. We shall prove the theorem for the case when $\min\{p_r,q_1\}=p_r$, the proof in other case follows along similar lines. As U satisfy a seminormal identity, by Result 2.3, S also satisfy a seminormal identity. We shall show that if U satisfies (12), then (12) is also satisfied for all $x_1,\ldots,x_r,y_1,\ldots,y_s,w_1,\ldots,w_\ell$ in S and $z_1,\ldots,z_p\in U$. If $x_1,\ldots,x_r,y_1,\ldots,y_s\in S$ and all of $w_1,\ldots,w_\ell,z_1,\ldots,z_p\in U$, then (12) holds by Lemma 3.1. So, assume first that not all of $w_1,\ldots,w_\ell$ are from U. We prove the equality

(12) by induction on the number k of arguments $w_1,\ldots,w_k$ of the word u in S, assuming that the remaining arguments $w_{k+1},\ldots,w_\ell\in U$. So assume that $w_1\in S$ and $w_2,\ldots,w_\ell$ are from U. When $w_1\in U$, then (12) is satisfied by Lemma 3.1. So assume that $w_1\in S\setminus U$. Let (2) be a zigzag of minimal length m over U with value w_1. Letting $x=x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}$ and $y=y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$, we have

$$
\begin{aligned}
&x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,\ldots,w_\ell,z_1,\ldots,z_p)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}\\
&= xu(y_ma_{2m},w_2,\ldots,w_\ell,z_1,\ldots,z_p)y \text{ (by the zigzag equations)}\\
&= x(y_m)^{|w_1|_u}u(a_{2m},w_2,\ldots,w_\ell,z_1,\ldots,z_p)y \text{ (by Corollary 3.3)}\\
&= x(y_m)^{|w_1|_u}v(a_{2m},w_2,\ldots,w_\ell)y\\
&\quad \text{(by Proposition 3.4 as } y_m\in S\setminus U \text{ and } |w_1|_u\geq p_r)\\
&= x(y_m)^{|w_1|_u}v(a_{2m-1}t_m,w_2,\ldots,w_\ell)y \text{ (by the zigzag equations)}\\
&= x(y_m)^{|w_1|_u}(t_m)^{|w_1|_v}v(a_{2m-1},w_2,\ldots,w_\ell)y \text{ (by Corollary 3.3)}\\
&= x(y_m)^{|w_1|_u}(t_m)^{|w_1|_v}u(a_{2m-1},w_2,\ldots,w_\ell,z_1,\ldots,z_p)y\\
&\quad \text{(by Proposition 3.4 as } t_m\in S\setminus U \text{ and } |w_1|_v\geq p_r)\\
&= x(t_m)^{|w_1|_v}u(y_ma_{2m-1},w_2,\ldots,w_\ell,z_1,\ldots,z_p)y\text{(by Corollary 3.3)}\\
&= x(t_m)^{|w_1|_v}u(y_{m-1}a_{2m-2},w_2,\ldots,w_\ell,z_1,\ldots,z_p)y \text{ (by the zigzag equations)}\\
&\ \vdots\\
&= x(t_2)^{|w_1|_v}u(y_1a_2,w_2,\ldots,w_\ell,z_1,\ldots,z_p)y\\
&= x(y_1)^{|w_1|_u}(t_2)^{|w_1|_v}u(a_2,w_2,\ldots,w_\ell,z_1,\ldots,z_p)y \text{ (by Corollary 3.3)}
\end{aligned}
$$

$$= x(y_1)^{|w_1|_u}(t_2)^{|w_1|_v}v(a_2, w_2, \ldots, w_\ell)y$$
(by Proposition 3.4 as $t_2 \in S \setminus U$ and $|w_1|_v \geq p_r$)

$$= x(y_1)^{|w_1|_u}v(a_2t_2, w_2, \ldots, w_\ell)y \text{ (by Corollary 3.3)}$$

$$= x(y_1)^{|w_1|_u}v(a_1t_1, w_2, \ldots, w_\ell)y \text{ (by the zigzag equations)}$$

$$= x(y_1)^{|w_1|_u}(t_1)^{|w_1|_v}v(a_1, w_2, \ldots, w_\ell)y \text{ (by Corollary 3.3)}$$

$$= x(y_1)^{|w_1|_u}(t_1)^{|w_1|_v}u(a_1, w_2, \ldots, w_\ell, z_1, \ldots, z_p)y$$
(by Proposition 3.4 as $t_1 \in S \setminus U$ and $|w_1|_v \geq p_r$)

$$= x(t_1)^{|w_1|_v}u(y_1a_1, w_2, \ldots, w_\ell, z_1, \ldots, z_p)y \text{ (by Corollary 3.3)}$$

$$= x(t_1)^{|w_1|_v}u(a_0, w_2, \ldots, w_\ell, z_1, \ldots, z_p)y \text{ (by the zigzag equations)}$$

$$= x(t_1)^{|w_1|_v}v(a_0, w_2, \ldots, w_\ell)y \text{ (by Proposition 3.4 as } t_1 \in S \setminus U \text{ and } |w_1|_v \geq p_r)$$

$$= xv(a_0t_1, w_2, \ldots, w_\ell)y \text{ (by Corollary 3.3)}$$

$$= x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}v(w_1, \ldots, w_\ell)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$$
(by the zigzag equations and as $x = x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}$ and $y = y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}$)

as required.

Next, assume inductively that (12) holds for all $x_1, \ldots, x_r, y_1, \ldots, y_s$ in S and all of $w_1, \ldots, w_{k-1} \in S$, and $w_k, w_{k+1}, \ldots w_\ell, z_1, \ldots, z_p \in U$. From this, we shall prove that (12) holds for all $x_1, \ldots, x_r, y_1, \ldots, y_s \in S$, $w_1, \ldots, w_{k-1}, w_k \in S$ and $w_{k+1}, \ldots w_\ell, z_1, \ldots, z_p$ in U. If $w_k \in U$, then the equality (12) follows by the inductive hypothesis. So, assume that $w_k \in S \setminus U$. Let (2) be a zigzag of minimum length m over U with value w_k. Now, we have

$$
\begin{aligned}
& x_1^{p_1}x_2^{p_2}\cdots x_r^{p_r}u(w_1,\ldots,w_k,\ldots,w_\ell,z_1,\ldots,z_p)y_1^{q_1}y_2^{q_2}\cdots y_s^{q_s}\\
= {} & xu(w_1,\ldots,y_ma_{2m},w_{k+1},\ldots,w_\ell,z_1,\ldots,z_p)y \text{ (by the zigzag equations)}\\
= {} & x(y_m)^{|w_k|_u}u(w_1,\ldots,a_{2m},w_{k+1},\ldots,w_\ell,z_1,\ldots,z_p)y \text{ (by Corollary 3.3)}\\
= {} & x(y_m)^{|w_k|_u}v(w_1,\ldots,a_{2m},w_{k+1},\ldots,w_\ell)y\\
& \text{(by the inductive hypothesis and Proposition 3.4 as } y_m\in S\setminus U,\ |w_k|_u\geq p_r)\\
= {} & x(y_m)^{|w_k|_u}v(w_1,\ldots,a_{2m-1}t_m,w_{k+1},\ldots,w_\ell)y \text{ (by the zigzag equations)}\\
= {} & x(y_m)^{|w_k|_u}(t_m)^{|w_k|_v}v(w_1,\ldots,a_{2m-1},w_{k+1},\ldots,w_\ell)y \text{ (by Corollary 3.3)}\\
= {} & x(y_m)^{|w_k|_u}(t_m)^{|w_k|_v}u(w_1,\ldots,a_{2m-1},w_{k+1},\ldots,w_\ell,z_1,\ldots,z_p)y\\
& \text{(by the inductive hypothesis and Proposition 3.4 as } t_m\in S\setminus U,\ |w_k|_v\geq p_r)\\
= {} & x(t_m)^{|w_k|_v}u(w_1,\ldots,y_ma_{2m-1},w_{k+1},\ldots,w_\ell,z_1,\ldots,z_p)y \text{ (by Corollary 3.3)}\\
= {} & x(t_m)^{|w_k|_v}u(w_1,\ldots,y_{m-1}a_{2m-2},w_{k+1},\ldots,w_\ell,z_1,\ldots,z_p)y\\
& \text{(by the zigzag equations)}\\
& \vdots\\
= {} & x(t_2)^{|w_k|_v}u(w_1,\ldots,y_1a_2,w_{k+1},\ldots,w_\ell,z_1,\ldots,z_p)y\\
= {} & x(y_1)^{|w_k|_u}(t_2)^{|w_k|_v}u(w_1,\ldots,a_2,w_{k+1},\ldots,w_\ell,z_1,\ldots,z_p)y \text{ (by Corollary 3.3)}\\
= {} & x(y_1)^{|w_k|_u}(t_2)^{|w_k|_v}v(w_1,\ldots,a_2,w_{k+1},\ldots,w_\ell)y\\
& \text{(by the inductive hypothesis and Proposition 3.4 as } t_2\in S\setminus U,\ |w_k|_v\geq p_r)\\
= {} & x(y_1)^{|w_k|_u}v(w_1,\ldots,a_2t_2,w_{k+1},\ldots,w_\ell)y \text{ (by Corollary 3.3)}
\end{aligned}
$$

$$= x(y_1)^{|w_k|_u} v(w_1, \ldots, a_1 t_1, w_{k+1}, \ldots, w_\ell) y \text{ (by the zigzag equations)}$$

$$= x(y_1)^{|w_k|_u} (t_1)^{|w_k|_v} v(w_1, \ldots, a_1, w_{k+1}, \ldots, w_\ell) y \text{ (by Corollary 3.3)}$$

$$= x(y_1)^{|w_k|_u} (t_1)^{|w_k|_v} u(w_1, \ldots, a_1, w_{k+1}, \ldots, w_\ell, z_1, \ldots, z_p) y$$
(by the inductive hypothesis and Proposition 3.4 as $t_1 \in S \setminus U,\ |w_k|_v \geq p_r$)

$$= x(t_1)^{|w_k|_v} u(w_1, \ldots, y_1 a_1, w_{k+1}, \ldots, w_\ell, z_1, \ldots, z_p) y \text{ (by Corollary 3.3)}$$

$$= x(t_1)^{|w_k|_v} u(w_1, \ldots, a_0, w_{k+1}, \ldots, w_\ell, z_1, \ldots, z_p) y \text{ (by the zigzag equations)}$$

$$= x(t_1)^{|w_k|_v} v(w_1, \ldots, a_0, w_{k+1}, \ldots, w_\ell) y$$
(by the inductive hypothesis and Proposition 3.4 as $t_1 \in S \setminus U,\ |w_k|_v \geq p_r$)

$$= xv(w_1, \ldots, a_0 t_1, w_{k+1}, \ldots, w_\ell) y \text{ (by Corollary 3.3)}$$

$$= x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} v(w_1, \ldots, w_k, \ldots, w_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$$
(by the zigzag equations and as $x = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r}$ and $y = y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$)

as required.

This completes the proof of the proposition. □

Proposition 3.7. Let U be a semigroup satisfying a seminormal identity and dense in S. Let u and v be any words in $w_1, w_2, \ldots, w_\ell,\ z_1, z_2 \ldots, z_p$ $(\ell, p \geq 1)$ and $w_1, w_2 \ldots, w_\ell$ $(\ell \geq 1)$ respectively such that for each $i \in \{1, \ldots, \ell\}$ and $j \in \{1, \ldots, p\}$, $\min\{|w_i|_u, |w_i|_v, |z_j|_u\} \geq \min\{p_r, q_1\}$. If U satisfies

$$x_1^{p_1} \cdots x_r^{p_r} u(w_1, \ldots, w_\ell, z_1, \ldots, z_p) y_1^{q_1} \cdots y_s^{q_s} = x_1^{p_1} \cdots x_r^{p_r} v(w_1, \ldots, w_\ell) y_1^{q_1} \cdots y_s^{q_s} \quad (13)$$

then so does S.

Proof. We shall prove the theorem in the Case when $\min\{p_r, q_1\} = p_r$, the proof in other case follows along similar lines. As U satisfy a seminormal identity, by Result 2.3, S also satisfy a seminormal identity. We shall show that if U satisfies (13), then so does S. If $x_1, \ldots, x_r, y_1, \ldots, y_s$ in S and all of $w_1, \ldots, w_\ell, z_1, \ldots, z_p \in U$, then (13) holds by Lemma 3.1, and if all of $x_1, \ldots, x_r, y_1, \ldots, y_s, w_1, \ldots, w_\ell \in S$ and $z_1, \ldots, z_p \in U$, then (13) holds by Proposition 3.6. So assume that not all of $z_1, \ldots, z_p \in U$. We prove the equality (13) by induction on the number k of arguments $z_1, \ldots, z_k$ of u in S, assuming that the remaining arguments $z_{k+1}, \ldots, z_p$ in U. First assume that $z_1 \in S$ and $z_2, \ldots, z_p \in U$. When $z_1 \in U$, then (13) is satisfied by Proposition 3.6. So assume that $z_1 \in S \setminus U$. By Result 2.4, let (2) be a zigzag of minimal length m over U with value z_1. Letting $x = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r}$ and $y = y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$, we have

$$
\begin{aligned}
& x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} u(w_1, \ldots, w_\ell, z_1, \ldots, z_p) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s} \\
= \;& x u(w_1, \ldots, w_\ell, y_m a_{2m}, z_2, \ldots, z_p) y \text{ (by the zigzag equations)} \\
= \;& x (y_m)^{|z_1|_u} u(w_1, \ldots, w_\ell, a_{2m}, z_2, \ldots, z_p) y \text{ (by Corollary 3.3)} \\
= \;& x (y_m)^{|z_1|_u} v(w_1, \ldots, w_\ell) y \text{ (by Proposition 3.6 as } |z_1|_u \geq p_r) \\
= \;& x (y_m)^{|z_1|_u} u(w_1, \ldots, w_\ell, a_{2m-1}, z_2, \ldots, z_p) y \text{ (by Proposition 3.6 as } |z_1|_u \geq p_r) \\
= \;& x u(w_1, \ldots, w_\ell, y_m a_{2m-1}, z_2, \ldots, z_p) y \text{ (by Corollary 3.3)} \\
= \;& x u(w_1, \ldots, w_\ell, y_{m-1} a_{2m-2}, z_2, \ldots, z_p) y \text{ (by the zigzag equations)} \\
= \;& x (y_{m-1})^{|z_1|_u} u(w_1, \ldots, w_\ell, a_{2m-2}, z_2, \ldots, z_p) y \text{ (by Corollary 3.3)} \\
= \;& x (y_{m-1})^{|z_1|_u} v(w_1, \ldots, w_\ell) y \text{ (by Proposition 3.6 as } |z_1|_u \geq p_r)
\end{aligned}
$$

$$
\begin{aligned}
&= x(y_{m-1})^{|z_1|_u} u(w_1, \dots, w_\ell, a_{2m-3}, z_2, \dots, z_p)y \text{ (by Proposition 3.6 as } |z_1|_u \geq p_r)\\
&\vdots\\
&= x(y_1)^{|z_1|_u} u(w_1, \dots, w_\ell, a_2, z_2, \dots, z_p)y\\
&= x(y_1)^{|z_1|_u} v(w_1, \dots, w_\ell)y \text{ (by Proposition 3.6 as } |z_1|_u \geq p_r)\\
&= x(y_1)^{|z_1|_u} u(w_1, \dots, w_\ell, a_1, z_2, \dots, z_p)y \text{ (by Proposition 3.6 as } |z_1|_u \geq p_r)\\
&= xu(w_1, \dots, w_\ell, y_1 a_1, z_2, \dots, z_p)y \text{ (by Corollary 3.3)}\\
&= xu(w_1, \dots, w_\ell, a_0, z_2, \dots, z_p)y \text{ (by the zigzag equations)}\\
&= x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} v(w_1, \dots, w_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}\\
&\quad \text{(by Proposition 3.6 and as } x = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} \text{ and } y = y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}) \text{ as required.}
\end{aligned}
$$

Next, assume inductively that (13) holds for all $z_k, z_{k+1}, \dots, z_p \in U$ and $x_1, x_2, \dots, x_r, y_1, y_2, \dots, y_s, w_1, \dots, w_\ell, z_1, \dots, z_{k-1} \in S$. From this, we shall prove that (13) holds for all $x_1, \dots, x_r, y_1, \dots, y_s, w_1, \dots, w_\ell, z_1, \dots, z_{k-1}, z_k$ in S and $z_{k+1}, \dots, z_p$ in U. If $z_k \in U$, then the equality (13) follows by the inductive hypothesis. So, assume that $z_k \in S \setminus U$. Let (2) be a zigzag of minimal length m over U with value z_k. Now

$$
\begin{aligned}
&x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} u(w_1, \dots, w_\ell, z_1, \dots, z_k, \dots, z_p) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}\\
&= xu(w_1, \dots, w_\ell, z_1, \dots, y_m a_{2m}, z_{k+1}, \dots, z_p)y \text{ (by the zigzag equations)}\\
&= x(y_m)^{|z_k|_u} u(w_1, \dots, w_\ell, z_1, \dots, a_{2m}, z_{k+1}, \dots, z_p)y \text{ (by Corollary 3.3)}\\
&= x(y_m)^{|z_k|_u} v(w_1, \dots, w_\ell)y \text{ (by the inductive hypothesis as } |z_k|_u \geq p_r)
\end{aligned}
$$

$$= x(y_m)^{|z_k|_u} u(w_1, \dots, w_\ell, z_1, \dots, a_{2m-1}, z_{k+1}, \dots, z_p)y$$
(by the inductive hypothesis as $|z_k|_u \geq p_r$)

$$= xu(w_1, \dots, w_\ell, z_1, \dots, y_m a_{2m-1}, z_{k+1}, \dots, z_p)y \text{ (by Corollary 3.3)}$$

$$= xu(w_1, \dots, w_\ell, z_1, \dots, y_{m-1} a_{2m-2}, z_{k+1}, \dots, z_p)y \text{ (by the zigzag equations)}$$

$$= x(y_{m-1})^{|z_k|_u} u(w_1, \dots, w_\ell, z_1, \dots, a_{2m-2}, z_{k+1}, \dots, z_p)y \text{ (by Corollary 3.3)}$$

$$= x(y_{m-1})^{|z_k|_u} v(w_1, \dots, w_\ell)y \text{ (by the inductive hypothesis as } |z_k|_u \geq p_r)$$

$$= x(y_{m-1})^{|z_k|_u} u(w_1, \dots, w_\ell, z_1, \dots, a_{2m-3}, z_{k+1}, \dots z_p)y$$
(by the inductive hypothesis as $|z_k|_u \geq p_r$)

$\vdots$

$$= x(y_1)^{|z_k|_u} u(w_1, \dots, w_\ell, z_1, \dots, a_2, z_{k+1}, \dots, z_p)y$$

$$= x(y_1)^{|z_k|_u} v(w_1, \dots, w_\ell)y \text{ (by the inductive hypothesis as } |z_k|_u \geq p_r)$$

$$= x(y_1)^{|z_k|_u} u(w_1, \dots, w_\ell, z_1, \dots, a_1, z_{k+1}, \dots, z_p)y$$
(by the inductive hypothesis as $|z_k|_u \geq p_r$)

$$= xu(w_1, \dots, w_\ell, z_1, \dots, y_1 a_1, z_{k+1}, \dots, z_p)y \text{ (by Corollary 3.3)}$$

$$= xu(w_1, \dots, w_\ell, z_1, \dots, a_0, z_{k+1}, \dots, z_p)y \text{ (by the zigzag equations)}$$

$$= x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r} v(w_1, \dots, w_\ell) y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$$
(by the inductive hypothesis and as $x = x_1^{p_1} x_2^{p_2} \cdots x_r^{p_r}$ and $y = y_1^{q_1} y_2^{q_2} \cdots y_s^{q_s}$)

as required. Thus the proof of the Proposition is completed. □

Now combining Propositions 3.6 and 3.7, we get the following.

Theorem 3.8. *All heterotypical identities of the forms*

$$x_1^{p_1}\cdots x_r^{p_r}u(w_1,\ldots,w_\ell,z_1,\ldots,z_p)y_1^{q_1}\cdots y_s^{q_s}=x_1^{p_1}\cdots x_r^{p_r}v(w_1,\ldots,w_\ell)y_1^{q_1}\cdots y_s^{q_s},$$

where u and v be any words in $w_1,\ldots,w_\ell,z_1,\ldots,z_p(\ell,p\geq 1)$ and $w_1,\ldots,w_\ell(\ell\geq 1)$ such that for each $i\in\{1,\ldots,\ell\}$ and $j\in\{1,\ldots,p\}$, $min\{|w_i|_u,|w_i|_v,|z_j|_u\}\geq min\{p_r,q_1\}$ are preserved under epis in conjunction with a seminormal identity.

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