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QUASI-IDEMPOTENTS IN FINITE SEMIGROUP OF FULL ORDER-PRESERVING TRANSFORMATIONS

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Abstract. Let X_n be the finite set $\{1,2,\ldots,n\}$ and $\mathscr{O}_{n,r} = \{\alpha \in \mathscr{O}_n : |\text{im}\alpha| \leq \text{r}\}$ (where $1 \leq r \leq n-1$) be the ideals of a semigroup of full order-preserving transformations on the finite set. In this article, a study of quasi-idempotent elements via idempotents and generating set in the ideals of finite semigroup of full order-preserving transformations is carried out. The quasi-idempotent elements are characterised in this semigroup and that it is quasi-idempotent generated via idempotents. Moreover, the minimum length of a factorisations into products of quasi-idempotent elements is obtained.

Keywords: full order-preserving; quasi-idempotent; generating set; ideals; length formula.

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1. Introduction

One of the most important generalizations of group theory and in fact, the leading area of research in modern algebra is the theory of semigroups. The theory become an interesting field in modern abstract algebra and the work done by Howie [1] in the full transformation semigroup $\mathscr{T}_n$ consisting of all mappings from a set X_n into itself, formed a major breakthrough and indeed the basis for further investigations into the area which has assumed an enviable place in the

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theory of semigroup. Since then, there have been many articles concerned with this idea in $\mathscr{T}_n$ (see for example, [2, 3, 4, 5]).

For any $\alpha \in \mathscr{O}_n$(the semigroup of full order-preserving transformations), if $\alpha = \alpha^2$ then α is called an idempotent; and if $\alpha \neq \alpha^2 = \alpha^4$ then α is called a quasi-idempotent.

We now begin with a finite generating set. Let S be a semigroup and let $\emptyset \neq A \subseteq S$. The smallest subsemigroup of S containing A is called the subsemigroup generated by A and is denoted by $\langle A \rangle$. Clearly, $\langle A \rangle$ is the set of all finite products of elements of A.

If there exists a non-empty subset A of S such that $\langle A \rangle = S$, then A is called a generating set of S. Also, the *rank* of a finitely generated semigroup S is defined by

$$\text{rank}(S) = \min\{|A| : A \subseteq S \text{ and } \langle A \rangle = S\}.$$

That is, the cardinality of a minimum generating set. If S is generated by the set E of idempotents, then the *idempotent rank* of S is defined by

$$\text{idrank}(S) = \min\{|A| : A \subseteq E \text{ and } \langle A \rangle = S\}.$$

Generation of finite transformation subsemigroups includes the work of Gomes and Howie in [7] proved that both the rank and idempotent rank of $\text{Sing}_n = \mathscr{T}_n \setminus \mathscr{S}_n$, where $\mathscr{S}_n$ is the symmetric group, are equal to $\frac{n(n-1)}{2}$. This was generalised by Howie and McFadden in [8] considered the semigroup $K(n,r) = \{\alpha \in \text{Sing}_n : |\text{im}(\alpha)| \leq r\}$ where $(2 \leq r \leq n-1)$, and showed that both the rank and idempotent rank are equal to $S(n,r)$, the Sterling number of the second kind. Gomes and Howie in [9] investigated the rank of the semigroups $\mathscr{O}_n$ and $\mathscr{P}\mathscr{O}_n$, (the semigroup of order-preserving full transformation and order-preserving partial transformations on X_n). It was shown that the rank of $\mathscr{O}_n$ is n and $\mathscr{P}\mathscr{O}_n$ is $(2n-1)$, and the idempotent rank of $\mathscr{O}_n$ is $(2n-2)$ and $\mathscr{P}\mathscr{O}_n$ is idempotent-generated and its idempotent rank is $(3n-2)$. Madu and Garba studied the subsemigroup $\mathscr{I}\mathscr{O}_n$ of $\mathscr{P}\mathscr{O}_n$ in [10]. Garba [14] in generalizing the work of [9, 7] considered the semigroup $L(n,r) = \{\alpha \in \mathscr{O}_n : |\text{im}(\alpha)| \leq r\}$ where $(2 \leq r \leq n-2)$, and showed that both the rank and idempotent rank are equal to $\binom{n}{r}$ and the rank and idempotent rank of $M_{n,r} = \{\mathscr{P}\mathscr{O}_n : |\text{im}(\alpha)| \leq r\}$ where $(2 \leq r \leq n-2)$ are both $\sum_{k=r}^{n} \binom{n}{k}\binom{k-1}{r-1}$.

In particular, if there exists a generating set A of S consisting of entirely quasi-idempotents, then A is called a quasi-idempotent generating set of S, and the *quasi-idempotent rank* of S is defined by

$$\mathrm{qrank}(S) = \min\{|A| : A \subseteq QE \text{ and } \langle \mathrm{A} \rangle = \mathrm{S}\}.$$

Umar [20] used quasi-idempotent elements to generate the semigroup

$$\mathscr{I}_n^- = \{\alpha \in \mathscr{I}_n : (\forall x \in \mathrm{dom}(\alpha)), x\alpha \leq x\},$$

of all partial one-to-one order-decreasing transformations in finite symmetric inverse semigroup and proved that the semigroup $\mathscr{I}_n^-$ is quasi-idempotent generated and its rank is equal to $\frac{n(n+1)}{2}$. Madu and Garba [10] showed that each element in the semigroup $\mathscr{I}\mathscr{O}_n$ is expressible as a product of quasi-idempotents of defect one in $\mathscr{I}\mathscr{O}_n$, and that the quasi-idempotent rank and depth of $\mathscr{I}\mathscr{O}_n$ are $2(n-1)$ and $(n-1)$ respectively. Similarly, [17] extended the work of [10] to the ideals of $\mathscr{I}\mathscr{O}_n$ and proved that it is quasi-idempotent generated and that its quasi-idempotent rank is $2(\binom{n}{r}-1)$. Garba et al [11] proved that Sing_n is quasi-idempotent generated and that the quasi-idempotent rank of Sing_n is $\frac{n(n-1)}{2}$. Garba and Imam [15] proved that the semigroup $\mathscr{S}\mathscr{I}_n$ (of all strictly partial one-to-one maps on X_n) is generated by quasi-idempotents of defect one and the best possible global lower bound for the number of quasi-idempotents (of defect and shift equal to one) required to generate $\mathscr{S}\mathscr{I}_n$ is equal to $\lceil\frac{3(n-1)}{2}\rceil$. Bugay [12] proved among other results that for $n \geq 4$ the quasi-idempotent rank of $\mathscr{I}_n$ is 4. Bugay [13] proved that $\mathscr{I}(n,r) = \{\alpha \in \mathscr{I}_n : |\mathrm{im}(\alpha)| \leq r\}$ for $(1 \leq r \leq n-1)$ is quasi-idempotent generated and the quasi-idempotent rank of $\mathscr{I}(n,r)$ is $\binom{n}{2}$ if $r = 2$ and $\binom{n}{r}+1$ if $r \geq 3$. Recently, Imam et al [18] showed that the semigroup $\mathscr{O}_n$ is generated by Quasi-idempotent of defect one and the upper bound for the quasi-idempotent rank is $\lceil\frac{3(n-2}{2}\rceil$. In the same manner, [**?**] investigated the product and rank of quasi-idempotents in the semigroups of Partial order-preserving mappings of a finite chain and obtained the upper bound for its rank to be $\lceil\frac{5n-4}{2}\rceil$.

In this article, we have characterized quasi-idempotents, investigate the product of quasi-idempotents and obtain obtained a length formula for quasi-idempotents (for $n \geq 4$) of $\mathscr{O}_{n,r}$.

2. PRELIMINARIES

In this section, we present a characterisation of quasi-idempotent in $\mathscr{O}_{n,r}$, this will enable us identify them among other elements of the ideals.

We begin with the following definition of quasi-idempotent.

Lemma 1. *Let $\alpha \in \mathscr{O}_{n,r}$. Then there exists $m \geq 1$ such that*

$$\alpha \neq \alpha^2 \neq \cdots \neq \alpha^{m-1} \neq \alpha^m = \alpha^{m+1} = \alpha^{m+2} = \cdots .$$

Proof. If α in an idempotent, then $\alpha = \alpha^2 = \alpha^3 = \cdots$ and so the case is trivial.
Suppose α is not an idempotent. Then
α contains both non-stationary and at least a stationary blocks.
Let $x \in X_n$. Then since X_n is finite, the sequence x, $x\alpha$, $x\alpha^2 \cdots$ must contains repetitions.
Let m_x be the first index to repeat and let $m_x + r_x$ be its first repetition.
That is,
$x\alpha^{m_x} = x\alpha^{m_x+r_x} = (x\alpha^{m_x})\alpha^{r_x}$.
If $r_x > 1$, then $x\alpha^{m_x}$, $x\alpha^{m_x+1}$, $x\alpha^{m_x+2}$,$\cdots$, $x\alpha^{m_x+r_x-1}$ are all distinct.
If $x\alpha^{m_x} < x\alpha^{m_x+r_x-1}$
$\Longrightarrow x\alpha^{m_x+1} < x\alpha^{m_x+r_x} = x\alpha^{m_x}$
$\Longrightarrow x\alpha^{m_x+2} < x\alpha^{m_x+1}$
$\Longrightarrow x\alpha^{m_x+3} < x\alpha^{m_x+2}$
$\Longrightarrow \cdots$
$\Longrightarrow x\alpha^{m_x+r_x-2} < x\alpha^{m_x+r_x-1}$
$\Longrightarrow x\alpha^{m_x+r_x-1} < x\alpha^{m_x+r_x} = x\alpha^{m_x}$, we arrive at a contradiction.
On the other hand,
if $x\alpha^{m_x+r_x-1} < x\alpha^{m_x}$
$\Longrightarrow x\alpha^{m_x} = x\alpha^{m_x+r_x} < x\alpha^{m_x+1}$
$\Longrightarrow x\alpha^{m_x+1} < x\alpha^{m_x+2}$
$\Longrightarrow x\alpha^{m_x+2} < x\alpha^{m_x+3}$
$\Longrightarrow \cdots$
$\Longrightarrow x\alpha^{m_x+r_x-1} < x\alpha^{m_x+r_x-2}$

$\Longrightarrow x\alpha^{m_x+r_x} < x\alpha^{m_x+r_x-1}$

$\Longrightarrow x\alpha^{m_x} = x\alpha^{m_x+r_x} < x\alpha^{m_x+r_x-1}$, we arrive at a contradiction again.

In both cases $r_x = 1$, so that

$x\alpha \neq x\alpha^2 \neq \cdots \neq x\alpha^{m_x} = x\alpha^{m_x+1} = x\alpha^{m_x+2} = \cdots$

now, let

$$\mathrm{m} = \max_{\mathrm{x}\in\mathrm{X_n}}\{\max : \mathrm{x} \in \mathrm{X_n}\} = \max_{\mathrm{x}\in\mathrm{X_n}}\{\max : \mathrm{x} \in \mathrm{X_n}\}.$$

$\forall x \in X_n$, $x\alpha \neq x\alpha^2 \neq x\alpha^3 \neq \cdots \neq x\alpha^m = x\alpha^{m+1} = x\alpha^{m+2} = \cdots$.

In which we have $\alpha \neq \alpha^2 \neq \alpha^3 \neq \cdots \neq \alpha^m = \alpha^{m+1} = \alpha^{m+2} = \cdots$. □

Corollary 2. *An element $\alpha \in \mathscr{O}_{n,r}$ is a quasi-idempotent if and only if every non-stationary block is mapped into the stationary block of α.*

An element $\alpha \in \mathscr{O}_{n,r}$ is written in array notation as $\alpha = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_r \\ b_1 & b_2 & b_3 & \cdots & b_r \end{pmatrix}$, where

$$im(\alpha) = \{b_1, b_2, b_3, \ldots, b_r\}$$

and

$$Ker(\alpha) = \{A_1, A_2, A_3, \ldots, A_r\},$$

where each $im(\alpha)$ is called image of α and $Ker(\alpha)$-class is called *block* of α.

Let $\alpha \in \mathscr{O}_{n,r}$. Then a block A_i of α is called stationary block if $a_i \in A_i$ otherwise it is called non-stationary block.

Theorem 3. *A non-idempotent element $\alpha \in \mathscr{O}_{n,r}$ is a quasi-idempotent if and only if the restriction of α to its image, is an idempotent.*

That is, $\alpha|_{im\alpha} = e$.

Proof. Suppose that $\alpha|_{im\alpha}$ is an idempotent. Then,

$\forall x \in X_n$, $(x\alpha^2)\alpha = x\alpha^2$ showing that $\alpha^3 = \alpha^2$.

An so, $\alpha^4 = \alpha^3\alpha = \alpha^2\alpha = \alpha^3 = \alpha^2$. Thus, α is a quasi-idempotent.

Similarly, suppose $\alpha|_{im\alpha}$ is not an idempotent and so,

for some $x \in dom(\alpha)$, $(x\alpha^2)\alpha \neq x\alpha^2$

therefore, $\alpha^3 \neq \alpha^4$.

Thus, by lemma 4.1.1 , $\alpha \neq \alpha^4$.

In which we have α is not a quasi-idempotent. □

Example 4. *Let* $\alpha = \begin{pmatrix} 12 & 34 & 56 & 789 & 10,11 & 12,13 \\ 1 & 2 & 5 & 6 & 10 & 11 \end{pmatrix}$ *in* $\mathcal{O}_{13,6}$.

Then $\alpha|_{im\alpha} = \begin{pmatrix} 12 & 56 & 10,11 \\ 1 & 5 & 10 \end{pmatrix}$

3. Main Results

In this section, we study the nature of quasi-idempotent elements in finite semigroup of full order-preserving transformations. In particlar, we show that the semigroup $\mathcal{O}_{n,r}$ is quasi-idempotent generated.

Theorem 5. [18], *Theorem 4.2.1] For* $n \geq 4$, $\mathcal{O}_n = \langle QE_1 \rangle$.

Lemma 6. [14], *Theorem 4.2.1] Every element* $\alpha \in J_r$ $(r \leq n-2)$ *is expressible as a product of idempotents in* J_{r+1}.

Theorem 7. *For* $n \geq 4$ *and* $1 \leq r \leq n-1$, $\langle QE \rangle = \mathcal{O}_{n,r}$.

Proof.

$$\alpha = \begin{pmatrix} A_1 & A_2 \cdots & A_r \\ a_1 & a_2 \cdots & a_r \end{pmatrix},$$

be an idempotent of height r. Where $a_i \in A_i$, $i = 1,2,\ldots,r$.

Case 1. If $a_{i-1} \neq minA_{i-1}$ for some $i = 2,3,\ldots,r$, then

$$\alpha = \begin{pmatrix} A_1 & A_2 \cdots & A_r \\ a_1 & a_2 \cdots & a_r \end{pmatrix} = \begin{pmatrix} A_1 & A_2 & A_{r-2} & A_{r-1} \cdots A_r \\ a_1 & a_2 & a_{r-2} & a_{r-1}-1 \cdots a_r \end{pmatrix}$$

$$\begin{pmatrix} A_1 & A_2 & \cdots A_{r-2} & \{minA_{r-1} \cdots a_{r-1}-1\} & \{a_{r-1} \cdots maxA_{r-1}\} \cup A_r \\ a_1 & a_2 & \cdots a_{r-2} & a_{r-1} & a_r \end{pmatrix}$$

$$= \begin{pmatrix} A_1 & A_2 \cdots A_{i-2} & A_{i-1} & A_i & A_{i+2} \cdots & A_r \\ a_1 & a_2 \cdots a_{i-2} & a_{i-1}-1 & a_{i-1} & a_{i+2} \cdots & a_r \end{pmatrix}$$

$$\begin{pmatrix} A_1 & A_2\cdots & A_{i-2} & \{minA_{i-1}\cdots a_{i-1}-1\} & \{a_{i-1}\cdots, maxA_{i-1}\}\cup A_i & A_{i+1}\cdots & A_r \\ a_1 & a_2\cdots & a_{i-2} & a_{i-1} & a_i & a_{i+1}\cdots & a_r \end{pmatrix}.$$

Case 2. If $a_i = minA_i$ for all $i = 2,3,\ldots,r$, then

$$\alpha = \begin{pmatrix} A_1 & A_2\cdots & A_r \\ a_1 & a_2\cdots & a_r \end{pmatrix} = \begin{pmatrix} A_1 & A_2\cdots & A_{r+1} & A_{r+2}\cdots & A_r \\ a_1 & a_2\cdots & a_{r+1}-1 & a_{r+2}\cdots & a_r \end{pmatrix}$$

$$\begin{pmatrix} A_1 & A_2\cdots & A_r\setminus\{a_{r+1}-1\} & a_{r+1}-1 & A_{r+1}\cup A_{r+2} & A_{r+3}\cdots & A_r \\ a_1 & a_2\cdots & a_r & a_{r+1} & a_{r+2} & a_{r+3}\cdots & a_r \end{pmatrix}$$

$$= \begin{pmatrix} A_1 & A_2\cdots A_i & A_{i+1} & A_{i+2}\cdots & A_r \\ a_1 & a_2\cdots a_i & a_{i+1}-1 & a_{i+2}\cdots & a_r \end{pmatrix}$$

$$\begin{pmatrix} A_1 & A_2\cdots & A_i\setminus\{a_{i+1}-1\} & a_{i+1}-1 & A_{i+1}\cup A_{i+2} & A_{i+3}\cdots & A_r \\ a_1 & a_2\cdots & a_i & a_{i+1} & a_{i+2} & a_{i+3}\cdots & a_r \end{pmatrix}$$

where $i < r$

$$\alpha = \begin{pmatrix} 1 & 2 & 3\cdots r-1 & \{r, r+1,\cdots,n\} \\ 1 & 2 & 3\cdots r-1 & r \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3\cdots & r-1 & \{r, r+1,\cdots,n\} \\ 1 & 2 & 3\cdots & r & r+1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3\cdots & \{r, r+1,\cdots,n\} & \{r+1, r+2,\cdots,n\} \\ 1 & 2 & 3\cdots & r-1 & r \end{pmatrix}$$

where $i = r$.

Case 3. If $a_r \neq minA_r$ and $a_i = minA_i$ for $i = 1,2,\ldots,r-1$, then

(1) $$\alpha = \begin{pmatrix} A_1 & A_2\cdots & A_r \\ a_1 & a_2\cdots & a_r \end{pmatrix} = \begin{pmatrix} A_1 & A_2\cdots A_{r-2} & A_{r-1} & A_r \\ a_1 & a_2\cdots a_{r-2} & \{minA_r\} & a_r \end{pmatrix}$$

$$\begin{pmatrix} A_1 & A_2\cdots A_{r-2} & A_{r-1}\cup\{minA_r\} & A_r\setminus\{minA_r\} \\ a_1 & a_2\cdots a_{r-2} & a_{r-1} & a_r \end{pmatrix}$$

□

Remark 8. *: If $a_{i+1}-1$ is not in the set of images before $a_{i+1}-1$. That is, $a_{i+1}-1$ not in a_k, where $k \leq r-1$. The previous blocks of α_s are non-singletons.*

Example 9. Let $n = 8$, $r = 4$.

$$\alpha = \begin{pmatrix} 12 & 34 & 56 & 78 \\ 2 & 3 & 6 & 7 \end{pmatrix} = \begin{pmatrix} 12 & 34 & 56 & 78 \\ 2 & 3 & 5 & 6 \end{pmatrix} \begin{pmatrix} 12 & 34 & 5 & 678 \\ 2 & 3 & 6 & 7 \end{pmatrix}$$

Example 10.

$$\alpha = \begin{pmatrix} 12 & 34 & 56 & 78 \\ 1 & 3 & 5 & 7 \end{pmatrix} = \begin{pmatrix} 12 & 34 & 56 & 78 \\ 1 & 2 & 5 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3456 & 78 \\ 1 & 3 & 5 & 7 \end{pmatrix}$$

Example 11. Let $n = 8$, $r = 4$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5678 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5678 \\ 1 & 2 & 3 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 45 & 678 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$$

Example 12.

$$\alpha = \begin{pmatrix} 12 & 34 & 56 & 78 \\ 1 & 3 & 5 & 8 \end{pmatrix} = \begin{pmatrix} 12 & 34 & 56 & 78 \\ 1 & 2 & 5 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 34567 & 8 \\ 1 & 3 & 5 & 8 \end{pmatrix}$$

4. LENGTH FORMULA OF QUASI-IDEMPOTENTS IN $\mathscr{O}_{n,r}$

For $\mathscr{O}_{1,1}$, $\mathscr{O}_{2,1}$ and $\mathscr{O}_{2,2}$ are $\mathscr{O}_n$, but if $n \geq 3$ and $r \leq n-2$ cannot contain any permutation and hence is a subsemigroup of $\mathscr{O}_n$. An element α of a semigroup is called Quasi-idempotent depth, if k the is least integer such that $\alpha = e_1e_2\ldots e_k$ where $e_i \in QE(S)$, $i = 1,2,\ldots,k$. The Quasi-idempotent depth of a semigroup S, denoted by $\triangle(S)$ is the maximum value of the depth of its element (if it exists).

The global version of this is given as $\triangle(S) = k$, if $QE^k(S) = S \neq QE^{k-1}(S)$. An arbitrary element α of a semigroup S is called idempotent if its elements range identically. In this case, we say $\alpha = \alpha^2$. The full transformation semigroup does contain quasi-idempotent elements, that is, a non-idempotent element whose square is an idempotent. We now consider the question of the Quasi-idempotent of $\mathscr{O}_{n,r}$. We shall consider general case which will cover all the semigroup $\mathscr{O}_{n,r}$, for $n \geq 3$.

Lemma 13. *An element α in $\mathscr{O}_{n,r}$ is a Quasi-idempotent if and only if it consists of an acyclic and trivial orbits.*

Proof. If $n \geq 3$ and $r \geq 1$, then every element must contain element of at least one shift. Therefore, in $\mathscr{O}_{n,r}$ there is no standard and cyclic orbits, only trivial and acyclic orbits exist, which each must have a fixed point. □

Theorem 14. *Let QE be the set of all Quasi-idempotents in $\mathscr{O}_{n,r}$ and let $\triangle(\langle QE\rangle)$ be the unique k for which $\langle QE\rangle = QE \cup QE^2 \cup \cdots \cup QE^k \neq QE \cup QE^2 \cup \cdots \cup QE^{k-1}$.*
Then $\triangle(\langle QE\rangle) = 2$, for all $n \geq 3$.

Example 15.

$$\alpha = \begin{pmatrix} 12 & 34 & 5 & 678 \\ 5 & 6 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 12 & 34 & 5 & 678 \\ 2 & 3 & 7 & 8 \end{pmatrix} \begin{pmatrix} 12 & 3456 & 7 & 8 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

Example 16.

$$\alpha = \begin{pmatrix} 12 & 34 & 56 & 78 & 9,10,11,12 \\ 3 & 8 & 9 & 10 & 12 \end{pmatrix} = \begin{pmatrix} 12 & 34 & 56 & 78 & 9,10,11,12 \\ 2 & 4 & 6 & 11 & 12 \end{pmatrix} \begin{pmatrix} 123 & 45 & 6789 & 10,11 & 12 \\ 3 & 8 & 9 & 10 & 12 \end{pmatrix}$$

Example 17.

$$\alpha = \begin{pmatrix} 12 & 34 & 5 & 67 & 8 \\ 1 & 2 & 4 & 5 & 7 \end{pmatrix} = \begin{pmatrix} 12 & 34 & 5 & 67 & 8 \\ 1 & 3 & 5 & 6 & 7 \end{pmatrix} \begin{pmatrix} 1 & 23 & 45 & 6 & 78 \\ 1 & 2 & 4 & 5 & 7 \end{pmatrix}$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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