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GREEN'S RELATIONS ON TERNARY SEMIGROUPS

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Abstract. Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces and the like. This provides sufficient motivation to researchers to review various concepts and results. The theory of ternary algebraic system was introduced by D. H. Lehmer [11]. He investigated certain ternary algebraic systems called triplexes which turn out to be commutative ternary groups. The notion of ternary semigroups was introduced by Banach S. He showed by an example that a ternary semigroup does not necessarily reduce to an ordinary semigroups.

In another hand, in mathematics, Green's relations characterise the elements of a semigroup in terms of the principal ideals they generate. John Mackintosh Howie, a prominent semigroup theorist, described this work as so all-pervading that, on encountering a new semigroup, almost the first question one asks is "What are the Green relations like?" (Howie 2002). The relations are useful for understanding the nature of divisibility in a semigroup.

In this paper we study Green's relations on ternary semigroup in view of those obtained in binary semigroups. Many interesting results (essentially analogous of Green's lemmas for semigroups [4] and [13]) can be derived for our case. We are also interested in the quality of idempotents with respect to the Green's relations. The particular case of ternary inverse semigroup has been studied and a relationship between the existence of idempotents and the inverse elements has been caracterized.

Keywords: ternary operation, ternary inverse semigroup,Grenn's relations, idempotent, equivalence classes..

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2. Preliminaries

Definition 0.1. *A nonempty set S is called a ternary semigroup if there exists a ternary operation;* $. : S \times S \times S \longrightarrow S$, *written as* $(a, b, c) \longmapsto a.b.c$ *satisfying the following identity for any* $a, b, c, d, e \in S$,

$$(a.b.c).d.e = a.(b.c.d).e = a.b(c.d.e).$$

Notation 0.2. *In the sequel the element $a.b.c$ will be simply denoted abc.*

Definition 0.3. *An element $1 \in S$ is called a unity if:*

$$\forall x, y \in S;\ 1xy = x1y = xy1 \text{ and } 11x = x.$$

From now, our ternary semigroups are supposed to have a unity which we always denote by 1.

Definitions 0.4. .

(1) *An element $a \in S$ is an inverse of an element $b \in S$ if $aba = a$ and $bab = b$.*
(2) *An element is then said to be regular if it has at least one inverse.*
(3) *An element $b \in S$ is a weak inverse of an element a if $aba = a$.*

Definition 0.5. *Let S be a ternary semigroup. An element a of S is said to be von Neumann regular if it has at least an weak inverse or equivalently; $a = axa$ for some $x \in S$; and S is called a von Neumann regular semigroup if every element of S is von Neumann regular.*

Definition 0.6. *Let S be a ternary semigroup. An element a of S is said to be an idempotent if $a.a.a = a$.*

Remark 0.7. *It is clear that an idempotent element is invertible and has itself as an inverse.*

Definition 0.8. *A ternary semigroup S is said to be*

(1) *commutative if $abc = acb = cba = bac\ \forall a, b, c \in S$.*
(2) *cyclicly commutative if $abc = bca = cab\ \forall a, b, c \in S$.*

Remark 0.9. *If S is commutative, then S is cyclicly commutative. The converse is false.*

Definition 0.10. *Let S be a ternary semigroup, the center of S denoted by $\mathcal{Z}(S)$ is the set defined by:*

$$\mathcal{Z}(S) = \{x \in S \ /\ xyz = yzx) \text{ for all } y, z \in S\}.$$

3. Main results

1. Green's relations on S

Let S be a ternary semigroup. By S^1 we denote the set $S \cup \{1\}$ where 1 is the identity for the ternary operation. We define five equivalence relations on S^1 which we call Green's relations by:

$$\forall a, b \in S^1;\ a\mathcal{L}^S b \iff \exists x, y, u, v \in S^1 \text{ such } a = x.y.b \text{ and } b = u.v.a.$$

$$\forall a, b \in S^1;\ a\mathcal{R}^S b \iff \exists x, y, u, v \in S^1 \text{ such } a = b.x.y \text{ and } b = a.u.v.$$

$$\forall a, b \in S^1;\ a\mathcal{I}^S b \iff \exists x, y, u, v \in S^1 \text{ such } a = x.b.y \text{ and } b = u.a.v.$$

$$\forall a, b \in S^1;\ a\mathcal{H}^S b \iff a\mathcal{L}^S b \text{ and } a\mathcal{R}^S b.$$

Now we define the relation $\mathcal{D}^S$ to be the least equivalence relation containing both $\mathcal{L}^S$ and $\mathcal{R}^S$.

For any $a \in S^1$, $L_a^S, R_a^S, H_a^S, D_a^S$ and I_a^S will denote the equivalence classes of a modulo respectively $\mathcal{L}^S, \mathcal{R}^S, \mathcal{H}^S, \mathcal{D}^S$ and $\mathcal{I}^S$. These relation will be denoted if there is no confusion on the ternary semigroup by: $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{I}$. The corresponding classes of an element $a \in S^1$, will be denoted by L_a, R_a, H_a, D_a and I_a.

Theorem 1.1. *On the ternary semigroup S one has $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.*

Proof.

Let $x\ (\mathcal{L} \circ \mathcal{R})\ y$ then there exists $z \in S$ such $x\mathcal{L}z$ and $z\mathcal{R}y$. So $x = abz, z = cdx, z = yuv$ and $y = zwt$ for some a, b, c, d, u, v, w and t in S^1.

Set $\alpha = (abz)wt$, then $x = abz = ab(yuv) = ab[(zwt)uv] = [(abz)wt]uv = \alpha uv$ and then $x \leq_{\mathcal{R}} \alpha$ (1).

In the other hand; $\alpha = (abz)wt = ab(zwt) = aby$ and then $y = zwt = (cdx)wt = [cd(abz)]wt = cd[(abz)wt] = cd\alpha$ and then $y \leq_{\mathcal{L}} \alpha$ (2).

By construction $\alpha \leq_{\mathcal{R}} x$ and $\alpha \leq_{\mathcal{L}} y$ so $x\mathcal{R}\alpha$ and $\alpha\mathcal{L}y$ and then $(x, y) \in \mathcal{R} \circ \mathcal{L}$.

□

Proposition 1.2. *For any $a \in \mathcal{Z}(S)$;*

(1) $X \in R_a \implies X \leq_{\mathcal{L}} a$.
(2) $X \leq_{\mathcal{L}} a$ *and* $a = Xuv \implies X \in R_a$.
(3) $X \in L_a \implies X \leq_{\mathcal{R}} a$.
(4) $X \leq_{\mathcal{R}} a$ *and* $a = uvX \implies X \in L_a$.

Proof.

The proof is trivial. □

Proposition 1.3. *$\mathcal{D} = \mathcal{R} \circ \mathcal{L}$ is the smallest equivalence relation that contains both $\mathcal{R}$ and $\mathcal{L}$.*

Proof.

If $x\mathcal{R}y$ we always have $y\mathcal{L}y$ so $x\mathcal{D}y$. If $x\mathcal{L}y$ we also have $x\mathcal{R}x$, so $x\mathcal{D}y$. Finally; $\mathcal{R} \subseteq \mathcal{D}$ and $\mathcal{L} \subseteq \mathcal{D}$.

$\forall x \in S^1; x\mathcal{L}x$ and $x\mathcal{R}x$ so $\mathcal{D}$ is reflexive.

The symmetry is a consequence of the previous theorem.

Let $x, y, z \in S^1$ such $x\mathcal{D}y$ and $y\mathcal{D}z$. $\exists\ a, b \in S^1$ such $x\mathcal{L}a,\ a\mathcal{R}y,\ y\mathcal{L}b,\ b\mathcal{R}z$.

For the transitivity it suffices to remark that

$$(\mathcal{L}\circ\mathcal{R})\circ(\mathcal{L}\circ\mathcal{R}) = \mathcal{L}\circ(\mathcal{R}\circ\mathcal{L})\circ\mathcal{R} = \mathcal{L}\circ(\mathcal{L}\circ\mathcal{R})\circ\mathcal{R} = (\mathcal{L}\circ\mathcal{L})\circ(\mathcal{R}\circ\mathcal{R}) = \mathcal{L}\circ\mathcal{R}.$$

Now suppose that $\mathcal{C}$ is an equivalence relation that contains both $\mathcal{L}$ and $\mathcal{R}$. If $(x, y) \in \mathcal{D}$, then there exists $u \in S^1$ such $(x, u) \in \mathcal{L}$ and $(u, y) \in \mathcal{R}$. In this case one has $(x, u) \in \mathcal{C}$ and $(u, y) \in \mathcal{C}$. So $(x, y) \in \mathcal{C}$ and $\mathcal{D} \subseteq C$. □

Proposition 1.4. *If $a \in S$ is regular then any element of D_a^S is regular.*

Proof.

Let a such $a = a\lambda a$ and let $b \in D_a^S$. Then there is $c \in S$ such

$$b = xyc, c = x'y'b, c = a\alpha\beta \text{ and } a = c\alpha'\beta'.$$

So

$$b = xyc = xy(a\alpha\beta) = xy((a\lambda a)\alpha\beta) = ((xy(c\alpha'\beta'))\lambda a)\alpha\beta = (((xyc)\alpha'\beta')\lambda a)\alpha\beta =$$

$$((b\alpha'\beta')\lambda a)\alpha\beta = b((\alpha'\beta'\lambda)a)\alpha\beta = b(\alpha'\beta'\lambda)(a\alpha\beta) = b(\alpha'\beta'\lambda)c =$$

$$b(\alpha'\beta'\lambda)(x'y'b) = b((\alpha'\beta'\lambda)x'y')b$$

and then b is regular. □

If T is a subsemigroup the Green's relations on T^1 will be denoted by $\mathcal{L}^T, \mathcal{R}^T, \mathcal{H}^T, \mathcal{D}^T$ and $\mathcal{I}^T$.

It is easy to prove that

$\mathcal{L}^T \subset \mathcal{L}^S \cap (T\times T), \mathcal{R}^T \subset \mathcal{R}^S \cap (T\times T), \mathcal{H}^T \subset \mathcal{H}^S \cap (T\times T), \mathcal{D}^T \subset \mathcal{D}^S \cap (T\times T)$ and $\mathcal{I}^T \subset \mathcal{I}^S \cap (T\times T)$.

But in general the Green's relations on a subsemigroup need not be the restrictions of Green's relations on a semigroup. In this sense we have the following facts:

Proposition 1.5. *If $aba = a$ then $ab1\mathcal{D}1ba$.*

Proof.

We have $a = aba = (aba)11 = (ab1)1a \implies a \leq_{\mathcal{R}} ab1$.

$ab1 = a.b1 \implies ab1 \leq_{\mathcal{R}} a$. So $ab1\mathcal{R}a$. (I)

$a = aba = a(ba1)1 = a1(1ba) \implies a \leq_{\mathcal{R}} 1ba$.

$1ba = 1b.a$ so $1ba \leq_{\mathcal{L}} a$, and then $1ba\mathcal{L}a$. (II)

(I) and (II) imply that $ab1\mathcal{D}ab1$. □

Theorem 1.6. *let a be an element of a ternary semigroup S. One has the following implications:*

(1) *1) a is Von Neumann regular $\iff$ 2) $aba = a$ for some $b \in S$ $\implies$ 3) $1ab$ and $1ba$ are idempotents.*

(2) *2) $aba = a$ for some $b \in S$ $\implies$ 5) L_a contains an idempotent and 4) R_a contains an idempotent.*

Proof.

(1) 1) $\iff$ 2) is by definition.

2) $\implies$ 3) If $aba = a$ for some $b \in S$,then $(1ab)(1ab)(1ab) = 1[ab(1ab)](1ab) = 1[(ab1)ab](1ab) = 1[(1ab)ab](1ab) = 1[1(aba)b](1ab) = 1[1ab](1ab) = 11(ab(1ab)) = ab(1ab) = (ab1)ab = (1ab)ab = 1(aba)b = 1ab$. $1ab$ is an idempotent,

If $aba = a$ for some $b \in S$,then $(1ba)(1ba)(1ba) = 1[ba(1ba)](1ba) = 1[(ba1)ba](1ba) = 1[(1ba)ba](1ba) = 1[1(bab)a](1ba) = 1[1ba](1ba) = 11(ba(1ba)) = ba(1ba) = (ba1)ba = (1ba)ba = 1(bab)a = 1ba$. $1ab$ is an idempotent,

(2) $ab1 \in R_a$ and by the previous proposition $ab1$ is an idempotent.

(3) $1ba \in L_a$ and by the previous proposition $1ba$ is an idempotent.

□

Definition 1.7. *Let S be a ternary semigroup. We define on S the following preorder relations:*

$$a \leq_{\mathcal{L}} b \iff a = xyb \text{ for some } x, y \in S.$$

$$a \leq_{\mathcal{R}} b \iff a = bxy \text{ for some } x, y \in S.$$

$$a \leq_{\mathcal{I}} b \iff a = xby \text{ for some } x, y \in S.$$

$$a \leq_{\mathcal{H}} b \iff a \leq_{\mathcal{L}} b \text{ and } a \leq_{\mathcal{R}} b.$$

Proposition 1.8. *Let S be a ternary semigroup.*

(1) *Let $a \in S$ be an idempotent and b be an element of S. Then*

$$b \leq_{\mathcal{R}} a \iff b = aab.$$

$$b \leq_{\mathcal{L}} a \iff b = baa.$$

(2) *If $a \leq_{\mathcal{R}} axy$, then $a\mathcal{R}axy$.*

(3) *If $a \leq_{\mathcal{L}} xya$, then $a\mathcal{L}xya$.*

Proof.

(1) $b \leq_{\mathcal{R}} a \implies b = axy$ for some $x, y \in S^1$. It follows that $aab = aa(axy) = (aaa)xy = axy = b$.

Conversely $b = aab \iff b \leq_{\mathcal{R}} a$.

The same proof can be make to obtain the other equivalence.

(2) In one hand we have $a \leq_{\mathcal{R}} axy$ and in the other the expression $axy = a.xy \implies axy \leq_{\mathcal{R}} a$. So $a\mathcal{R}axy$.

(3) With the same arguments as in 2), the expression $xya = xy.a$ implies $xya \leq_{\mathcal{L}} a$ but $a \leq_{\mathcal{L}} xya$ so $a\mathcal{L}xya$.

$\square$

Definition 1.9. *Let T be a subset of a ternary semigroup. We say that T is left (resp. right, twosided) (S,S)-stable or ideal if*

$$\forall x, y \in S,\ \forall a \in T;\ x.y.a \in T\ (resp.\ a.x.y \in T, a \in T;\ x.y.a \in T\ and\ a.x.y \in T).$$

Theorem 1.10. *Let T be a Von Neumann regular subset of S.*

(1) *If T is left (S,S)-stable, then $\mathcal{L}^T = \mathcal{L}^S \cap (T \times T)$.*
(2) *If T is right (S,S)-stable, then $\mathcal{R}^T = \mathcal{R}^S \cap (T \times T)$.*
(3) *If T is twosided (S,S)-stable, then $\mathcal{H}^T = \mathcal{H}^S \cap (T \times T)$*

Proof.

Let $(a,b) \in \mathcal{L}^S \cap (T \times T)$ then $a, b \in T$ and $\exists x, y, u, v \in S$ such $a = x.y.b$ and $b = u.u.a$. T is Von Neumann regular so $\exists \alpha, \beta \in T$ such $a = a.\alpha.a$ and $b = b.\beta.b$; By replacing a and b by their values we get $a = x.y.(b.\beta.b)$ and $b = u.v.(a.\alpha.a)$. and then $a = (x.y.b).\beta.b$ and $b = (u.v.a).\alpha.a$. T is left (S,S)-stable then $c = (x.y.b) \in T$ and $d = (u.v.a) \in T$. Finally there exists $c, \beta, d, \alpha \in T$ such $a = c.\beta.b$ and $b = d.\alpha.a$ and this means that $(a,b) \in \mathcal{L}^T$.

We obtain the second assertion by the using similar arguments. The third assertions is a consequence of the first and the second assertions. $\square$

Green's relations on ternary semigroups are stable under morphisms.

Proposition 1.11. *Let $\varphi : S \longrightarrow T$ be a morphism and $\mathcal{R}$ be one of the relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}, \mathcal{I}$. If $a\ \mathcal{R}^S b$ then $\varphi(a)\ \mathcal{R}^T \varphi(b)$.*

Proposition 1.12. *If a, b are two idempotent elements, the following conditions are equivalent:*

(1) $a \leq_{\mathcal{H}} b$,
(2) $aab = a = baa$,
(3) $bab = a$.

Proof.

(1) $(1 \implies 2)$

$a \leq_{\mathcal{H}} b \iff a \leq_{\mathcal{R}} b$ and $a \leq_{\mathcal{L}} b \iff a = bxy$ and $a = uvb$ for some x, y, u, v in S. So $bba = bb(bxy) = (bbb)xy = bxy = a$ and $abb = (uvb)bb = uv(bbb) = uvb = a$.

(2) $(2 \implies 3)$

$aab = a \implies ba(aab) = baa \iff b(aaa)b = baa \iff bab = baa = a.$

If we use the other equality we get the same equality; that is:

$baa = a \implies (baa)ab = aab \iff b(aaa)b = aab \iff bab = aab = a.$

(3) $(2 \implies 1)$

Trivial.

□

Proposition 1.13. *If α is an inverse of a, then*

(1) $a = (a\alpha a)(\alpha a\alpha)(a\alpha a)$,

(2) $\alpha = (\alpha a\alpha)(a\alpha a)(\alpha a\alpha)$.

And then $(a\alpha a)$ is an inverse of $(\alpha a\alpha)$.

Proof.

(1) $(a\alpha a)(\alpha a\alpha)(a\alpha a) = a(\alpha a\alpha)[(a\alpha a)\alpha a] = a\alpha(a\alpha a) = a\alpha a = a$,

(2) $(\alpha a\alpha)(a\alpha a)(\alpha a\alpha) = \alpha[(a\alpha a)(\alpha a\alpha)a]\alpha = \alpha(a\alpha a)\alpha = \alpha a\alpha = \alpha$,

□

Proposition 1.14. *If $E(S)$ denotes the set of all idempotent elements of S the the restriction of the preorder $\leq_{\mathcal{H}}$ to $E(S)$ is an order, called the naturel order on $E(S)$ and denoted $\leq$.*

Proof.

The symmetry results from the previous proposition. □

Definition 1.15. *Let S be a ternary semigroup and T be a subsemigroup of S. T is called a $\mathcal{G}-$subsemigroup if*

$$\mathcal{L}^T = \mathcal{L}^S \cap (T \times T), \mathcal{R}^T = \mathcal{R}^S \cap (T \times T), \mathcal{H}^T = \mathcal{H}^S \cap (T \times T),$$

$$\mathcal{D}^T = \mathcal{D}^S \cap (T \times T), \mathcal{I}^T = \mathcal{I}^S \cap (T \times T).$$

Proposition 1.16 (Grenn's lemma)**.** *Let a, b be two $\mathcal{R}-$equivalent elements of a ternary semigroup S. If $a = buv$ and $b = acd$ for some $u, v, c, d \in S^1$, then the map $\varphi : x \longmapsto xuv$ is a bijection from $L(b)$ onto $L(a)$ and the map $\psi : x \longmapsto xcd$ is a bijection from $L(a)$ onto $L(b)$. Further, these bijections are inverse each other and are such, for $\alpha, \beta \in S$:*

$$\alpha\mathcal{L}\beta \iff \varphi(\alpha)\mathcal{L}\varphi(\beta) \quad \textit{and} \quad \alpha\mathcal{L}\beta \iff \psi(\alpha)\mathcal{L}\psi(\beta)$$

.

Lemma 1.17. *The relation $\mathcal{L}$ is right $S.S-$stable and the relation $\mathcal{R}$ is left $S.S-$stable.*

Proof.

The proof is trivial. □

Proof. (Grenn's lemma.)

Let $n \in L(a)$. Since $\mathcal{L}$ is right $S.S-$stable then $ncd \in L(acd)$. But $n = xya$ so $(ncd)uv = [(xya)cd]uv = [xy(acd)]uv = [xyb]uv = xy(buv) = xya = n$. In the other hand if $m \in L(b)$ with the same argument we can prove that $(muv)cd = m$ so the maps $x \longmapsto xuv$ and $x \longmapsto xcd$ are inverse of each other then they are bijections between the mentioned sets.

Let $\alpha\mathcal{L}\beta$ then $\alpha = xy\beta$ and $\beta = x'y'\alpha$. $\varphi(\alpha) = \alpha uv = (xy\beta)uv = xy(\beta uv) = xy\varphi(\beta)$ and $\psi(\alpha) = \alpha uv = (xy\beta)uv = xy(\beta uv) = xy\psi(\beta)$; so

$$\alpha\mathcal{L}\beta \implies \varphi(\alpha)\mathcal{L}\varphi(\beta) \text{ and } \psi(\alpha)\mathcal{L}\psi(\beta)$$

. Conversely; suppose that $\varphi(\alpha)\mathcal{L}\varphi(\beta)$, then $\psi(\varphi(\alpha))\mathcal{L}\psi(\varphi(\beta))$ and so $\alpha\mathcal{L}\beta$. For the other implication we use the same argument. □

The next dual version of the proposition is proved similarly.

Proposition 1.18 (Green's lemma)**.** *Let a, b be two $\mathcal{L}-$equivalent elements of a ternary semigroup S. If $a = uvb$ and $b = cda$ for some $u, v, c, d \in S^1$, then the map $\varphi : x \longmapsto uvx$ is a bijection from $R(b)$ onto $R(a)$ and the map $\psi : x \longmapsto cdx$ is a bijection from $R(a)$ onto $R(b)$. Further, these bijections preserve the $\mathcal{R}-$classes and are inverse each other, that is, for $\alpha, \beta \in S$:*

$$\alpha\mathcal{R}\beta \iff \varphi(\alpha)\mathcal{R}\varphi(\beta) \text{ and } \alpha\mathcal{R}\beta \iff \psi(\alpha)\mathcal{R}\psi(\beta).$$

Proof.

The proof is exactly the same as in proposition 2.8, which we adapt to the right classes. □

Corollary 1.19. *If a, b are $\mathcal{H}-$equivalent then, $\forall\alpha, \beta \in S$:*

$$\alpha\mathcal{H}\beta \iff \varphi(\alpha)\mathcal{H}\varphi(\beta) \text{ and } \alpha\mathcal{H}\beta \iff \psi(\alpha)\mathcal{H}\psi(\beta).$$

Proposition 1.20. *Let $x, y \in S$. If $R(y) \cap L(x)$ contains an idempotent e then $xey \in R(x) \cap L(y)$.*

Proof.

If $e \in R(y) \cap L(x)$ then $eey = y$ and $xee = x$ (Hint:$e \in R(y) \implies y = eab \implies eey = ee(eab) = (eee)ab = eab = y$).

$e\mathcal{R}y \implies xee\mathcal{R}xey$ and the $x\mathcal{R}xey$.

$e\mathcal{L}x \implies eey\mathcal{L}xey$ and then $y\mathcal{L}xey$.

Finally $xey \in L(y) \cap R(x)$. □

Proposition 1.21. *Let $e, f \in E(S)$ the set of all idempotents of S. For all $x \in R(e) \cap L(f)$ there exists $y \in R(f) \cap L(e)$ such $xfy = e$ and $yex = f$.*

Proof.

If $x \in R(e) \cap L(f)$ then $x = eex$ and $x = xff$. There also are $u, v, a, b \in S^1$ such $e = xuv$ and $f = abx$. Let $y = fuv$ then:

$$f = abx = ab(eex) = ab((xuv)ex) = ((abx)uv)ex = (fuv)ex = yex.$$

and

$$e = xuv = (xff)uv = xf(fuv) = xfy.$$

In the other hand:

$y = fuv$ and $y = yex$ imply that $y \in R(f)$.

$e = xfy$ and $y = fuv = (abx)uv = ab(xuv) = abe$ imply $y \in L(e)$, and so $y \in R(f) \cap L(e)$. □

Corollary 1.22. *Let e be an idempotent. For all $x \in H(e)$ there exists $y \in H(e)$ such $xey = e = yex$. So xef and yex are in E_S.*

Proof.

Take $e = f$ in the previous proposition. □

2. Inverse semigroups

Definition 2.1. *A ternary semigroup S is called a ternary inverse semigroup, if each element $x \in S$ has a unique inverse element denoted x^{-1}; that is*

$$xx^{-1}x = x \text{ and } x^{-1}xx^{-1} = x^{-1}x^{-1}$$

Proposition 2.2. *Let S be a ternary inverse semigroup and e, f, g be in $E(S)$. If $x = (efg)^{-1}$, then*

$$(xee)(efg)(xee) = xee,$$

$$(ggx)(efg)(ggx) = ggx.$$

And finally; $x = xee = ggx$

Proof.

$efg = (efg)x(efg) = ef[gx(efg)] = ef[(ggg)x(efg)] = (efg)(ggx)(efg)$

$efg = (efg)x(efg) = ef[(gxe)fg] = ef[(gx(eee))fg] = ef[(g(xee)e)fg] = (efg)(xee)(efg)$.

But $(xee)(efg)(xee) = x((eee)fg)(xee) = x(efg)(xee) = (x(efg)x)ee = xee$ and

$(ggx)(efg)(ggx) = gg[x(efg)(ggx)] = gg[(xef)gx] = gg[x(efg)x] = ggx$. So xee and ggx are inverses of egf and by the unicity of this inverse, we conclude that

$$(efg)^{-1} = xee = ggx.$$

□

Definition 2.3. *A nonempty set G endowed with a ternary operation is a ternary group if:*

(1) $\forall x, y, z, a, b \in G;\ (xyz)ab = x(yza)b = xy(zab)$*;*
(2) *there exists an element* $1 \in S$ *such* $11x = 1x1 = x11 = x,\ \forall x \in S$*;*
(3) $\forall x \in S,\ \exists! x' \in S$ *such* $xx'x = x$ *and* $x'xx' = x'$*.*

Proposition 2.4. *Let S be a ternary inverse semigroup with unity. If $E(s)$ is commutative then $E(S)$ is a ternary group.*

Proof.

If $E(S)$ is commutative then it is stable under the ternary operation, that is if e, f, g are idempotents then:$(efg)(efg)(efg) = (eee)(fff)(ggg) = efg$.

1 is an idempotent.

Every idempotent is invertible and its unique inverse is then itself. □

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