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REGULAR PROPER *-EMBEDDING OF PROPER *-SEMIGROUPS AND RINGS

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Abstract. In this paper, it is shown that a cancellative semigroup is embeddable in an inverse semigroup. It is shown that finite proper *-semigroup is regular and any finite commutative proper *-semigroup is a union of groups. Also it is shown that a finite cyclic proper * semigroup is a group while an infinite one is *-embedded in a proper*-group, and any finite maximal proper*- semigroup has a proper *-extension ring. It is shown that there is a nonregular proper *-ring that cannot be *-embedded in any regular proper *-ring. Also it is shown that an Artinian proper *-ring is a finite direct product of matrix rings over skew fields. It is shown that a commutative proper * and cancellative semigroup is *-embeddable in a regular proper *-semigroup.

Keywords: proper * semigroups and proper *rings, mp*–semigroups, strongly regular proper *-semigroup, *-embedding and *-extension, regular semigroups and rings, formally complex *-rings, union of groups.

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1. Introduction

Let $(S, *)$ be a *-semigroup with involution *. Then $(S, *)$ is called a *proper *-semigroup* (p*-semigroup) if for every a, b in $S, aa^* = ab^* = bb^*$ implies that $a = b$. A proper*-semigroup which is a union of groups each of which is closed under the involution * is called a *strongly proper* *-semigroup (sp*-semigroup). A ring with involution (*-ring)

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$(R,*)$ is called *a proper *-ring (p*-ring)* if for a in R , $aa^*=0$ implies that $a=0$. Let $(R,*)$ be a *-ring and n be a positive integer. We say that $(R,*)$ is *n-formally complex* if for every $r_1,...,r_n$ in R, $\sum r_ir_i^*=0$ implies that all r_i are 0. Let $(S,*)$ be a proper*-semigroup and let s,t,u be elements in S such that $tss^*=uss^*$. Thus $ss^*t^*=ss^*u^*$. Then $ts=us$. This is called the **-cancellation law* and can be seen by noticing that, under the hypothesis, $(ts)(ts)^*=t.ss^*t^*=t.ss^*u^*=(ts)(us)^*=tss^*.u^*=uss^*.u^*=(us)(us)^*$. Then by using properness of * we get $ts=us$.

A *-semigroup $(S,*)$ is called a maximal proper *-semigroup (*mp*-semigroup)* if for every distinct elements $s_1,...,s_n$ in S, there exists an s_i such that $s_is_i^*\neq s_is_j^*, j\neq i$, and such that if $s_is_i^*=s_ks_l^*$ then $s_i^*s_k=s_i^*s_l; k,l=1,...,n$. For example an inverse semigroup is an mp*-semigroup under the inverse involution. The converse need not be true, see [6]. Let $(S,*)$ be a p*-semigroup and $(R,*)$ be a p*-ring. We say that $(S,*)$ is **-embeddable* in $(R,*)$, or $(R,*)$ **-embeds* $(S,*)$ if there is a semigroup **-embedding* $f:S\rightarrow R$. Thus f is injective and for every x,y in $S, f(xy)=f(x)f(y), f(x^*)=(f(x))^*$. Let $(S,*)$ be a p*-semigroup and let x be an element of S. We denote by $S_x=<xx^*>$ the set $\{(xx^*)^n:n\in N\}$. In general if x is an element in a semigroup S then $<x>$ denotes the set of all positive exponents of x. A semigroup S is *cyclic* if there is an element $a\in S$ such that $S=<a>$. Let S be a semigroup. An element x in a semigroup S is called *regular* if there is y in S such that $xyx=x$. If x is regular for all elements x in S we say that S is regular. Let S be a regular semigroup and $x\in S$. Thus there is $y\in S$ such that $xyx=x$. Then we notice that $x.yxy.x=x, yxy.x.yxy=yxy$. Denoting yxy by z we see that x has an inverse z such that $xzx=x, zxz=z$. . A semigroup S is called a *0-group* if there is an element x such that $(S\backslash\{x\},.)$ is a group and $xg=gx=x$ for all g in S. Let $(S,*)$ be a semigroup. with involution. A *-congruence on S is an equivalence relation ˜ which is a *-congruence in the sense that whenever $a\tilde{}b$ in S then $a^*\tilde{}b^*$. Thus once $a\tilde{}b$ then $a^*\tilde{}b^*$ for all $a,b\in S$. Then S is partitioned into equivalence classes $S/\tilde{}=\{[a]:a\in S\}$. We define multiplication on $S/\tilde{}$ by setting $[a][b]=[ab]$ for all $a,b\in S$. We define an involution on $S/\tilde{}$ by setting $[x]^*=[x^*]$. If $(S,*)$ is a semigroup with involution then a similar proof to that given in ([2]) can be constructed to show that $S/\tilde{}$ is a semigroup with involution which is a *-homomorphic to $(S, *)$ under the *-homomorphism $f : (S, *) \to (S/\tilde{}, *), f(a) = [a]$. Thus $f(a^*) = (f(a))^* = ([a])^* = [a^*]$ for all $a \in S$.

It was proved in [4] that there is a proper *-semigroup that cannot be *-embedded in any p*-ring. The next question is: Given a p*-semigroup $(S, *)$ does there exist a regular p*-semigroup $(T, *)$ that *-embeds $(S, *)$?. A related question is that given a p*-ring $(R, *)$ does there exist a regular p*-ring $(T, *)$ that *-embeds $(R, *)$?

Malcev(see [3], p. 10) has exhibited a cancellative semigroup S which cannot be embedded in any group. We will show that a left cancellative semigroup S can be embedded in an inverse semigroup.

Remark 1. *Let S be a regular left cancellative semigroup. Then S is a group.*

For, let $a \in S$. There is $a' \in S$, $aa'a = a$. Then $aa'.aa' = aa'$. Now let $c \in S$. Then $aa'.aa'c = aa'c$. Cancelling aa' we get $aa'.c = aa'$ for all $c \in S$. Thus S has a left identity which can be any aa' for any $a \in S$. Thus for every $a \in S$ there is $a \in S$ and aa' is a left identity. It follows that S is a group.

Proposition 1. *(1) Let S be a left cancellative semigroup. Then S can be embedded in an inverse semigroup.*

(2) If $(S, *)$is a left cancellative p-semigroup then it can be *-embedded in a regular p*-semigroup.*

Proof. (1) If S is finite then it is a group and we are done. In general for every element $x \in S$ let l_x be the mapping from S to S given by $l_x(s) = xs$ for every element$s \in S$. Then l_x is an injective mapping on S. The family $L(S) = \{l_x : x \in S\}$ is a semigroup under composition. For if $x, y \in S$ then $l_x \circ l_y(s) = l_x(l_y(s)) = l_x(ys) = xy(s) = l_{xy}(s)$ for all $s \in S$. The mapping $f : x \to l_x$ is injective. For if $l_x = l_y$ then $l_x(y) = l_y(y)$ and so $xy = y^2$ which implies that $x = y$ since S is cancellative. The set $T(S)$ of all partial injective transformations on a subset of S under composition of mappings is an inverse semigroup. $T(S)$ contains all l_x such that $x \in S$. Thus the mapping f is a semigroup embedding of S into the inverse semigroup $T(S)$. (See [2]).

(2) If S has a proper involution * then $L(S)$ is again a semigroup with involution defined by $(l_x)^* = l_{x^*}$. This involution is proper for if $(l_x)(l_y)^* = (l_x)(l_x)^* = (l_y)(l_y)^*$then $xy^* = xx^* = yy^*$ and so $x = y$. This implies that $l_x = l_y$. Thus $(S,*)$ is *-embeddable in a regular proper *-semigroup. This completes the proof. □

There is a *finite regular* p*-semigroup $(S,*)$ that cannot be *-embedded in any p*ring(regular or not). (see [4]). We use this to show that there is a *non-regular infinite* p*-semigroup $(S,*)$ that cannot be *-embedded in any regular p*-ring.

Example 1. *Let N be the commutative semigroup of positive integers under multiplication. Consider a finite regular proper *-semigroup $(S,*)$ that cannot be *-embedded in any p*-ring and let * be the identity involution. Then $(N,*)$ is a non-regular p*-semigroup. Let $T = S \oplus N$ and define multiplication on T by $(s,n).(s\,',n\,') = (ss\,',nn\,')$ for all $s,s\,' \in S$ and for all $n,n\,'$ in N. Define * on T by $(s,n)^* = (s^*,n)$ for all $s \in S$ and for all $n \in N$. Then $(T,*)$ is a non-regular p*-semigroup. We will show that $(T,*)$ cannot be *-embedded in any regular p*-ring $(R,*)$. For, if there is such a proper *-ring $(R,*)$ then the p*-ring $(R,*)$ would contain an isomorphic copy of $(S,*)$, namely $(S \oplus \{1\},*)$ and we know that there is no p*-ring containing $(S,*)$. This is a contradiction.*

We prove below some properties of regular p*-semigroups and regular p*-rings.

Proposition 2. *Let $(S,*)$ be a finite p*-semigroup. Then*

(1) S is a regular p-semigroup.*

(2) If x is a non-zero element in S then $S_x =< xx^* >$ is a group.

(3) If S is cyclic p-semigroup then S is a cyclic group.*

Proof. (1) If x is a zero element of S then x is regular and $S_x = \{0\}$ is a group. Let x be a non zero element in S. Then xx^* and all of its powers are different from zero by properness of * and by *-cancellation. Then S_x, being the set of all positive powers of xx^*, is a finite cyclic subsemigroup. Let n be the first positive integer such that

$(xx^*)^n = (xx^*)^k, 1 \leq k < n$. The pair (k, n) must exist since $xx^* \neq 0$ and by properness of $*$. Then $(xx^*)^{n-k}(xx^*)^k = (xx^*)^k$. If we use the *-cancellation law repeatedly, we get $(xx^*)^{n-k+1} = (xx^*)$. If $k > 1$, we have a contradiction with the minimality of n and so $k = 1$.Thus $(xx^*)^n = (xx^*)$. Let $a = xx^*$. Then a^{n-1} acts as an identity e in S_x and $S_x = \{e, a, a^2, ..., a^{n-2}\}$ and so S_x is a cyclic group generated by a.

(2) Let x be an element in S. If $x = 0$ then $0.0.0 = 0$ and so x is regular. If x is a non-zero element in S then as shown above $< xx^* >$ is a finite group and so there is a positive integer $n > 1$ such that $(xx^*)^n = xx^*$. By *-cancellation $(xx^*)^{n-1}.x = x$ and so x is regular for all $x \in S$ and so S is regular.

(3) Let $S =< x >$ and let m be the number of elements in S. If $m = 1$ then S is a trivial group. Let $m > 1$. We will show that $x^{m+1} = x$. Since S is finite there is a pair $(n, k), 1 \leq k < n \leq m + 1$ such that $x^n = x^k$, and let this (n, k) be the first such pair. It follows that $k = m + 1$ for otherwise the number of elements in S would be less than n.Thus $(m+1, k)$ is the first pair. Assume $k > 1$. Now x is the only generator for S since $k > 1$. Thus $x^* = x$ because x^* is a generator for S and $S =< x >= S^* =< x^* >$. Now $x^m.x^m = x^{m+1}.x^{m-1} = x^k.x^{m-1} = x^{m+k-1} = x^{m+1}x^{k-2}$

$= x^k.x^{k-2} = x^{k-1}.x^{k-1}$. Also

$x^m.x^{k-1} = x^{m+1}.x^{k-2} = x^k.x^{k-2} = x^{k-1}.x^{k-1}$. Since $x = x^*$, it follows by properness of $*$ that $x^m = x^{k-1}$. This is a contradiction with the choice of $(m + 1, k)$ as a first pair rather than (m, k). Thus $k = 1$. Since $m > 1$, then x is not a zero element. Thus S is a finite cyclic group.

Here is another proof: Let $m > 1$. Then $x^* = x$, otherwise $*$ is not surjective. Since $x^n = x^k, n > k$ we can verify easily that $x^{k-1}(x^{k-1})^* = x^{k-1}(x^{n-1})^* = x^{n-1}.(x^{n-1})^*$. Thus if $*$ is proper then $x^{k-1} = x^{n-1}$ contrary to the choice of the pair (n, k) as a first pair with $x^n = x^k, n > k$. This completes the proof. □

Remark 2. *If $S =< x >$ is a finite cyclic group of order n with involution $*$. Let $x^* = x^m$. From $x^{**} = x$ we have $m^2 = 1 \mod n$. Thus m is a unit in the ring $Z_n$whose square is 1.*

Remark 3. *Not every regular p*-semigroup is a strongly p*-semigroup(i.e. a union of groups). For this to hold, it is necessary and sufficient that* $\forall x$ *in* $S, \exists n_x, x^{n_x} = x$.

We give below some counter examples.

Example 2. *Let* (R, t) *be the *-ring of* 2×2 *matrices over the ring* Z_7 *with the transpose involution* t. *Since* $a^2 + b^2 = 0$ *implies that* $a = 0 = b, \forall a, b$ *in* R, *it is easily checked that* (R, t) *is a p*-ring, and hence it is regular by Proposition 2. Let* $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. *It is easily checked that* $A^4 = A^2 \neq I$ *and that this is the first equality of two positive powers of* A. *Thus* A *cannot belong to a subgroup inside* R. *This example gives a finite regular p*-semigroup which is not strongly proper. This semigroup is not an inverse semigroup. To see this we take* $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

We notice that $BCB = B, CBC = C, BDB = B, DBD = D$. *Thus* B *has two inverses*

C, D. *Thus this example serves as an example of a finite regular proper *-semigroup which is neither an inverse semigroup nor an sp*- semigroup.*

Example 3. *Consider the following matrices in* $M_3(Z)$

$$x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, y = x^t, z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, u = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $S = \{x, y, z, u, v\}$. *Then* S *is a semigroup under multiplication. In fact it has the following multiplication table*

.	x	y	z	u	w
x	z	w	z	x	z
y	u	z	z	z	y
z	z	z	z	z	z
u	z	y	z	u	z
w	x	z	z	z	w

.

We notice that the only idempotents are $z, u,$ *and* w *and that these elements commute;*

thus S is an inverse semigroup. We notice that $x \neq x^2 = x^3$, and hence x cannot belong to a subgroup inside S. This serves as an example of a finite inverse (and hence a p-semigroup) which is not a strongly proper *-semigroup under its inverse involution.*

We now prove the following.

Proposition 3. *Let $(S, *)$ be a finite mp*-commutative semigroup. Then $(S, *)$ is a strongly p*-semigroup.*

Proof. Let $x \neq 0$ be an element of S. Then $< x, x^* >$ is finite. Let (k, n) be a pair of positive integers such that $1 \leq k < n$ and $x^k = x^n, x^{*k} = x^{*n}$. Such a pair must exists since S is finite. Let $a = xx^*$. Then from properness of * and commutativity $a \neq 0, a^k = a^n$. Then as in the proof of proposition 2, $k = 1$. Thus for every x in S there is a positive integer n such that $x^n = x$. Thus S is strongly proper. This completes the proof. □

Proposition 4. *(1) Let $(R, *)$ be an Artinean proper *-ring. Then R is a finite direct product of matrix rings over skew fields and so it is regular.*

*(2) If $(R, *)$ is finite then R is a finite direct product of matrix rings each of which is over a field. (3) If $(R, *)$ is a finite commutative proper *-ring then it is a finite direct product of finite fields.*

Proof. (1) R is nil-semisimple. For let A be an element in a nilpotent ideal I in R. Then AA^* is in I and so it is nilpotent. Thus there is a positive integer n such that $(AA^*)^n = 0$. By *-cancellation then $A = 0$. Thus R is nil-semisimple. Since it is Artinean then by Wedderburn's theorem it is a finite direct product of matrix rings each over a skew field and so R is regular.

(2) If R is finite then each matrix ring is over a finite skew field and hence a field.

(3) If $(R, *)$ is a finite commutative p*-ring then the corresponding matrix rings are all of size 1 by 1 owing to commutativity of R. This completes the proof. □

2. Regular *embedding of Some Proper *-Semigroup

Proposition 5. *Let $(S,*)$ be a finite mp*- semigroup of order m and let $(R,*)$ be an n-formally complex *-ring with $m \leq n$. Then $(R[S],*)$ is a p*-ring. If R is finite then $(R[S],*)$ is a regular p*-ring embedding of $(S,*)$.*

Proof. : Since $A = \sum_{i=1}^{k} r_i s_i$ in $(R[S],*)$ implies that $k \leq n$, the proof that $(R[S],*)$ is proper* is the same as the proof given in [7]. This completes the proof. □

Let $(S,*)$ be a finite semigroup with involution *. By Maschke's theorem we can choose a field F such that $F[S]$ is regular, see for example Clifford and Preston book [1]. Let * be any involution on F. Define an involution * on $F[S]$ by $(\sum a_i s_i)^* = \sum a_i^* s_i^*$. Then $(F[S],*)$ is a regular *-ring which *-embeds $(S,*)$. But this involution, although extends that on S, may not be proper.

In spite of this, there is a finite p*-semigroup not *-embeddable in any p*-ring (regular or not). See [4]. Also refer to examples 4 and 6.

Proposition 6. . *Let $(S,*)$ be a proper-*cyclic semigroup. Then $(S,*)$ is *-embeddable in a regular p*-semigroup.*

Proof. If S is finite then it is regular as has been shown in 3 of proposition 2. Let S be infinite and let x be an element in S. Let $x^* = x^m, m > 0$. If $m > 1$. Let $y = xx^*$. Then $y = y^*$. But $x^{m+1} = y = y^* = (x^*)^{m+1} = x^{m(m+1)}$. Thus S is finite and this is a contradiction. Thus $m = 1$ and so * is the identity involution. Then $(S,*) \approx (Z^+,+,id)$ and the latter can be *-embedded in $(Z,+,id)$ which is a group. The identity mapping is a proper involution on $(Z,+)$. This completes the proof. □

Proposition 7. *Let $(S,*)$ be a proper*-semigroup and let x be an element in S such that $S_x = \langle xx^* \rangle$ is finite or such that $\langle x \rangle$ is finite and x commutes with x^*. Then x is regular.*

Proof. Let x be an element such that S_x be finite and let $a = xx^*$. Then $S_x = \langle a \rangle$ and $a = a^*$. Let (k,n) be a pair of positive integers, $1 \leq k < n$, be such that $a^k = a^n$. Then be *-cancellation it follows that $a^{n-k+1} = a$ and so S_x is a group. Then using the same argument as used in the previous proposition 6 it follows that x is regular.

If $< x >$ is finite and x commutes with x^* then $xx^* \neq 0$ and $< x^* >$ is finite. It follows that $< xx^* >$ is finite and we use the same argument above to deduce that x is regular. This completes the proof. □

Proposition 8. *Let $(S, *)$ be a commutative 0- cancellative p*-semigroup. Then there is a regular p*-semigroup $(T, *)$ which *-embeds $(S, *)$.*

Proof. Since for all $s \in S, s^{**} = s$, it follows that $S^* = S$. Since for all $s \in S, 0^* = (0.s)^* = s^*.0^*$ it follows that $0^* = 0^*s$ for all $s \in S$. Thus $0^* = 0^*.0 = 0$. Thus for all $s \in S, s^* = 0 \Leftrightarrow s = 0$. Since S has no zero divisors, $T = S\backslash\{0\}$ is a subsemigroup closed under * and so it is a proper *-subsemigroup of S. Let $W = T \otimes S$. We define multiplication on W by $(t_1, s_1)(t_2, s_2) = (t_1t_2, s_1s_2)$ for all $t_1, t_2 \in T$ and for all $s_1, s_2 \in S$. We define involution on W by $(s, t)^* = (s^*, t^*)$ for all $(s, t) \in W$. We notice that this involution is proper. For let $(t_1, s_1), (t_2, s_2) \in W$ be such that $(t_1, s_1)(t_1, s_1)^* = (t_1, s_1)(t_2, s_2)^* = (t_2, s_2)(t_2, s_2)^*$. Then $t_1t_1^* = t_1t_2^* = t_2t_2^*, s_1s_1^* = s_1s_2^* = s_2s_2^*$. Since * is proper in S it follows that $t_1 = t_2, s_1 = s_2$ as required. Thus $(W, *)$ is a proper *-semigroup. Next we define a relation ˜ on W be declaring that $(t_1, s_1)\tilde{}(t_2, s_2)$ if and only if $t_1s_2 = s_1t_2$. Then ˜ is reflexive and symmetric. Let $(t_1, s_1)\tilde{}(t_2, s_2), (t_2, s_2)\tilde{}(t_3, s_3)$. Then $t_1s_2 = t_2s_1, t_2s_3 = t_3s_2$. Then $t_1s_2t_2s_3 = t_2s_1t_3s_2$. Then by cancelling t_2, $t_1s_2s_3 = s_1t_3s_2$. Now if $s_2 = 0$ then $t_2s_1 = 0 = t_2s_3$. Since $t_2 \neq 0, s_1 = 0 = s_3$. This implies that $t_1s_3 = s_1t_3$, and so $(t_1, s_1)\tilde{}(t_3, s_3)$. On the other hand if $s_2 \neq 0$ then from $t_1s_2s_3 = s_1t_3s_2$, by cancellation $t_1s_3 = s_1t_3$. Thus again $(t_1, s_1)\tilde{}(t_3, s_3)$. Thus ˜ is transitive. We show that ˜ is a congruence. Let $(t_1, s_1)\tilde{}(t_2, s_2), (t, s) \in W$. Then $t_1s_2 = t_2s_1$. We need to show that $(t_1t, s_1s)\tilde{}(t_2t, s_2s)$, or $t_1t.s_2s = s_1s.t_2t$ and this is true. Let the class $[(a, b)]$ in W be denoted by a/b for all $(a, b) \in W$. Thus $W/\tilde{} = \{a/b : b \in T, a \in S\}$ is a semigroup. We define an involution * on $W/\tilde{}$ by $(a/b)^* = a^*/b^*$ for all $a/b \in W/\tilde{}$. This is well-defined. For let $a/b = c/d$. Then $ad = bc$ and so $d^*a^* = c^*b^*$. Thus $a^*/b^* = c^*/d^*$. Also $(a/b)^{**} = a/b, (a/b.c/d)^* = (ac/bd)^* = (ac)^*/(bd)^* = (c^*a^*)/(d^*b^*)$

$= c^*/d^*.a^*/b^* = (c/d)^*.(a/b)^*$, for all $a/b, c/d \in W/\tilde{}$. This involution is proper. For let $(a/b.a/b)^* = (a/b.c/d)^* = (c/d.c/d)^*, b, d \neq 0$. Then $aa^*/bb^* = ac^*/bd^* = cc^*/dd*$. Then $aa^*bd^* = bb^*ac^*, ac^*dd^* = bd^*cc^*, aa^*dd^* = bb^*cc^*$. We need to show that $a/b = c/d$ or $ad = bc$. By cancellation $aa^*\ d^* = b^*ac^*$. If $a = 0$ then $bb^*cc^* = 0, b \neq 0$. Then $cc^* = 0$ and so $c = 0$. It follows that $ad = bc$ as required. On the other hand if $a \neq 0$ then from

$aa^*\ d^* = b^*ac^*$ we get $a^*\ d^* = b^*c^*$ and so $da = cb$ as required. Thus $(W/\tilde{}, *)$ is a proper *-semigroup. Now we see that $W/\tilde{}$ is regular. For let $a/b \in W/\tilde{}$. If $a = 0$ then $a/b.a/b.a/b = a/b$. On the other hand if $a \neq 0$ then $a/b.b/a.a/b = aba/bab = a/b$ since $abab = abab$. Finally we show that there is a *-embedding $f : (S, *) \to (W/\tilde{}, *)$ given by $f(a) = ab/b$ where b is some fixed non zero element in S. For if $f(a) = f(c)$ then $ab/b = cb/b$ and so $abb = cbb$ from which we get $a = b$. Also $f(ac) = acb/b = ab/b.cb/b$ $= f(a)f(c)$ because $acb^3 = acb^3$. And $f(a^*) = a^*b/b = a^*b^*/b^* = (f(a))^*$ because $a^*bb^* = a^*bb^*$. This completes the proof. □

3. Two Counter Examples

Example 4. *Let S be the multiplicative group generated by the matrix* $A = \begin{pmatrix} -1 & 2i \\ 2i & 3 \end{pmatrix}$. *We notice that $A^{-1} = A^t$. Now $(S, *)$ under the inverse involution (which is the transpose involution) is a proper *-semigroup: $aa^{-1} = ab^{-1} = bb^{-1}$ implies that $a = b$ for all $a, b \in S$. It is to be noticed that S is an infinite cyclic group. This is a $Z-$module with involution defined by $(\sum m_i A^{m_j})^* = \sum m_i A^{-m_j}, m_i \in Z$. This is the same as the semigroup ring with involution $(Z[S], *)$ where $*$ is as defined above. Since Z is a formally complex ring under the identity involution and since $(S, *)$ is an inverse it follows that $(Z[S], *)$ is a proper *-ring. This not regular if we take $2A$ then there is no element in $Z[S]$ of the $C = \sum m_i A^{m_j}$ such that $2AC.2A = 2A$ then taking absolute values of determinants of both sides would give $2^8k = 2^4$ where k is a positive integer and this is impossible. We claim that the proper *-ring $(Z[S], *)$ cannot be *-embedded in any regular proper *-ring . For assume that $(Z[S], *)$ is *-embedded in a regular proper *-ring $(R, *)$. We notice that*

$A^3 = \begin{pmatrix} -5 & 6i \\ 6i & 7 \end{pmatrix}, A^4 = \begin{pmatrix} -7 & 8i \\ 8i & 9 \end{pmatrix}, B = A^3 - A^4 = \begin{pmatrix} 2 & -2i \\ -2i & -2 \end{pmatrix}$. *Then in* $(R, *)$ *we have* $BB^* = 0$. *This implies that* $A^3 = A^4$ *in* $(R, *)$ *although* $A^3 \neq A^4$*in* $(Z[S], *)$ *and this is a contradiction.*

Example 5. *Consider the set* S *of matrices of kind* $\begin{pmatrix} a & b \\ ka & kb \end{pmatrix}, a, b, k \in Z[x]$. *We restrict* a *to be a polynomial of nonzero constant term and b to be a polynomial without constant term. Then* S *is closed under multiplication as can be easily verified. We notice that the two columns as well as the two rows in any of these matrices are linearly dependent and that* S *is closed under the transposition involution. Also we notice that this involution is proper for there is no nonzero matrix* A *in* S *such that* $AA^* = AA^t = 0$. *Thus* (S, t) *is a proper *-semigroup. Now consider the multiplicative semigroup* $T = M_2(Q(i)(x))$ *of all 2 by 2 matrices with entries as rational functions in* x *and with coefficients from the field* $Q[i]$. *This semigroup* (S, t) *is a semigroup with the transpose involution* t. *But this involution is not proper.*

*Assume that there is a smallest proper *semigroup* (W, t) *in* (T, t) *that contains* (S, t). *Consider the matrix* $A = \begin{pmatrix} 1+x & ix \\ 0 & 0 \end{pmatrix} \in S$. *Then there is a matrix* $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$ *such that* $ABA = A, BAB = B$. *Thus* $\begin{pmatrix} 1+x & ix \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1+x & ix \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+x & ix \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1+x & ix \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. *Carrying out the necessary calculations we find that* $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \in W$. *But we notice that* $BB^t = 0$ *and so* (W, t), *although regular and contains* (S, t), *is not a proper *-semigroup.*

Example 6. *In this example we exhibit a p*-semigroup which is not regular and can be *-embedded in a regular p*-semigroup yet it cannot be *-embedded in any regular p*-ring. Let* $R = M_2([2Z][i])$ *be the ring of all* 2×2 *matrices with entries from the ring* $[2Z][i]$.

Let S be the subsemigroup of R (under multiplication) generated by the elements ae_{ij}, where $a = 0, 2$ or $2i$ and e_{ij} is the matrix with 0 everywhere except the $ij-$entry, which is 1. Let $s_1 = 2e_{11}$ and $s_2 = 2ie_{12}$. Then s_1, s_2 are in S. Let t be the transpose involution on S. Then $(S, *)$ is a non-regular p-semigroup. Let T be the set $\{qe_{ij}, q \in Q\}$. Then (T, t) is a regular p*-semigroup which *–embeds (S, t). We claim that (S, t) cannot be *-embedded in any p*-ring (regular or not). Otherwise, let f be a *-embedding of (S, t) into a proper*-ring $(W, *)$. Then (R, t) has a ring homomorphic image in $(W, *)$, because the elements of S form a basis for the free Z-module R and W is a Z-module which contains S. Let us call such a homomorphism by f^-. Then f^- is $[2Z][i]-$linear and it extends f. The involution * on W extends t in the sense that $f^-(A^t) = (f^-(A))^*$. Now t is not a proper involution on R since if A is the non zero matrix with first row being $(2, -2i)$, and the second row being the zero row then $A = s_1 - s_2 \in R$ and $AA^t = 0$. Thus $f^-(AA^t) = 0 = f^-(A)(f^-(A))^*$ in W. Since $(W, *)$ is a p*-ring it follows that $\overline{f}(A) = 0$ in W. Thus $0 = f^-(s_1) - f^-(s_2)$ in W. But then $f(s_1) = f(s_2)$ in W and this is a contradiction.*

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