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RESULTS OF SEMIGROUP OF LINEAR OPERATORS ON FAVARD CLASS AND INTERPOLATION SPACE

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Abstract. This paper presents the results of ω-order preserving partial contraction mapping ($\omega - OCP_n$) initiating a Favard class and interpolation. The study investigated some detailed properties of the Favard class concerning interpolation and perturbation theory on a Banach space. Then it was shown that the semigroup is bounded, closed, $\odot$-reflexive, and the limes superior of the Favard class can be replaced by a limes inferior on C_0-semigroup.

Keywords: interpolation; c_0-semigroup; closed linear operator.

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1. Introduction

Let $T(t)$ be C_0-semigroup on X. Define its Favard class by

$$Fav(T(t)) := \left\{ x \in X : \limsup_{t\to 0} \frac{1}{t} \|T(t)x - x\| < \infty \right\}.$$

In other words, $Fav(T(t))$ consists of those x whose orbits are Lipschitz continuous in neighbourhood of $t = 0$. We have below that the limes superior in the definition can be replaced by a

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limes interior. If $\|T(t)\| \leq Me^{\omega t}$, then we can easily check that

$$\|x\|_{Fav(T(t))} := \|x\| + \sup_{t>0} \frac{1}{t} \|e^{-\omega t} T(t)x - x\|$$

turns $Fav(T(t))$ into a Banach space.

Assume that X is a Banach space, $X_n \subseteq X$ is a finite set, $\omega - OCP_n$ the ω-order preserving partial contraction mapping, M_m be a matrix, $L(X)$ be a bounded linear operator on X, P_n a partial transformation semigroup, $\rho(A)$ a resolvent set, $\sigma(A)$ a spectrum of A and $A \in \omega - OCP_n$ is a generator of C_0-semigroup. This paper consist of results of ω-order preserving partial contraction mapping generating some results of Favard class and interpolation space.

Akinyele *et al.* [1], obtained some results of semigroup of linear operator in spectra theory. Batty *et al.* [2], showed some asymptotic behavior of semigroup of operators. Balakrishnan [3], obtained an operator calculus for infinitesimal generators of semigroup. Banach [4], established and introduced the concept of Banach spaces. Chill and Tomilov [5], deduced some resolvent approach to stability operator semigroup. Davies [6], introduced linear operators and their spectra. Engel and Nagel [7], obtained one-parameter semigroup for linear evolution equations. Nagel *et al.* [8], identified extrapolation spaces for unbounded operators. Neerven [9], presented some results on adjoint of semigroup of linear operators. Omosowon *et al.* [10], proved some analytic results of semigroup of linear operator with dynamic boundary conditions, and also in [11], Omosowon *et al.*, deduced dual Properties of ω-order Reversing Partial Contraction Mapping in Semigroup of Linear Operator. Rauf and Akinyele [12], introduced ω-order preserving partial contraction mapping and established its properties, also in [13], Rauf *et al.* presented some results of stability and spectra properties on semigroup of linear operator. Vrabie [14], proved some results of C_0-semigroup and its applications. Yosida [15], obtained some results on differentiability and representation of one-parameter semigroup of linear operators.

2. Preliminaries

Definition 2.1 (C_0-Semigroup) [14]

A C_0-Semigroup is a strongly continuous one parameter semigroup of bounded linear operator on Banach space.

Definition 2.2 (ω-OCP_n) [12]

A transformation $\alpha \in P_n$ is called ω-order preserving partial contraction mapping if $\forall x, y \in \text{Dom}\alpha : x \leq y \implies \alpha x \leq \alpha y$ and at least one of its transformation must satisfy $\alpha y = y$ such that $T(t+s) = T(t)T(s)$ whenever $t, s > 0$ and otherwise for $T(0) = I$.

Definition 2.3 (Interpolation space) [9]

An intermediate space X is called an interpolation space of (X_0, X_1) if each admissible operator leaves X invariant. In this case, the restrictions to X are easily seen to be bounded on X.

Definition 2.4 (closed linear operator) [15]

Let X, Y be two Banach spaces. A linear operator $A : D(A) \subseteq X \to Y$ is closed if for every sequence x_n in $D(A)$ converging to x in X such that $Ax_n \to y \in Y$ as $n \to \infty$, one has $x \in D(A)$ and $Ax = y$. Equivalently, A is closed if its graph is closed in the direct sum $X \oplus Y$.

Example 1

2×2 matrix $[M_m(\mathbb{N} \cup \{0\})]$

Suppose that

$$A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{I} \\ e^{t} & e^{2t} \end{pmatrix}.$$

Example 2

3×3 matrix $[M_m(\mathbb{N} \cup \{0\})]$

Suppose that

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA}$, then

$$e^{tA} = \begin{pmatrix} e^{2t} & e^{2t} & e^{3t} \\ e^{2t} & e^{2t} & e^{2t} \\ e^{t} & e^{2t} & e^{2t} \end{pmatrix}.$$

Example 3

3×3 matrix $[M_m(\mathbb{C})]$, we have for each $\lambda > 0$ such that $\lambda \in \rho(A)$ where $\rho(A)$ is a resolvent set on X.

Suppose that we have

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 2 & 2 \end{pmatrix}$$

and let $T(t) = e^{tA_\lambda}$, then

$$e^{tA_\lambda} = \begin{pmatrix} e^{2t\lambda} & e^{2t\lambda} & e^{3t\lambda} \\ e^{2t\lambda} & e^{2t\lambda} & e^{2t\lambda} \\ e^{t\lambda} & e^{2t\lambda} & e^{2t\lambda} \end{pmatrix}.$$

Theorem 2.1 Hille-Yoshida [14]

A linear operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator for a C_0-semigroup of contraction if and only if

i. A is densely defined and closed,

ii. $(0, +\infty) \subseteq \rho(A)$ and for each $\lambda > 0$, we have

$$\|R(\lambda, A)\|_{L(X)} \leq \frac{1}{\lambda}. \tag{1}$$

3. Main Results

This section present results of semigroup of linear operators in Favard class and extrapolation space generated by ω-OCP_n by showing the bounded, closure and $\odot$-reflexive nature of the semigroup:

Theorem 3.1

Assume that $A : D(A) \subseteq X \to X$ is the generator of a C_0-semigroup of a Favard class such that $A \in \omega - OCP_n$. Let $x^* \in X^*$, then the following assertions are equivalent:

(i) $x^* \in D(A^*)$;

(ii) $\limsup_{t\to 0} t^{-1}\|T^*(t)x^* - x^*\| < \infty$;

(iii) $\liminf_{t\to 0} t^{-1}\|T^*(t)x^* - x^*\| < \infty$;

(iv) $Fav(T^{\odot}(t)) = D(A^*)$;

(v) $Fav(T(t)) = D(A^{\odot *}) \cap X$.

Proof:

It is trivial that (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii). Now to show (iii), assume there is a sequence $t_j \to 0$ and a finite constant C such that

$$\frac{1}{t}\|T^*(t_j)x^* - x^*\| \leqq C, \quad \forall j \in \mathbb{N}.$$

Define a linear form y^* on $D(A)$ by

$$y^*(x) := \langle x^*, Ax \rangle, \quad \forall x \in D(A) \text{ and } A \in \omega - OCP_n. \tag{2}$$

From

$$\begin{aligned} |y^*(x)| = |\langle x^*, Ax \rangle| &= \left| \lim_{j\to\infty} \frac{1}{t_j} \langle x^*, T(t_j)x - x \rangle \right| \\ &= \left| \lim_{j\to\infty} \frac{1}{t_j} \langle T^*(t_j)x^* - x^*, x \rangle \right| \leqq C\|x\|. \end{aligned} \tag{3}$$

It follows that y^* is bounded. Hence, by definition of A^* we have $x^* \in D(A^*)$, and

$$A^* x^* = y^*. \tag{4}$$

To prove (iv), suppose that $x^\odot \in Fav(T^\odot(t))$, then $x^\odot \in D(A^*)$ by (i), (ii) and (iii).

Conversely, if $x^* \in D(A^*)$, then $x^* \in X^\odot$, so that $x^* \in Fav(T^\odot(t))$, and this archived (iv).

To prove (v), from (iv) we deduced that

$$Fav(T^{\odot\odot}(t)) = D(A^{\odot *}).$$

But it is a trivial consequence of the defnition of the Favard class that

$$Fav(T(t)) = Fav(T^{\odot\odot}(t)) \cap X, \tag{5}$$

where X is identified with its image jX in $X^{\odot *}$. Hence, the prove is completed.

Theorem 3.2

Suppose that $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup of a Favard class such that $A \in \omega - OCP_n$. Then we have the following inclusions and equivalent assertions.

(i) $\overline{R(\lambda, A)B_{(X,\|\cdot\|)}} \subset R(\lambda, A^{\odot *})B_{X^{\odot *}} \cap X \subset \bigcup_{x\in\mathbb{N}} n \cdot \overline{R(\lambda, A)B_{(X,\|\cdot\|)}}$;

(ii) $x \in Fav(T(t))$;

(iii) There exists a bounded sequence $(y_n) \subset X$ such that $\lim_n R(\lambda, A)y_n = x$;

(iv) There exists a bounded sequence $(y_n) \subset X$ and an integer $n \in \mathbb{N}$ such that

$$\lim_{n} R(\lambda, A)^{n+1} y_n = R(\lambda, A)^n x.$$

Proof:

Since $B_{X^{\odot *}}$ is weak*-compact, then $R(\lambda, A^{\odot *})B_{X^{\odot *}}$ is also weak*-compact in a particular norm-closed. The first inclusion now follows easily from the fact that $j : X \to X^{\odot *}$ is an isometry from $(X, \|\cdot\|')$ into $X^{\odot *}$. The second inclusion follows from the equality

$$\frac{1}{t}\int_0^t T(\eta)xd\eta = R(\lambda, A)\left(\frac{\lambda}{t}\int_0^t T(\eta)xd\eta - \frac{1}{t}(T(t)x - x)\right). \tag{6}$$

In fact, we have

$$R(\lambda, A^{\odot *})B_{X^{\odot *}} \cap X \subset D(A^{\odot *}) \cap X = Fav(T(t)). \tag{7}$$

Therefore if $x \in R(\lambda, A^{\odot *})B_{X^{\odot *}} \cap X$, then

$$\frac{\lambda}{t}\int_0^t T(\eta)xd\eta - \frac{1}{t}(T(t)x - x)$$

remains bounded as $t \to 0$ whereas the left hand side converges to x. Since $B_X \subset B_{(X,\|\cdot\|)} \subset MB_X$ for some M, one may replace $\|\cdot\|'$ by $\|\cdot\|$, and this proves (i) .

To prove (ii), we have that the implication (ii) $\Longrightarrow$ (iii) is immediate from (i). (iii) $\Longrightarrow$ (iv) is trivial.

(iv) $\Longrightarrow$ (ii): Suppose $n = 0$, then (ii) follows from (i). Therefore, assume $n > 0$. From (iv) it follows that for any $x^{\odot} \in X^{\odot}$, we have

$$\lim_{n\to\infty} \langle R(\lambda, A^*)x^{\odot}, R(\lambda, A)^n y_n\rangle = \langle R(\lambda, A^*)x^{\odot}, R(\lambda, A)^{n-1}x\rangle. \tag{8}$$

By a density argument, this implies that

$$R(\lambda, A)^n y_n \to R(\lambda, A)^{n-1}x$$

in the $\sigma(X, X^{\odot})$-topology. Repeating this argument, it follows that $R(\lambda, A)y_n \to x$ in the $\sigma(X, X^{\odot})$-topology. Therefore x belongs to the $\sigma(X, X^{\odot})$-closure $KR(\lambda, A)B_X$ for some constant K which by results is equal to the norm-closure of $KR(\lambda, A)B_X$. Hence, $x \in Fav(T(t))$ by (i). Note that (iii) is equivalent to: there is a sequence $(x_n) \subset D(A)$ such that $x_n \to x$, and $\sup_n \|Ax\| < \infty$, and this complete the proof.

Theorem 3.3

Let $A \in \omega - OCP_n$ be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t \geqq 0}$ on c_0 with the property that $Fav(T(t)) = D(A)$, then:

(i) if and only if $R(\lambda, A)B_{(X,\|\cdot\|')}$ is norm-closed;

(ii) suppose that X is $\odot$-reflexive with respect to $T(t)$. The following are equivalent:

(a) $Fav(T(t)) = D(A)$;

(b) j maps X onto $X^{\odot *}$;

(c) $R(\lambda, A)B_{(X,\|\cdot\|')}$ is weakly compact;

(d) $R(\lambda, A)B_{(X,\|\cdot\|')}$ is $\sigma(X, X^{\odot})$-compact;

(e) $B_{(X,\|\cdot\|')}$ is $\sigma(X, X^{\odot})$-compact.

Proof:

First, assume that

$$Fav(T(t)) = D(A).$$

Let $y \in \overline{R(\lambda, A)B_{(X,\|\cdot\|')}}$, that is $y = R(\lambda, A^{\odot *})x^{\odot *}$ for some $x^{\odot x} \in B_{X^{\odot}}$ and $A \in \omega - OCP_n$, using (i) of Theorem 3.2. Since

$$Fav(T(t)) = D(A^{\odot *}) \cap X$$

by (v) of Theorem 3.1, $y \in Fav(T(t))$ and hence by assumption there is an $x \in X$ such that

$$y = R(\lambda, A)x$$

but

$$R(\lambda, A)x = R(\lambda, A^{\odot *})jx$$

and since $R(\lambda, A^{\odot *})$ is injective, we have $jx = x^{\odot *}$. But j is an isometry from $B_{(X,\|\cdot\|')}$ into $B_{X^{\odot *}}$ which forces $x \in B_{(X,\|\cdot\|')}$. Hence, $y \in R(\lambda, A)B_{(X,\|\cdot\|')}$ as was to be shown.

Conversely, if $R(\lambda, A)B_{(X,\|\cdot\|')}$ is closed, then by (i) of Theorem 3.2 we have

$$R(\lambda, A^{\odot *})B_{X^{\odot *}} \cap X \subset \bigcup_{n \in \mathbb{N}} n \cdot R(\lambda, A)B_{(X,\|\cdot\|')} = D(A). \tag{9}$$

Since $Fav(T(t)) = D(A^{\odot *}) \cap X$, it follows that $Fav(T(t)) \subset D(A)$, as was to be shown.

To prove (ii),

(a) $\Longrightarrow$ (c): since $\|\cdot\|'$ is an equivalent norm, this follows from (iv) of Theorem 3.1 and (i) of

Theorem 3.3.

(c) $\Longrightarrow$ (a): by assumption X is $\odot$-reflexive and $R(\lambda,A)B_{(X,\|\cdot\|')}$ is closed. Hence, from (i) of Theorem 3.3 and from the inclusions

$$D(A^{\odot *}) \subset X^{\odot\odot} = X \tag{10}$$

we have

$$D(A^{\odot *}) = D(A^{\odot *}) \cap X = Fav(T(t)) = D(A) = D(A^{\odot\odot}). \tag{11}$$

Since $A^{\odot\odot}$ is the part of $A^{\odot *}$ in $X^{\odot\odot}$ if follows that

$$X^{\odot\odot} = X^{\odot *}.$$

Since X is $\odot$-reflexive with respect to $T(t)$, this is the desired result.

(b) $\Longrightarrow$ (e): $B_{X^{\odot *}}$ is weak*-compact. By assumption, we may identify $B_{X^{\odot *}}$ with $B_{(X,\|\cdot\|')}$ and (e) follows.

(e) $\Longleftrightarrow$ (d) $\Longleftrightarrow$ (c): Since a weak $T(t)$-equicontinuous in X is weakly convergent if it is $\sigma(X,X^{\odot})$-convergent. Suppose (x_n) is $\sigma(X,X^{\odot})$-convergent to x. we put $G = \{x_n\}_{n=1}^{\infty} \cup \{x\}$. Then G is weakly $T(t)$-equicontinuous as well. If V is a weakly open neighbourhood of x in X. Then $V \cap G$ is relatively weakly open in G, hence relatively $\sigma(X,X^{\odot})$-open in G. If H is bounded, then $\overline{R(\lambda,A)H}$ is $T(t)$-equicontinuous. Assume

$$T(t)R(\lambda,A)h - R(\lambda,A)h - R(\lambda,A)h = \int_0^t T(\sigma)AR(\lambda,A)hd\sigma$$

and use that the operator $AR(\lambda,A)$ is bounded. In particular if H is bounded and convex, then $\overline{R(\lambda,A)H}$ is $\sigma(X,X^{\odot})$-closed. We start by observing that a $\sigma(X,X^{\odot})$-compact set G is norm bounded. Indeed by regarding G as a subset of $X^{\odot *}$, G is weak*-compact, and the boundedness in $X^{\odot *}$ follows from the uniform boundedness theorem. Similarly, every $\sigma(X,X^{\odot})$-sequentially compact set is norm bounded. Then we have that a bounded set G is $\sigma(X,X^{\odot})$-compact if $R(\lambda,A)G$ is $\sigma(X,X^{\odot})$-compact. Hence, the prove is complete.

Theorem 3.4

Suppose that A generalized Hille-Yosida of type $(M,0)$. For all $x_0 \in X_0$, $t > 0$ and $A \in \omega - OCP_n$, then we have

$$K(x_0,t;X,D(A)) \leq K(x_0,t;X_0,D(A_0)) \leq MK(x_0,t;X,D(A)).$$

Proof:

Since $D(A_0) \subset D(A)$ and $X_0 \subset X$ are closed subspaces (the first with respect to the graph norm on $D(A)$), from the definition of K-functional we trivially have inequality

$$K(x_0,t;X,D(A)) \leq K(x_0,t;X_0,D(A_0)). \quad (x_0 \in X \text{ and } A \in \omega - OCP_n).$$

The proof of the other inequality is done in two steps. Fix $\varepsilon > 0$ and $x_0 \in X_0$ arbitrary.

Step 1. Chose $\mu > 0$ so large that $\|x_0 - x_\mu\| \leqq \varepsilon$, where

$$x_\mu := \mu R(\mu, A) x_0.$$

Fix $t > 0$ arbitrary. Choose $\phi \in X_0$ and $\Psi \in D(A_0)$ such that $\phi + \Psi = x_\mu$ and

$$\|\phi\|_{X_0} + t\|\Psi\|_{D(A_0)} \leq K(x_\mu,t;X_0,D(A_0)) + \varepsilon.$$

Noting that

$$x_0 = (\phi + x_0 - x_\mu) + \Psi \in X_0 + D(A_0). \tag{12}$$

It follows that

$$K(x_0,t;X_0,D(A_0)) \leq \|\phi + x_0 - x_\mu\| + t\|\Psi\|_{D(A_0)} \leq K(x_\mu,t;X_0,D(A_0)) + 2\varepsilon.$$

Step 2. Let μ be as in step 1. Choose $\phi \in X$, $\Psi \in D(A)$ and $A \in \omega - OCP_n$ such that

$$\phi + \Psi = x_0$$

and

$$\|\phi\|_X + t\|\Psi\|_{D(A)} \leq K(x_0,t;X,D(A)) + \varepsilon.$$

Next, note that the maps

$$\phi \mapsto \phi_\mu,$$

$$\Psi \mapsto \Psi_\mu$$

and bounded as maps $X \to X_0$ and $D(A) \to D(A_0)$ respectively of norm $\leqq M$.

Since $\phi_\mu \in X_0$, $\Psi_\mu \in D(A)$, $A \in \omega - OCP_n$ and

$$x_\mu = \phi_\mu + \Psi_\mu$$

by step 1 we have

$$\begin{aligned} K(x_0,t;X_0,D(A_0)) &\leq K(x_\mu,t;X_0,D(A_0)) + 2\varepsilon \\ &\leq \|\phi_\mu\|_{X_0} + t\|\Psi_\mu\|_{D(A_0)} + 2\varepsilon \\ &\leq M\|\phi\|_X + tM\|\Psi\|_{D(A)} + 2\varepsilon \\ &\leq MK(x_0,t;X,D(A)) + 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then the prove is complete.

CONCLUSION

In this paper, it has been established that ω-order preserving partial contraction mapping generate some results of semigroup of linear operators in Favard class and interpolation space.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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