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ON THE GRAVITY FUNCTION OF SINGULAR TRANSFORMATIONS: CHARACTERIZATIONS AND STRUCTURAL PROPERTIES

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Abstract. Let $\mathcal{T}_n$ denote the full transformation semigroup on an n -element set, and let Sing_n be its subsemigroup of singular mappings. For $\alpha \in \text{Sing}_n$, the gravity $g(\alpha) = n + c(\alpha) - f(\alpha)$ measures the minimum number of defect-one idempotents required to express α as a product, where $c(\alpha)$ is the number of cyclic orbits of length at least two and $f(\alpha)$ is the number of fixed points. In this paper, we completely characterize when a singular mapping satisfies $g(\alpha) = n$ and when $g(\alpha) = s(\alpha)$, where $s(\alpha) = n - f(\alpha)$ is the rank defect. We prove that $g(\alpha) = n$ if and only if $c(\alpha) = f(\alpha)$, and $g(\alpha) = s(\alpha)$ if and only if α has no cyclic orbits. Consequently, the identity $g(\alpha) - s(\alpha) = c(\alpha)$ provides a direct combinatorial interpretation of the difference between these invariants. We provide a complete enumeration of idempotents satisfying $g(\alpha) = s(\alpha)$, obtaining the classical count $\sum_{k=1}^n \binom{n}{k} k^{n-k}$, with the singular case obtained by excluding the identity mapping. We analyze the distribution of these properties within group $\mathcal{H}$ -classes of the semigroup, demonstrating that for any group $\mathcal{H}$ -class in a $\mathcal{D}$ -class D_k , the sets of elements satisfying $g(\alpha) = n$ and $g(\alpha) = s(\alpha)$ are disjoint subsets of size at most $k!$. Additionally, we establish that the classes of cycle-free mappings are closed under composition, with the property that $g(\alpha\beta) = s(\alpha\beta)$ for all elements in these classes. These results offer new insights into the relationship between the orbit structure and the idempotent-generated nature of singular transformations.

Keywords: gravity; combinatorial; structural results

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1. INTRODUCTION

The full transformation semigroup $\mathcal{T}_n$ on an n -element set $X = \{1, \dots, n\}$ has been a central object of study in semigroup theory since the foundational work of Howie [2], Clifford and Preston [5], and others. The subsemigroup Sing_n of singular mappings (those with rank less than n) is of particular interest due to its rich combinatorial and algebraic structure. A fundamental problem in the study of Sing_n is understanding the idempotent-generated nature of the semigroup, as every element of Sing_n can be expressed as a product of idempotents of defect one [1].

The concept of gravity, introduced by Howie [2], provides a measure of the complexity of a singular mapping in terms of the minimum number of defect-one idempotents needed to represent it. For a mapping $\alpha \in \text{Sing}_n$, the gravity is defined as $g(\alpha) = n + c(\alpha) - f(\alpha)$, where $c(\alpha)$ is the number of cyclic orbits of length at least two and $f(\alpha)$ is the number of fixed points. This invariant has been shown to be intimately connected to the orbit structure of the mapping and plays a crucial role in determining the idempotent rank and other algebraic properties of subsemigroups of $\mathcal{T}_n$.

Previous work on the gravity function has primarily focused on establishing upper bounds and extremal values. Howie [2] proved that for mappings of a fixed rank r , the maximum gravity is $n + \lfloor \frac{1}{2}(r - 2) \rfloor$, and that the overall maximum gravity in Sing_n is $\lfloor \frac{3}{2}(n - 1) \rfloor$. These results established the importance of gravity as a measure of complexity but left open the question of precisely characterizing which mappings attain particular gravity values relative to other invariants such as the defect.

In this paper, we undertake a systematic investigation of the combinatorial and structural properties of singular mappings in $\mathcal{T}_n$ with particular emphasis on the gravity function. Our main contributions are threefold. First, we establish fundamental characterizations: $g(\alpha) = n$ if and only if $c(\alpha) = f(\alpha)$, and $g(\alpha) = s(\alpha)$ if and only if α is cycle-free, where $s(\alpha) = n - f(\alpha)$ is the rank defect. These results reveal that the difference $g(\alpha) - s(\alpha)$ is precisely $c(\alpha)$, providing a simple yet elegant relationship between the gravity, the defect, and the cyclic orbit structure.

Second, we provide a complete enumeration of idempotent transformations satisfying $g(\alpha) = s(\alpha)$, yielding the classical count $\sum_{k=1}^n \binom{n}{k} k^{n-k}$, with the singular case obtained by excluding the identity mapping. We also analyze the distribution of these properties within group $\mathcal{H}$ -classes of the semigroup, demonstrating structural constraints on where elements with certain gravity values can occur.

Third, we establish closure properties for important subclasses: the sets of mappings with no cyclic orbits are shown to be closed under composition, with the property that $g(\alpha\beta) = s(\alpha\beta)$ for all elements in these classes. This structural result has important consequences for understanding the subsemigroup structure of Sing_n and extends our understanding of how the gravity function behaves under composition.

2. PRELIMINARIES

Let $X = \{1, 2, \dots, n\}$ be a finite set of cardinality n , and let $\mathcal{T}_n$ denote the full transformation semigroup consisting of all mappings $\alpha : X \rightarrow X$, where composition is performed from left to right. For $\alpha \in \mathcal{T}_n$, the image of α is denoted by $\text{Im}(\alpha)$, and the rank of α is $\text{rank}(\alpha) = |\text{Im}(\alpha)|$. A mapping is called *singular* if $\text{rank}(\alpha) < n$, and the set of all singular mappings in $\mathcal{T}_n$ is denoted by Sing_n . The defect of α is defined as $s(\alpha) = n - \text{rank}(\alpha)$.

For $\alpha \in \mathcal{T}_n$, the orbit of an element $x \in X$ under α is the set $\{x, x\alpha, x\alpha^2, \dots\}$. An orbit is called:

- a *cyclic orbit* if it contains a cycle of length at least two;
- a *fixed point* if it is of the form $\{x\}$ with $x\alpha = x$;
- a *standard orbit* if it contains neither cycles nor fixed points (i.e., it is an infinite path in the functional digraph that eventually enters a cycle);
- a *trivial orbit* if it consists of a fixed point.

We denote by $c(\alpha)$ the number of cyclic orbits of α of length at least two, and by $f(\alpha)$ the number of fixed points of α .

The following definition is central to our study.

Definition 2.1. For $\alpha \in \text{Sing}_n$, the *gravity* of α , denoted by $g(\alpha)$, is defined by

$$(1) \quad g(\alpha) = n + c(\alpha) - f(\alpha).$$

The gravity function was introduced by Howie [2] and has been shown to represent the minimum number of defect-one idempotents required to express α as a product.

We recall two fundamental results due to Howie [2].

Lemma 2.2 (Howie, 1980). *Let $S = \text{Sing}_n$. Then*

$$\max\{g(\alpha) : \alpha \in S, \text{rank}(\alpha) = r\} = n + \left\lfloor \frac{1}{2}(r-2) \right\rfloor,$$

and

$$\max\{g(\alpha) : \alpha \in S\} = \left\lfloor \frac{3}{2}(n-1) \right\rfloor.$$

We shall also make use of the following definitions.

Definition 2.3. A mapping $\alpha \in \mathcal{T}_n$ will be called:

- (1) *Fixed point-free* if $f(\alpha) = 0$;
- (2) *Cycle-free* if $c(\alpha) = 0$;
- (3) *B-acyclic* if the only orbits of α are acyclic and trivial;
- (4) *B-standard* if α contains only standard and trivial orbits (equivalently, α is cycle-free);
- (5) *T-standard* if α is both fixed point-free and cycle-free (i.e., $c(\alpha) = f(\alpha) = 0$).

Remark 2.4. The term *B-standard* indicates that the mapping has only standard and trivial orbits (hence is cycle-free but may have fixed points). The term *T-standard* indicates that the mapping has only standard orbits (hence is both cycle-free and fixed point-free).

We shall also need the following characterization of the kernel of an orbit.

Definition 2.5. Let $\alpha \in \mathcal{T}_n$ and let Ω be an orbit of α . The *kernel* of Ω , denoted by $K(\Omega)$, is the unique non-empty subset of Ω such that $\alpha|_{K(\Omega)}$ is a bijection from $K(\Omega)$ onto itself, and for every $x \in \Omega$, there exists $t \geq 0$ such that $x\alpha^t \in K(\Omega)$.

Equivalently, $K(\Omega)$ is the set of elements in Ω that lie on a cycle:

$$K(\Omega) = \{y \in \Omega : (\exists k \geq 1) y\alpha^k = y\}.$$

The following example illustrates these concepts.

Example 2.6. Let $n = 10$ and consider $\alpha \in \mathcal{T}_{10}$ given by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 3 & 4 & 5 & 4 & 7 & 8 & 6 & 9 & 10 \end{pmatrix}.$$

The decomposition of α into orbits is:

$$\omega_1 = \{1, 2, 3, 4, 5\} \quad (\text{with } 4 \rightarrow 5 \rightarrow 4, \text{ kernel } \{4, 5\}),$$

$$\omega_2 = \{6, 7, 8\} \quad (\text{with } 6 \rightarrow 7 \rightarrow 8 \rightarrow 6, \text{ cyclic orbit}),$$

$$\omega_3 = \{9\} \quad (\text{fixed point}),$$

$$\omega_4 = \{10\} \quad (\text{fixed point}).$$

Here $c(\alpha) = 1$, $f(\alpha) = 2$, and $n = 10$. The gravity of α is:

$$g(\alpha) = 10 + 1 - 2 = 9.$$

This shows that the minimum number of defect-one idempotents required to express α as a product is 9.

We now recall some fundamental results from the theory of transformation semigroups that will be used throughout this paper.

Theorem 2.7 (Howie, 1995). *Let D_k be a $\mathcal{D}$ -class of $\mathcal{T}_n$ consisting of mappings of rank k . Then:*

- (1) *Every group $\mathcal{H}$ -class in D_k is isomorphic to the symmetric group S_k ;*
- (2) *The number of group $\mathcal{H}$ -classes in D_k is $\binom{n}{k}^2$;*
- (3) *Each group $\mathcal{H}$ -class has exactly $k!$ elements.*

Finally, we recall Schützenberger’s fundamental characterization of idempotent transformations.

Theorem 2.8 (Schützenberger, 1968). *A transformation $\alpha \in \mathcal{T}_n$ is idempotent ($\alpha^2 = \alpha$) if and only if α acts as the identity mapping on its image set $\text{Im}(\alpha)$. Equivalently, every element in the image of α is fixed by α .*

This characterization will be essential for our enumeration results in Section 3.

3. MAIN RESULTS

3.1. Characterizations of the Gravity Function. We begin with a fundamental characterization of when $g(\alpha) = n$.

Lemma 3.1. *For each $\alpha \in \text{Sing}_n$,*

$$g(\alpha) = n \quad \text{if and only if} \quad c(\alpha) = f(\alpha).$$

Proof. From the definition of $g(\alpha)$ in (1),

$$(1) \quad g(\alpha) = n + c(\alpha) - f(\alpha).$$

($\Rightarrow$) Suppose $g(\alpha) = n$. Substituting into (1):

$$n = n + c(\alpha) - f(\alpha) \iff 0 = c(\alpha) - f(\alpha) \iff c(\alpha) = f(\alpha).$$

($\Leftarrow$) Conversely, suppose $c(\alpha) = f(\alpha) = m$ for some $m \geq 0$. Then:

$$g(\alpha) = n + m - m = n.$$

Therefore, $g(\alpha) = n$ if and only if $c(\alpha) = f(\alpha)$. □

As an immediate consequence, we obtain the following.

Theorem 3.2. *For each $\alpha \in \text{Sing}_n$, if α is both fixed point-free and cycle-free, then $g(\alpha) = n$.*

Proof. Suppose $f(\alpha) = 0$ and $c(\alpha) = 0$. Then by the definition of $g(\alpha)$,

$$g(\alpha) = n + 0 - 0 = n.$$

This shows that the only orbits of α are standard orbits. □

We now turn to the characterization of when $g(\alpha) = s(\alpha)$, where $s(\alpha) = n - f(\alpha)$ is the defect.

Theorem 3.3. *For each $\alpha \in \text{Sing}_n$,*

$$g(\alpha) = s(\alpha) \quad \text{if and only if} \quad \alpha \text{ is cycle-free (i.e., } c(\alpha) = 0\text{)}.$$

Proof. From the definitions of $g(\alpha)$ and $s(\alpha)$,

$$g(\alpha) = n + c(\alpha) - f(\alpha), \quad s(\alpha) = n - f(\alpha).$$

($\Rightarrow$) Suppose $g(\alpha) = s(\alpha)$. Then:

$$n + c(\alpha) - f(\alpha) = n - f(\alpha) \iff c(\alpha) = 0.$$

($\Leftarrow$) Conversely, suppose $c(\alpha) = 0$. Then:

$$g(\alpha) = n + 0 - f(\alpha) = n - f(\alpha) = s(\alpha).$$

Therefore, $g(\alpha) = s(\alpha)$ if and only if $c(\alpha) = 0$. $\square$

Remark 3.4. Lemma 3.1 and Theorem 3.3 together imply the elegant identity

$$g(\alpha) - s(\alpha) = c(\alpha).$$

This shows that the difference between the gravity and the defect is precisely the number of cyclic orbits, providing a direct combinatorial interpretation of the relationship between these two invariants.

For idempotent transformations, we have the following characterization.

Lemma 3.5. *Let $\alpha \in E$, where E is the set of idempotents in $\mathcal{T}_n$. Then $g(\alpha) = s(\alpha)$.*

Proof. Since α is an idempotent, by Theorem 2.8, α acts as the identity on its image set, and therefore the only orbits for α are fixed points. Hence $c(\alpha) = 0$ and

$$g(\alpha) = n + 0 - f(\alpha) = n - f(\alpha) = s(\alpha).$$

$\square$

The following example illustrates this result.

Example 3.6. Let $n = 3$ and consider the idempotent $\alpha \in \mathcal{T}_3$ given by

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}.$$

The functional digraph of α has $c(\alpha) = 0$ and $f(\alpha) = 2$. Thus,

$$g(\alpha) = 3 + 0 - 2 = 1 = s(\alpha).$$

Hence $g(\alpha) = s(\alpha)$ for this idempotent.

3.2. Enumeration Results. In this subsection, we provide complete enumerations of idempotent transformations satisfying $g(\alpha) = s(\alpha)$.

Theorem 3.7. *Let $\alpha \in \mathcal{T}_n$ be an idempotent. The total number of α satisfying $g(\alpha) = s(\alpha)$ is exactly*

$$N_t = \sum_{k=1}^n \binom{n}{k} k^{n-k}.$$

Proof. By Theorem 2.8, a transformation $\alpha \in \mathcal{T}_n$ is idempotent if and only if α acts as the identity mapping on its image set $\text{Im}(\alpha)$.

Let $I = \text{Im}(\alpha)$ be the image set of α , and suppose $|I| = k$ for some k with $1 \leq k \leq n$. Since every element of I is fixed, there are exactly k elements whose images are already determined. The number of ways to choose such a subset I of size k from the n elements is $\binom{n}{k}$.

For each of the remaining $n - k$ elements not in I , their images may be chosen independently to be any element of I (which has size k). Thus, there are k choices for each of these $n - k$ elements, giving k^{n-k} possible assignments.

Therefore, for a fixed image size k , the number of idempotent transformations is $\binom{n}{k} k^{n-k}$. Summing over all possible image sizes $k = 1, \dots, n$ gives the total number of idempotents in $\mathcal{T}_n$:

$$N_t = \sum_{k=1}^n \binom{n}{k} k^{n-k}.$$

For idempotents, $c(\alpha) = 0$, so by Theorem 3.3, $g(\alpha) = s(\alpha)$ for all idempotents. $\square$

We now obtain the count for singular idempotents.

Corollary 3.8. *Let $\alpha \in \text{Sing}_n$ be an idempotent. The total number of α satisfying $g(\alpha) = s(\alpha)$ is exactly*

$$N_{ef} := \sum_{k=1}^{n-1} \binom{n}{k} k^{n-k} - 1.$$

Proof. From Theorem 3.7, the total number of idempotents in $\mathcal{T}_n$ is $\sum_{k=1}^n \binom{n}{k} k^{n-k}$. This includes the identity mapping $\text{id} \in S_n$, which is the only idempotent permutation. Since $\text{Sing}_n = \mathcal{T}_n \setminus S_n$, we must exclude the identity. Therefore:

$$N_{\text{ef}} = \sum_{k=1}^{n-1} \binom{n}{k} k^{n-k} - 1.$$

Note that the subtraction of 1 removes the identity mapping, which corresponds to $k = n$ in the sum. $\square$

Example 3.9. Let $n = 4$. Then the total number of idempotents in $\mathcal{T}_4$ is:

$$N_t = \sum_{k=1}^4 \binom{4}{k} k^{4-k} = \binom{4}{1} \cdot 1^3 + \binom{4}{2} \cdot 2^2 + \binom{4}{3} \cdot 3^1 + \binom{4}{4} \cdot 4^0 = 4 + 24 + 12 + 1 = 41.$$

The number of singular idempotents is $N_{\text{ef}} = 41 - 1 = 40$.

3.3. Distribution in Group $\mathcal{H}$ -Classes. We now analyze the distribution of elements satisfying $g(\alpha) = n$ and $g(\alpha) = s(\alpha)$ within group $\mathcal{H}$ -classes of Sing_n .

Theorem 3.10. *Let α be an element of a group $\mathcal{H}$ -class of a $\mathcal{D}$ -class D_k in Sing_n . Then:*

- (1) *The number of elements in each group $\mathcal{H}$ -class satisfying $g(\alpha) = n$ is at most $k!$.*
- (2) *The number of elements in each group $\mathcal{H}$ -class satisfying $g(\alpha) = s(\alpha)$ is at most $k!$.*
- (3) *The sets of elements satisfying $g(\alpha) = n$ and $g(\alpha) = s(\alpha)$ are disjoint.*

Moreover, for any group $\mathcal{H}$ -class H in D_k , the exact counts depend on the orbit structure of the representative element of H .

Proof. Let H be a group $\mathcal{H}$ -class in the $\mathcal{D}$ -class D_k . By Theorem 2.7, every group $\mathcal{H}$ -class in D_k is isomorphic to the symmetric group S_k . For any $\alpha \in H$, we have $|\text{Im}(\alpha)| = k$, and α acts as a permutation on its image set.

For $g(\alpha) = n$, by Lemma 3.1, we require $c(\alpha) = f(\alpha)$. For $g(\alpha) = s(\alpha)$, by Theorem 3.3, we require $c(\alpha) = 0$.

Since $H \cong S_k$, there are exactly $k!$ elements in H . The elements satisfying $g(\alpha) = n$ are those where the number of cyclic orbits equals the number of fixed points. The elements satisfying $g(\alpha) = s(\alpha)$ are those with no cyclic orbits. These conditions are mutually exclusive because if

$c(\alpha) = 0$, then for $g(\alpha) = n$ we would need $f(\alpha) = 0$, but in a group $\mathcal{H}$ -class, every element has at least one fixed point. Therefore, the two sets are disjoint.

The exact number of elements satisfying each condition depends on the specific orbit structure of the representative element of H , specifically on how the k elements in the image are arranged into cycles and fixed points. In general, the counts are bounded above by $k!$ and are determined by the cycle type of elements in the corresponding symmetric group. $\square$

The following corollary provides a concrete case where the counts are known exactly.

Corollary 3.11. *For a group $\mathcal{H}$ -class H in D_k whose representative element has no cyclic orbits (i.e., acts as the identity on its image), the number of elements satisfying $g(\alpha) = s(\alpha)$ is exactly $k!$, and the number satisfying $g(\alpha) = n$ is 0.*

Proof. If the representative element has no cyclic orbits, then $c(\alpha) = 0$ for all $\alpha \in H$ because all elements in H have the same cycle structure. By Theorem 3.3, $g(\alpha) = s(\alpha)$ for all $\alpha \in H$. Since $|H| = k!$, the count is exactly $k!$. Since $c(\alpha) = 0$ for all $\alpha \in H$, the condition $c(\alpha) = f(\alpha)$ for $g(\alpha) = n$ would require $f(\alpha) = 0$, which is impossible because every element in H has at least one fixed point. Therefore, no element satisfies $g(\alpha) = n$. $\square$

Remark 3.12. Let $\alpha \in \mathcal{T}_n$ be in a group $\mathcal{H}$ -class. Then $g(\alpha) = n$ if and only if either:

- $c(\alpha) = f(\alpha) = 0$, or
- $c(\alpha) = f(\alpha) = r$ for some $r \geq 1$.

3.4. Closure Properties. We now establish important closure properties for classes of cycle-free mappings.

Theorem 3.13. *Let $S \subseteq \mathcal{T}_n$ consist of all T -standard mappings in $\mathcal{T}_n$. Then for all $\alpha, \beta \in S$, $g(\alpha\beta) = s(\alpha\beta)$.*

Proof. Let $\alpha, \beta \in S$. By Definition 2.3, both α and β have no cyclic orbits (i.e., $c(\alpha) = c(\beta) = 0$) and no fixed points (i.e., $f(\alpha) = f(\beta) = 0$). For any $x \in X$, $(\alpha\beta)(x) = \alpha(\beta(x))$. Since β has no cycles, $\beta(x)$ eventually maps to a non-periodic element under iteration. Composition of cycle-free mappings cannot create new cycles that were not already present in either α or β .

Therefore, $\alpha\beta$ has no cyclic orbits: $c(\alpha\beta) = 0$. By Theorem 3.3, this implies $g(\alpha\beta) = s(\alpha\beta)$. Hence the result. $\square$

Theorem 3.14. *Let $S \subseteq \mathcal{T}_n$ consist of all B -acyclic mappings in $\mathcal{T}_n$. The product of any two elements $\alpha, \beta \in S$ satisfies $g(\alpha\beta) = s(\alpha\beta)$.*

Proof. Let $\alpha, \beta \in S$. By Definition 2.3, both α and β have no cyclic orbits (i.e., $c(\alpha) = c(\beta) = 0$). Composition of cycle-free mappings cannot create new cycles. Therefore, $\alpha\beta$ has no cyclic orbits, and so $c(\alpha\beta) = 0$. By Theorem 3.3, $g(\alpha\beta) = s(\alpha\beta)$. $\square$

Remark 3.15. Let S be the subset of $\mathcal{T}_n$ consisting of all mappings that are B -standard. Then, for any $\alpha, \beta \in \mathcal{T}_n$ that are B -standard, the function g satisfies:

$$g(\alpha\beta) = n \quad \text{if and only if} \quad c(\alpha\beta) = f(\alpha\beta).$$

This follows directly from Lemma 3.1.

A particularly important special case is the product of idempotents.

Remark 3.16. For any $\alpha_i, \alpha_j \in \mathcal{T}_n$ where α_i and α_j are idempotents, then

$$g(\alpha_i\alpha_j) = s(\alpha_i\alpha_j).$$

We illustrate this with an example.

Example 3.17. Let $n = 3$ and consider the idempotents $\alpha_i, \alpha_j \in \mathcal{T}_3$ given by:

$$\alpha_i = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}.$$

Their product is:

$$\alpha_i\alpha_j = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}.$$

For α_i : $c(\alpha_i) = 0$, $f(\alpha_i) = 2$, so

$$g(\alpha_i) = 3 + 0 - 2 = 1 = s(\alpha_i).$$

For α_j : $c(\alpha_j) = 0$, $f(\alpha_j) = 2$, so

$$g(\alpha_j) = 3 + 0 - 2 = 1 = s(\alpha_j).$$

For $\alpha_i\alpha_j$: $c(\alpha_i\alpha_j) = 0$, $f(\alpha_i\alpha_j) = 1$, so

$$g(\alpha_i\alpha_j) = 3 + 0 - 1 = 2 = s(\alpha_i\alpha_j).$$

This confirms Remark 3.16: for idempotents, the product satisfies $g(\alpha_i\alpha_j) = s(\alpha_i\alpha_j)$.

Remark 3.18. Let $\alpha_i, \alpha_j \in \mathcal{T}_n$ be quasi-idempotents such that $g(\alpha_i) = s(\alpha_i)$ and $g(\alpha_j) = s(\alpha_j)$.

Then by Theorem 3.3, $c(\alpha_i) = c(\alpha_j) = 0$.

The product $\alpha_i\alpha_j$ may or may not have cyclic orbits:

- If $c(\alpha_i\alpha_j) = 0$, then $g(\alpha_i\alpha_j) = s(\alpha_i\alpha_j)$.
- If $c(\alpha_i\alpha_j) > 0$, then $g(\alpha_i\alpha_j) > s(\alpha_i\alpha_j)$.

The claim that $g(\alpha_i\alpha_j) = n$ is only true when $c(\alpha_i\alpha_j) = f(\alpha_i\alpha_j)$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

- [1] J.M. Howie, The subsemigroup generated by the idempotents of a full transformation semigroup, J. London Math. Soc. 41 (1966), 707–716.
- [2] J.M. Howie, Products of idempotents in finite full transformation semigroups, Proc. Roy. Soc. Edinburgh Sect. A 86 (1980), 243–254.
- [3] J.M. Howie, Fundamentals of semigroup theory, London Mathematical Society Monographs, No. 12, The Clarendon Press, Oxford University Press, 1995.
- [4] M.P. Schützenberger, A characterization of idempotents in semigroups, J. Combin. Theory 5 (1968), 370–373.
- [5] A.H. Clifford, G.B. Preston, The algebraic theory of semigroups, Vol. I, Mathematical Surveys, No. 7, American Mathematical Society, 1961.
- [6] O. Ganyushkin, V. Mazorchuk, Classical finite transformation semigroups: an introduction, Springer-Verlag, London, 2009.
- [7] G.U. Garba, B.A. Madu, A.T. Imam, Products and rank of quasi-idempotents in finite full transformation semigroups, J. Math. Sci. 2 (2011), 12–19.