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STRUCTURAL PROPERTIES OF THE POWER GRAPH OF THE FULL TRANSFORMATION SEMIGROUP

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Abstract. We study the power graph $\mathcal{P}(\mathcal{T}_n)$ of the full transformation semigroup $\mathcal{T}_n$ using functional digraphs. We characterize idempotent and nilpotent transformations and examine the structural roles of idempotent, quasi-idempotent, nilpotent, and permutation elements. We prove that distinct idempotents are non-adjacent, proper quasi-idempotents are adjacent to idempotents, and the nilpotent subgraph forms a forest rooted at constant transformations. For $n = 3$, we obtain a complete computational description of $\mathcal{P}(\mathcal{T}_3)$, consisting of ten connected components and 18 edges on 27 vertices. The results are verified using GAP, providing a basis for further investigation of power graphs of full transformation semigroups.

Keywords: idempotents; quasi-idempotent; nilpotents; power graph

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1. INTRODUCTION

Graph-theoretic techniques have become an effective tool for studying algebraic structures, especially semigroups and their variants. Among these, power-type graphs have attracted significant attention because they translate algebraic relations into adjacency relations, thereby revealing structural properties from a combinatorial viewpoint [1, 4, 11, 14]. In transformation semigroups, functional digraphs provide a natural model for describing the action of mappings

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under iteration, making them particularly useful for studying orbit structure, cycles, fixed points, and special elements such as idempotents and nilpotents [7, 10, 13]. This connection motivates the study of how algebraic classes of transformations shape associated graphs.

The undirected power graph of a semigroup has the semigroup as its vertex set, where two distinct elements are adjacent whenever one is a positive power of the other [5, 17]. Power graphs have been widely investigated for groups and semigroups, together with related variants such as reduced and enhanced power graphs. These studies show that power graphs capture important algebraic information, including connectivity, completeness, and other structural invariants [1, 5, 8, 16].

The full transformation semigroup T_n on a finite set plays a central role in semigroup theory, since every finite semigroup embeds into some transformation semigroup. Its internal structure has been studied from several perspectives, including idempotent generation, nilpotent subsemigroups, commutative substructures, and graph representations of transformations [7, 9, 12, 15]. These properties make T_n a natural setting for exploring interactions between algebra and graph theory.

Recent work has also produced characterizations of idempotent, quasi-idempotent, and nilpotent elements in transformation semigroups through algebraic conditions and functional digraph descriptions [2, 3, 9, 13]. Such results suggest that these distinguished elements should play an important role in the structure of the power graph. Motivated by this, the present paper investigates how these classes are positioned in $\mathcal{P}(T_n)$ and how they determine adjacency, connectivity, and component structure.

The power graphs of semigroups and, in particular, of transformation semigroups have been studied in several prior works [1, 5, 12]. The functional digraph representation of $\mathcal{T}_n$ is classical [10], and general facts about eventually periodic sequences in finite semigroups are standard [7]. The present paper makes the following contributions that go beyond the existing literature. First, we give *explicit algebraic characterizations* of idempotent, quasi-idempotent, and nilpotent elements of $\mathcal{T}_n$ in terms of their functional digraphs and use these to derive *global structural properties* of the power graph (independence of the idempotent set, adjacency of proper quasi-idempotents to idempotents, tree-like structure of nilpotent components). Second, while the

functional digraph of a single transformation is classical, the *power graph* $\mathcal{P}(\mathcal{T}_n)$ is a different object whose vertices are the transformations themselves and whose edges encode the power relation among them; its structural analysis requires different techniques. Third, we provide a complete, computationally verified description of $\mathcal{P}(\mathcal{T}_3)$, including the exact component structure (ten components), edge count (18 undirected edges), degree sequence, and diameters, going substantially beyond what is available in [12] for the case $n = 3$.

2. PRELIMINARIES

This section collects the notation, definitions, and auxiliary facts used throughout the paper in the investigation of the power graph of the full transformation semigroup. Background material may be found in standard texts on semigroup theory and graph theory.

2.1. Notation and Preliminaries. Let $X = \{1, 2, \dots, n\}$, where $n \geq 1$ is fixed and finite. Every map considered in this article is a self-map of X .

Definition 2.1. *The full transformation semigroup on X , written $\mathcal{T}_n$, is the collection of all mappings $\alpha : X \rightarrow X$, equipped with composition as the binary operation.*

Remark 2.2. *Since each of the n elements of X may be sent independently to any point of X , one has $|\mathcal{T}_n| = n^n$. For $n \geq 2$, this semigroup is finite and generally non-commutative.*

Definition 2.3. *For any $\alpha \in \mathcal{T}_n$, define:*

- *the image set $\text{Im}(\alpha) = \{\alpha(x) : x \in X\}$;*
- *the kernel relation, given by $x \equiv y \iff \alpha(x) = \alpha(y)$;*
- *the rank of α , $\text{rank}(\alpha) = |\text{Im}(\alpha)|$.*

Definition 2.4. *Let $\alpha \in \mathcal{T}_n$. Then α is called:*

- *idempotent whenever $\alpha^2 = \alpha$;*
- *quasi-idempotent if $\alpha^k = \alpha$ for some integer $k \geq 2$; note that every idempotent satisfies this condition (with $k = 2$), so we call α a proper quasi-idempotent if $\alpha^k = \alpha$ for some $k \geq 2$ but $\alpha^2 \neq \alpha$ (i.e. α is quasi-idempotent but not itself idempotent);*
- *nilpotent if α^k becomes a constant map for some $k \in \mathbb{N}$;*

- a constant transformation if there exists $c \in X$ such that $\alpha(x) = c$ for all $x \in X$.

Remark 2.5. Every constant transformation satisfies $\alpha^2 = \alpha$, and hence is idempotent. Moreover, a transformation in $\mathcal{T}_n$ is idempotent exactly when each point in its image is fixed by it.

Definition 2.6. For $\alpha \in \mathcal{T}_n$, the functional digraph associated with α is the directed graph $D(\alpha)$ whose vertex set is X and whose arcs are $\{(x, \alpha(x)) : x \in X\}$.

Theorem 2.7. Each connected component of $D(\alpha)$ contains precisely one directed cycle. Every remaining vertex of that component lies on a directed tree feeding into the cycle.

Proof. Starting from any vertex and repeatedly applying α , one obtains a sequence in the finite set X . Some value must recur, producing a directed cycle. Since each vertex has exactly one outgoing edge, no component can contain more than one cycle, and all non-cyclic vertices must eventually enter that cycle. $\square$

Definition 2.8. For a vertex $x \in X$, the cycle length of x is the size of the cycle reached by successive iteration of α starting from x .

Definition 2.9. For $\alpha \in \mathcal{T}_n$ and $k \in \mathbb{N}$, define the k -th power recursively by repeated composition: $\alpha^k = \alpha \circ \alpha \circ \dots \circ \alpha$ (k factors).

Lemma 2.10. For any $\alpha \in \mathcal{T}_n$, the sequence $\alpha, \alpha^2, \alpha^3, \dots$ is eventually periodic.

Proof. Because $\mathcal{T}_n$ is finite, infinitely many powers of α cannot all be distinct. Hence $\alpha^r = \alpha^s$ for some integers $r < s$, from which periodic repetition follows. $\square$

Definition 2.11. An element $\alpha \in \mathcal{T}_n$ is said to be eventually idempotent if α^k is idempotent for some positive integer k .

Definition 2.12. A graph $G = (V, E)$ consists of a vertex set V together with an edge set $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$.

Definition 2.13. A graph is said to be:

- simple if loops and multiple edges are absent;
- connected if every two vertices can be joined by a path;
- complete if every distinct pair of vertices is adjacent.

Definition 2.14. A path is a finite sequence of vertices in which consecutive terms are joined by edges.

Definition 2.15. The distance between two vertices is the minimum length among all paths connecting them.

Definition 2.16. Let S be a semigroup. The power graph $\mathcal{P}(S)$ is the simple graph whose vertex set is S , where two distinct vertices $a, b \in S$ are adjacent whenever $a = b^k$ or $b = a^m$ for some $k, m \in \mathbb{N}$.

Remark 2.17. Thus, the power graph records how elements of a semigroup are linked through positive powers.

Definition 2.18. If $a = b^k$ for some $k \geq 1$, then a is called a power descendant of b .

Lemma 2.19. Suppose $a, b, c \in S$ satisfy $a = b^k$ and $b = c^m$ for positive integers k, m . Then

$$a = c^{mk}.$$

Proof. Indeed,

$$a = b^k = (c^m)^k = c^{mk}.$$

□

3. MAIN RESULTS

The power graph $\mathcal{P}(\mathcal{T}_n)$ of the full transformation semigroup $\mathcal{T}_n$ encodes the power relations among its elements and serves as a fundamental object for understanding the interplay between algebraic and combinatorial structures. As $\mathcal{T}_n$ contains a rich variety of elements, including permutations, idempotents, and nilpotents, its power graph exhibits diverse structural features that reflect these distinctions. In particular, properties such as adjacency, connectivity, and component structure are closely tied to the behavior of iterated transformations and their

associated functional digraphs. This subsection is devoted to establishing the basic structural properties of $\mathcal{P}(\mathcal{T}_n)$, which will serve as a foundation for the more detailed analysis presented in subsequent sections.

Two related objects appear in this paper. The *directed power graph* $\vec{\mathcal{P}}(\mathcal{T}_n)$ has vertex set $\mathcal{T}_n$ with a directed arc $\alpha \rightarrow \beta$ whenever $\beta = \alpha^k$ for some $k \geq 2$ and $\beta \neq \alpha$; this is the structure produced computationally by GAP's `DigraphEdges`. The *undirected power graph* $\mathcal{P}(\mathcal{T}_n)$ is the underlying undirected graph: two distinct vertices α, β are adjacent if and only if $\alpha = \beta^k$ or $\beta = \alpha^m$ for some $k, m \in \mathbb{N}$. A pair of directed arcs $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$ collapses to a single undirected edge in $\mathcal{P}(\mathcal{T}_n)$. Unless stated otherwise, all graph-theoretic statements (adjacency, connectivity, components, degree, diameter) in this paper refer to the *undirected* power graph $\mathcal{P}(\mathcal{T}_n)$. The computational edge table (Table 1) lists directed arcs; their undirected counterpart has one fewer edge due to the reciprocal arc between the two 3-cycles.

Idempotent Elements in $\mathcal{P}(\mathcal{T}_n)$. Idempotent elements play a central role in the structure of the full transformation semigroup $\mathcal{T}_n$, and their influence extends naturally to the structure of its power graph $\mathcal{P}(\mathcal{T}_n)$. Since idempotents are precisely the elements that stabilize under iteration, they induce distinctive adjacency patterns and often serve as structural anchors within the graph. Their associated functional digraphs are characterized by fixed points and tree-like attachments, which significantly affect the local and global connectivity properties of $\mathcal{P}(\mathcal{T}_n)$. This subsection focuses on describing how idempotent elements are represented within the power graph and how their algebraic properties shape its structural features.

Proposition 3.1. *Every connected component of $\mathcal{P}(\mathcal{T}_n)$ contains an idempotent element.*

Proof. This follows directly from the eventual periodicity of powers in a finite semigroup. $\square$

Lemma 3.2. *Let $\alpha \in \mathcal{T}_n$. Then α is idempotent if and only if $\text{Im}(\alpha) \subseteq \text{Fix}(\alpha)$.*

Proof. $(\Rightarrow)$ Suppose $\alpha^2 = \alpha$. Let $y \in \text{Im}(\alpha)$. Then there exists $x \in X$ such that $y = \alpha(x)$. Hence

$$\alpha(y) = \alpha(\alpha(x)) = \alpha^2(x) = \alpha(x) = y,$$

so $y \in \text{Fix}(\alpha)$.

($\Leftarrow$) Conversely, assume $\text{Im}(\alpha) \subseteq \text{Fix}(\alpha)$. Then for any $x \in X$, $\alpha(x) \in \text{Im}(\alpha)$, and hence

$$\alpha^2(x) = \alpha(\alpha(x)) = \alpha(x).$$

Thus $\alpha^2 = \alpha$, and α is idempotent. $\square$

Proposition 3.3. *Let $\alpha \in \mathcal{T}_n$ be idempotent. Then for any $\beta \in \mathcal{T}_n$, if α and β are adjacent in $\mathcal{P}(\mathcal{T}_n)$, then $\alpha = \beta^k$ for some $k \geq 1$.*

Proof. Suppose α and β are adjacent. Then either $\alpha = \beta^k$ or $\beta = \alpha^m$ for some positive integers k, m .

If $\beta = \alpha^m$, then since α is idempotent, we have $\alpha^m = \alpha$ for all $m \geq 1$. Hence $\beta = \alpha$, contradicting the assumption that vertices are distinct. Therefore, the only possibility is $\alpha = \beta^k$. $\square$

Theorem 3.4. *The set of idempotents of $\mathcal{T}_n$ induces an edgeless subgraph in $\mathcal{P}(\mathcal{T}_n)$.*

Proof. Let $\alpha, \beta \in \mathcal{T}_n$ be distinct idempotents. Suppose, for contradiction, that they are adjacent. Then either $\alpha = \beta^k$ or $\beta = \alpha^m$ for some $k, m \geq 1$.

Since β is idempotent, $\beta^k = \beta$ for all $k \geq 1$. Thus $\alpha = \beta$. Similarly, $\beta = \alpha$ if $\beta = \alpha^m$. In either case, we obtain $\alpha = \beta$, contradicting the assumption that they are distinct.

Hence no two distinct idempotents are adjacent, and the induced subgraph is edgeless. $\square$

Corollary 3.5. *Every idempotent is a minimal vertex (with respect to the power relation) in its connected component of $\mathcal{P}(\mathcal{T}_n)$.*

Proof. Let α be idempotent. If β is adjacent to α , then by the proposition above, $\alpha = \beta^k$ for some $k \geq 1$. Thus α cannot be expressed as a proper power of any distinct element smaller than it in the power relation, establishing minimality. $\square$

Example 3.6 (Idempotents in $\mathcal{T}_3$ and the power graph $\mathcal{P}(\mathcal{T}_3)$). *Let $X = \{1, 2, 3\}$. We work inside the full transformation semigroup $\mathcal{T}_3$, whose power graph we denote $\mathcal{P}(\mathcal{T}_3)$.*

Step 1: Identifying the idempotents. *By the Lemma, $\alpha \in \mathcal{T}_3$ is idempotent if and only if $\text{Im}(\alpha) \subseteq \text{Fix}(\alpha)$. The idempotents of $\mathcal{T}_3$ are exactly the retractions onto their image; they are*

in bijection with pairs $(A, \text{partition of } X \text{ with transversal } A)$. A concrete list of representatives at each image-size is:

- **Image size 3** (the identity):

$$\varepsilon = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \text{Fix}(\varepsilon) = \{1, 2, 3\}, \quad \text{Im}(\varepsilon) = \{1, 2, 3\}.$$

- **Image size 2** (six idempotents, one for each 2-element subset and each choice of “glued” element):

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}, \quad \text{Im}(\alpha) = \{1, 2\} \subseteq \text{Fix}(\alpha) = \{1, 2\}$$

We work with α as our primary example below; the analysis is identical for the other five.

- **Image size 1** (three constant maps):

$$c_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}, \quad c_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}, \quad c_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}.$$

For c_j : $\text{Im}(c_j) = \{j\}$ and $c_j(j) = j$, so $\{j\} \subseteq \text{Fix}(c_j)$.

Step 2: Consider the element

$$\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix}.$$

We check adjacency with α :

$$\beta^2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \end{pmatrix} = \beta,$$

so β is itself idempotent; by the Theorem they are not adjacent (see Step 3).

Now take

$$\gamma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \end{pmatrix}.$$

Then

$$\gamma^2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} = c_1, \quad \gamma^3 = c_1.$$

Since c_1 is idempotent, γ and c_1 are adjacent in $\mathcal{P}(\mathcal{T}_3)$ (with $c_1 = \gamma^2$). The Proposition confirms that c_1 , being idempotent, can only be adjacent to γ via $c_1 = \gamma^k$ (here $k = 2$), never via $\gamma = c_1^m$ (since $c_1^m = c_1 \neq \gamma$).

Step 3: The induced subgraph on idempotents is edgeless

Take any two distinct idempotents, say α and c_1 . Suppose they were adjacent. Then either $\alpha = c_1^k = c_1$ or $c_1 = \alpha^m = \alpha$, both contradictions. Hence $\alpha \not\sim c_1$. The same argument applies to every pair of distinct idempotents; the 10 idempotents of $\mathcal{T}_3$ form an *independent set* in $\mathcal{P}(\mathcal{T}_3)$.

Step 4: Minimality of idempotents.

In the component containing c_1 , every vertex δ adjacent to c_1 satisfies $c_1 = \delta^k$ for some $k \geq 1$; for instance $c_1 = \gamma^2$. There is no element $\mu \neq c_1$ with $c_1 = \mu^0$ (trivially) and c_1 is not a proper power of itself via a *distinct* element, confirming c_1 's minimality in the power relation within its component.

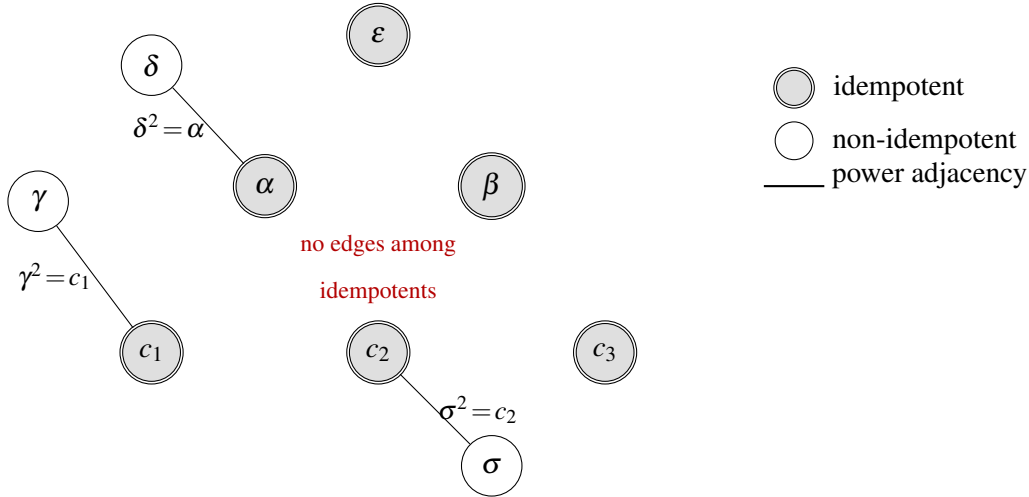


FIGURE 1. A portion of $\mathcal{P}(\mathcal{T}_3)$ illustrating the idempotent structure. Shaded double-ringed vertices are idempotents ($c_1, c_2, c_3, \alpha, \beta, \epsilon$). Edges connect elements that are related by a power; labels indicate which power witnesses the adjacency. Consistent with the Theorem, *no edge joins two idempotents*: the idempotents form an independent set. Consistent with the Corollary, each idempotent is minimal in the power relation within its component: every neighbour δ satisfies (idempotent) $= \delta^k$ for some $k \geq 1$.

Remark 3.7. Figure 1 displays only the vertices and edges most relevant to the idempotent results; the full graph $\mathcal{P}(\mathcal{T}_3)$ has 27 vertices. The identity ε is the unique idempotent of rank 3 and is adjacent to every permutation of $\mathcal{T}_3$ (since every permutation π satisfies $\pi^{|\pi|} = \varepsilon$), but is not adjacent to any other idempotent, in agreement with the Theorem.

Quasi-Idempotent Elements. Quasi-idempotent elements form a natural generalization of idempotents within the full transformation semigroup $\mathcal{T}_n$, capturing elements whose iterates stabilize after finitely many steps. Their behavior is closely linked to the structure of functional digraphs, where repeated composition leads to eventual stabilization of orbits and the emergence of constrained cycle–path configurations. In the power graph $\mathcal{P}(\mathcal{T}_n)$, quasi-idempotent elements exhibit characteristic adjacency patterns that reflect their eventual periodicity, distinguishing them from both idempotent and general transformations. This subsection examines the role of quasi-idempotent elements in shaping the structural and connectivity properties of $\mathcal{P}(\mathcal{T}_n)$.

Lemma 3.8. *Let $\alpha \in \mathcal{T}_n$. If $\alpha^k = \alpha$ for some $k \geq 2$, then α^{k-1} is idempotent.*

Proof. Let $\beta = \alpha^{k-1}$. Then

$$\beta^2 = \alpha^{2k-2} = \alpha^{k-1} \cdot \alpha^{k-1}.$$

Since $\alpha^k = \alpha$, we have $\alpha^{k+r} = \alpha^{r+1}$ for all $r \geq 0$. Taking $r = k - 2$, we obtain

$$\alpha^{2k-2} = \alpha^{k-1}.$$

Thus $\beta^2 = \beta$, so β is idempotent. □

Proposition 3.9. *Every proper quasi-idempotent element is adjacent to an idempotent element in $\mathcal{P}(\mathcal{T}_n)$.*

Proof. Let α be a proper quasi-idempotent, so that $\alpha^k = \alpha$ for some $k \geq 2$ with $\alpha^2 \neq \alpha$. By the previous lemma, α^{k-1} is idempotent. Since α^{k-1} is a positive power of α (with exponent $k - 1 \geq 1$), it follows directly from the definition of the power graph that α and α^{k-1} are adjacent in $\mathcal{P}(\mathcal{T}_n)$. □

Theorem 3.10. *Every proper quasi-idempotent lies in a connected component of $\mathcal{P}(\mathcal{T}_n)$ containing an idempotent.*

Proof. Let α be a proper quasi-idempotent. Then there exists $k \geq 2$ such that $\alpha^k = \alpha$ and $\alpha^2 \neq \alpha$. By the lemma above, α^{k-1} is idempotent, and α^{k-1} is a positive power of α , hence α is adjacent to α^{k-1} . Thus α lies in the same connected component as an idempotent. $\square$

Example 3.11. *Let $X = \{1, 2, 3, 4\}$ and consider the full transformation semigroup $\mathcal{T}_4$. We illustrate the preceding results by exhibiting quasi-idempotent transformations, identifying their associated idempotent powers, and describing the corresponding adjacency relations in the power graph $\mathcal{P}(\mathcal{T}_4)$.*

(i) **A quasi-idempotent transformation.** Define

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 3 \end{pmatrix}.$$

A direct computation gives

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 3 \end{pmatrix}, \quad \alpha^3 = \alpha.$$

Hence $\alpha^3 = \alpha$, so α is quasi-idempotent. Since

$$(\alpha^2)^2 = \alpha^4 = \alpha^2,$$

the element α^2 is idempotent. Moreover,

$$\text{Im}(\alpha^2) = \{1, 2, 3\} \subseteq \text{Fix}(\alpha^2) = \{1, 2, 3\},$$

which confirms the idempotency criterion.

Since α^2 is a positive power of α , the vertices α and α^2 are adjacent in $\mathcal{P}(\mathcal{T}_4)$.

(ii) Consider

$$\nu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 2 & 4 \end{pmatrix}.$$

Its powers satisfy

$$v^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 3 & 4 \end{pmatrix}, \quad v^3 = v.$$

Thus v is quasi-idempotent. Furthermore,

$$(v^2)^2 = v^4 = v^2,$$

so v^2 is idempotent. Since v^2 is obtained as a power of v , the vertices v and v^2 are adjacent in $\mathcal{P}(\mathcal{T}_4)$.

(iii) The previous computations show that each quasi-idempotent lies in a connected component containing an idempotent vertex:

$$\alpha \sim \alpha^2, \quad v \sim v^2.$$

Hence these quasi-idempotent elements are linked to idempotents by a single power edge.

(iv) Comparison with a non-quasi-idempotent element. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix},$$

the 4-cycle permutation. Then σ has order 4, so

$$\sigma^4 = \varepsilon,$$

where ε is the identity transformation. Although ε is idempotent, one has $\sigma^k \neq \sigma$ for $k = 2, 3$, and therefore σ is not quasi-idempotent.

This contrasts with the preceding examples: quasi-idempotent elements are directly adjacent to an idempotent, whereas a general element may only reach an idempotent after several power iterations.

Remark 3.12. *The key structural distinction revealed by Figure 2 is the distance from a vertex to the nearest idempotent in $\mathcal{P}(\mathcal{T}_n)$. Idempotents have distance 0; proper quasi-idempotents (those satisfying $\alpha^k = \alpha$ for some $k \geq 2$ but with $\alpha^2 \neq \alpha$) have distance exactly 1 from an idempotent (they are directly adjacent to α^{k-1} , which is idempotent); general elements may be arbitrarily far. The Lemma provides the algebraic mechanism: the power α^{k-1} is always*

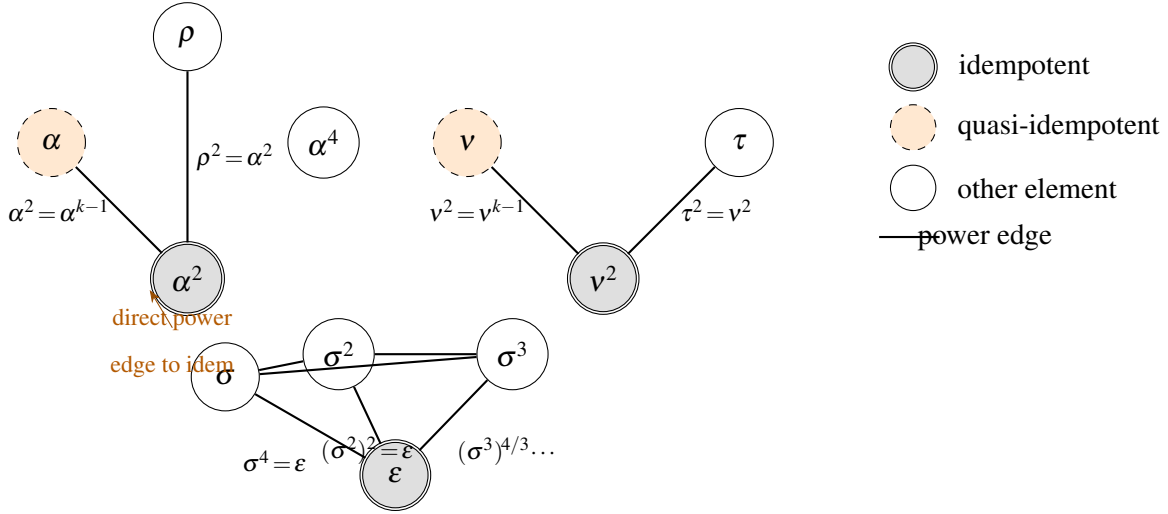


FIGURE 2. A portion of $\mathcal{P}(\mathcal{T}_4)$ illustrating the quasi-idempotent results. Orange dashed vertices (α , v) are quasi-idempotents; each is *directly adjacent* to its associated idempotent (α^2 , v^2 , shaded double-ringed), confirming the Proposition. Both quasi-idempotents therefore lie in components containing an idempotent, as guaranteed by the Theorem. By contrast, the permutation σ (bottom cluster) is not quasi-idempotent: it reaches the idempotent ϵ only after $|\sigma| = 4$ steps, with several intermediate power edges.

the witnessing idempotent, ensuring the distance is at most 1 whenever $\alpha^k = \alpha$ and α is not itself idempotent. Idempotents, being quasi-idempotent with $k = 2$ and $\alpha^2 = \alpha$, trivially satisfy $\alpha^{k-1} = \alpha$, so there is no distinct idempotent neighbour guaranteed by this lemma.

Nilpotent Elements. Nilpotent elements constitute an important class of transformations in $\mathcal{T}_n$, characterized by the eventual collapse of all points under repeated iteration to a single absorbing state. In terms of functional digraphs, they are represented by acyclic structures whose directed paths terminate at a fixed sink vertex, reflecting their annihilating behavior. Within the power graph $\mathcal{P}(\mathcal{T}_n)$, nilpotent elements exhibit distinctive connectivity patterns arising from their strict iterative vanishing property, which strongly influences the formation of components and adjacency relations [6]. This subsection focuses on the structural role of nilpotent elements and their impact on the overall architecture of $\mathcal{P}(\mathcal{T}_n)$.

Lemma 3.13. [3] *Let $\alpha \in \mathcal{T}_n$. Then α is nilpotent if and only if its functional digraph contains exactly one cycle, and that cycle is a fixed point.*

Proof. ($\Rightarrow$) Suppose α is nilpotent. Then there exists k such that α^k is a constant map. Hence all elements of X are eventually mapped to a single point $c \in X$, which must satisfy $\alpha(c) = c$. Thus the only cycle is the fixed point c .

($\Leftarrow$) Conversely, suppose the only cycle in the functional digraph of α is a fixed point c . Then every element eventually maps into c under iteration of α . Hence there exists k such that $\alpha^k(x) = c$ for all $x \in X$, so α^k is constant. Therefore, α is nilpotent. $\square$

Proposition 3.14. *Every nilpotent element is adjacent to a constant map in $\mathcal{P}(\mathcal{T}_n)$.*

Proof. Let α be nilpotent. Then there exists k such that $\alpha^k = c$, where c is a constant transformation. By definition of the power graph, α and c are adjacent. $\square$

Theorem 3.15. *The subgraph of $\mathcal{P}(\mathcal{T}_n)$ induced by the nilpotent elements decomposes into disjoint components, each containing exactly one constant map as its unique idempotent root. For a non-constant nilpotent α of index k (i.e., $\alpha^k = c$ is constant but α^{k-1} is not), the vertices $\{\alpha, \alpha^2, \dots, \alpha^k\}$ induce a connected subgraph in which α is adjacent to every α^j ($2 \leq j \leq k$), and $\alpha^i \sim \alpha^j$ whenever $i \mid j$ or $j \mid i$. In particular, if $k \geq 3$, the triple $\{\alpha, \alpha^2, c\}$ forms a triangle. Consequently, the nilpotent-induced subgraph is a forest if and only if every non-constant nilpotent in $\mathcal{T}_n$ has nilpotency index exactly 2; otherwise, cycles are inherent to the structure.*

Proof. Let $\alpha \in \mathcal{T}_n$ be nilpotent with nilpotency index k , so $\alpha^k = c$ is a constant map and $\alpha^{k-1} \neq c$. Since c is idempotent, $\alpha^m = c$ for all $m \geq k$. By the definition of the undirected power graph, α is adjacent to α^j for every $j \in \{2, \dots, k\}$ because α^j is a positive power of α . Moreover, for any $1 \leq i < j \leq k$, α^i and α^j are adjacent if and only if one is a positive power of the other, which occurs precisely when $i \mid j$ or $j \mid i$. Thus, the subgraph induced by the cyclic power sequence $\langle \alpha \rangle$ is isomorphic to the comparability graph of the divisibility poset on $\{1, 2, \dots, k\}$, with the constant map c adjacent to every vertex in the sequence.

If $k \geq 3$, then $\alpha \sim \alpha^2$ (since $1 \mid 2$), $\alpha \sim c$ (since $c = \alpha^k$), and $\alpha^2 \sim c$ (since $c = (\alpha^2)^k = \alpha^{2k} = (\alpha^k)^2 = c^2 = c$). Hence $\{\alpha, \alpha^2, c\}$ necessarily forms a triangle. This directly accounts for the

manuscript's example $\{\beta, \beta^2, c_3\}$, which corresponds to the case $k = 3$ and demonstrates why the original claim that the induced subgraph is always a tree is incorrect for the ordinary power graph.

Distinct constant maps are pairwise non-adjacent, as they form an independent set of idempotents. Therefore, each connected component of the nilpotent-induced subgraph contains exactly one constant map. Every non-constant nilpotent α reaches its unique constant root c via its power sequence, placing it in the component of c . If two distinct non-constant nilpotents α, γ satisfy $\gamma = \alpha^j$ or $\alpha = \gamma^m$ for some $j, m \geq 1$, they share the same constant root and are directly adjacent, potentially merging directed paths into additional undirected cycles. When no such mutual-power pairs exist and every non-constant nilpotent has index $k = 2$, each power sequence reduces to the single edge $\alpha \sim c$, yielding a star graph (a tree) rooted at c . It follows that the nilpotent-induced subgraph is a forest if and only if all non-constant nilpotents have index 2; otherwise, cycles arising from higher indices or mutual power relations are unavoidable. $\square$

Example 3.16 (Nilpotent elements in $\mathcal{T}_4$ and their role in $\mathcal{P}(\mathcal{T}_4)$). *Let $X = \{1, 2, 3, 4\}$ and consider the full transformation semigroup $\mathcal{T}_4$. We present nilpotent transformations, determine their nilpotency indices, and describe their adjacency relations in the power graph $\mathcal{P}(\mathcal{T}_4)$.*

(i) *A nilpotent element of index 2. Let*

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 3 \end{pmatrix}.$$

Then

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix} = c_3,$$

where c_3 denotes the constant transformation sending every element of X to 3. Since α is not constant while α^2 is constant, α is nilpotent of index 2.

The functional digraph of α has the form

$$1 \rightarrow 2 \rightarrow 3, \quad 4 \rightarrow 3, \quad 3 \rightarrow 3,$$

so it has a unique cycle, namely the fixed point 3.

(ii) **A nilpotent element of index 3.** Consider

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 1 \end{pmatrix}.$$

Successive powers yield

$$\beta^2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 2 \end{pmatrix}, \quad \beta^3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 \end{pmatrix} = c_3.$$

Since β^2 is not constant but β^3 is constant, β is nilpotent of index 3.

Its functional digraph is

$$4 \rightarrow 1 \rightarrow 2 \rightarrow 3, \quad 3 \rightarrow 3,$$

again with a unique cycle consisting of the fixed point 3.

(iii) **Adjacency to constant transformations.** Since

$$\alpha^2 = c_3 \quad \text{and} \quad \beta^3 = c_3,$$

both α and β are adjacent to c_3 in $\mathcal{P}(\mathcal{T}_4)$. Thus each nilpotent element lies in a component containing a constant map.

(iv) **Induced nilpotent structure.** The vertices α , β , β^2 , and c_3 satisfy

$$\beta \sim \beta^2, \quad \beta \sim c_3, \quad \beta^2 \sim c_3, \quad \alpha \sim c_3.$$

Hence the constant transformation c_3 acts as a root vertex to which the nilpotent elements are connected, illustrating the rooted-tree structure of the nilpotent subgraph.

(v) **Comparison with a non-nilpotent element.** Let

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$$

Its functional digraph contains the cycles (12) and (34), so no power of γ is constant. Therefore γ is not nilpotent and is not adjacent to any constant transformation in $\mathcal{P}(\mathcal{T}_4)$.

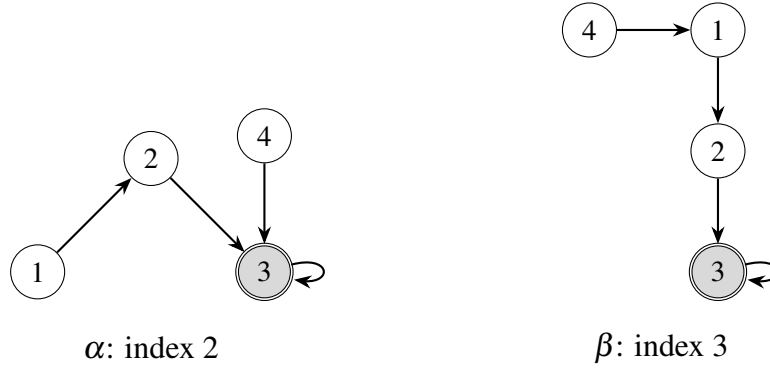


FIGURE 3. Functional digraphs of the nilpotent elements α (index 2, left) and β (index 3, right) in $\mathcal{T}_4$. In both cases the only cycle is the fixed point $\{3\}$ (shaded double-ringed), and every other element follows a directed path of finite length into this sink, confirming the Lemma. The depth of the deepest path equals the nilpotency index minus one.

Remark 3.17. *The nilpotency index of $\alpha \in \mathcal{T}_n$ equals the length of the longest directed path in the functional digraph of α (i.e., the depth of the deepest node above the fixed-point sink). In $\mathcal{P}(\mathcal{T}_n)$, this index controls the distance from α to the constant-map root of its component: an element of index m lies at graph-distance $m - 1$ from the root (since $\alpha^m = c$ but $\alpha^{m-1} \neq c$, yet α^{m-1} and c are adjacent via $(\alpha^{m-1})^2 = \alpha^{2m-2}$, which collapses to c once $2m - 2 \geq m$, i.e. $m \geq 2$).*

4. COMPUTATIONAL VERIFICATION AND EXACT EDGE STRUCTURE OF $\mathcal{P}(\mathcal{T}_3)$

This section provides a complete, computationally verified description of the power graph $\mathcal{P}(\mathcal{T}_3)$ obtained via the computer algebra system **GAP** (Groups, Algorithms, Programming) [18], augmented with the **Semigroups** and **Digraphs** packages. The case $n = 3$ is the smallest non-trivial instance: $|\mathcal{T}_3| = 27$, and the directed power graph has exactly 19 arcs, which collapse to 18 undirected edges in $\mathcal{P}(\mathcal{T}_3)$ (the reciprocal arc pair between the two 3-cycles yields a single undirected edge). All arcs are enumerated below and matched to the theoretical results of the preceding sections. Using the **GAP** command `DigraphEdges(D)`, we extracted all 19 directed edges $\alpha \rightarrow \beta$ (meaning $\beta = \alpha^k$ for some $k \geq 2$, $\beta \neq \alpha$) of the power graph. Table 1 classifies each edge by the ranks of its endpoints and identifies the algebraic relationship.

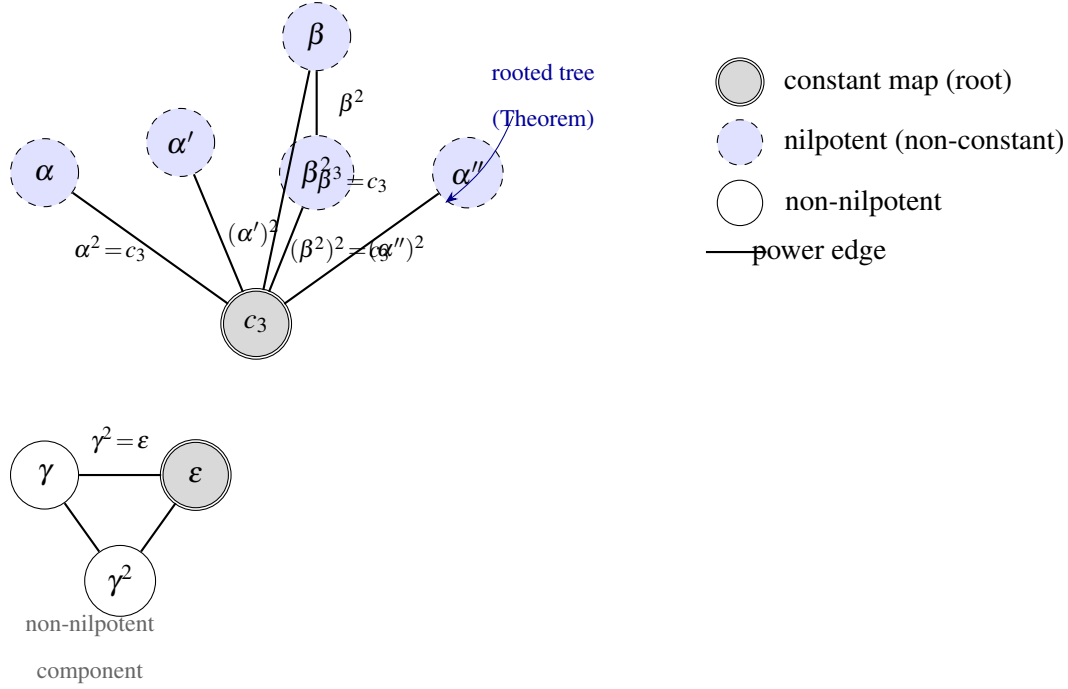


FIGURE 4. A portion of $\mathcal{P}(\mathcal{T}_4)$ showing a nilpotent component rooted at the constant map c_3 . The elements form a rooted tree of depth 2, while the non-nilpotent element γ lies in a separate component rooted at the idempotent ϵ .

Remark 4.1. Every edge in Table 1 terminates at an idempotent at the level of $\mathcal{T}_3$. In particular, the six nilpotent non-constant elements (edges 1, 2, 7, 11, 15, 17) each reach a constant map in exactly two steps, while the six quasi-idempotents (edges 4, 5, 10, 12, 18, 19) each square directly to a rank-2 idempotent.

Exact Power Graph of $\mathcal{T}_3$. We now state and prove several propositions that go beyond the general theory and are specific to the exact structure revealed by Table 1.

Proposition 4.2. Every non-permutation element of $\mathcal{T}_3$ reaches an idempotent in at most two steps, i.e. α^2 is idempotent for every non-permutation $\alpha \in \mathcal{T}_3$.

Proof. The non-permutation elements of $\mathcal{T}_3$ fall into two categories: nilpotents (rank ≤ 2 with a unique fixed-point cycle) and quasi-idempotents (rank 2, non-idempotent). For nilpotents α

TABLE 1. All 19 directed power arcs of $\mathcal{P}(\mathcal{T}_3)$, obtained from **GAP**. The reciprocal pair between the two 3-cycles (rows 8 and 13) collapses to one undirected edge, giving 18 undirected edges in the undirected power graph. Here “rk” denotes rank, “Idem” = idempotent, “Nilp” = nilpotent, “Perm” = permutation, “QI” = quasi-idempotent.

No.	Source α	Target $\beta = \alpha^k$	Type	Index k
1	(1, 1, 2)	(1, 1, 1) = c_1	Nilp $\rightarrow$ Idem	2
2	(1, 3, 1)	(1, 1, 1) = c_1	Nilp $\rightarrow$ Idem	2
3	(1, 3, 2)	ε	Perm $\rightarrow$ Idem	2
4	(2, 1, 1)	(1, 2, 2)	QI $\rightarrow$ Idem	2
5	(2, 1, 2)	(1, 2, 1)	QI $\rightarrow$ Idem	2
6	(2, 1)	ε	Perm $\rightarrow$ Idem	2
7	(2, 2, 1)	(2, 2, 2) = c_2	Nilp $\rightarrow$ Idem	2
8	(2, 3, 1)	(3, 1, 2)	Perm $\rightarrow$ Perm	2
9	(2, 3, 1)	ε	Perm $\rightarrow$ Idem	3
10	(2, 3, 2)	(3, 2, 3)	QI $\rightarrow$ Idem	2
11	(2, 3, 3)	(3, 3, 3) = c_3	Nilp $\rightarrow$ Idem	2
12	(3, 1, 1)	(1, 3, 3)	QI $\rightarrow$ Idem	2
13	(3, 1, 2)	(2, 3, 1)	Perm $\rightarrow$ Perm	2
14	(3, 1, 2)	ε	Perm $\rightarrow$ Idem	3
15	(3, 1, 3)	(3, 3, 3) = c_3	Nilp $\rightarrow$ Idem	2
16	(3, 2, 1)	ε	Perm $\rightarrow$ Idem	2
17	(3, 2, 2)	(2, 2, 2) = c_2	Nilp $\rightarrow$ Idem	2
18	(3, 3, 1)	(1, 1, 3)	QI $\rightarrow$ Idem	2
19	(3, 3, 2)	(2, 2, 3)	QI $\rightarrow$ Idem	2

we have $\alpha^2 = c_j$ for some $j \in \{1, 2, 3\}$, and every constant map c_j is idempotent. For quasi-idempotents α , inspection of Table 1 (edges 4, 5, 10, 12, 18, 19) confirms α^2 is a rank-2 idempotent in every case. Hence α^2 is idempotent for all non-permutation $\alpha \in \mathcal{T}_3$. $\square$

Proposition 4.3. The two 3-cycles (2, 3, 1) and (3, 1, 2) form a 2-element strongly connected subdigraph (a directed 2-cycle) in the directed power graph of $\mathcal{T}_3$, and both are also adjacent to the identity ε .

Proof. From Table 1 (edges 8 and 13): $(2, 3, 1)^2 = (3, 1, 2)$ and $(3, 1, 2)^2 = (2, 3, 1)$. Hence each is a power of the other, giving directed edges in both directions, i.e. a directed 2-cycle.

Moreover, edges 9 and 14 show $(2, 3, 1)^3 = \varepsilon$ and $(3, 1, 2)^3 = \varepsilon$, so both are also adjacent to ε in the undirected power graph. $\square$

Theorem 4.4. The power graph $\mathcal{P}(\mathcal{T}_3)$ has exactly ten connected components:

- (i) Three nilpotent components, one for each constant map c_j ($j = 1, 2, 3$), namely

$$\{c_1, (1, 1, 2), (1, 3, 1)\}, \quad \{c_2, (2, 2, 1), (3, 2, 2)\}, \quad \{c_3, (2, 3, 3), (3, 1, 3)\}.$$

- (ii) Six rank-2 pair components, each consisting of a rank-2 idempotent and its unique proper quasi-idempotent:

$$\{(1, 1, 3), (3, 3, 1)\}, \quad \{(1, 2, 2), (2, 1, 1)\}, \quad \{(1, 2, 1), (2, 1, 2)\},$$

$$\{(3, 2, 3), (2, 3, 2)\}, \quad \{(2, 2, 3), (3, 3, 2)\}, \quad \{(1, 3, 3), (3, 1, 1)\}.$$

- (iii) One permutation component containing the identity ε , the three transpositions $(1, 3, 2)$, $(2, 1, 3)$, $(3, 2, 1)$, and the two 3-cycles $(2, 3, 1)$, $(3, 1, 2)$.

Proof. We verify component membership by tracing power sequences. *Nilpotent components*

(i). By Proposition 4.2, every non-constant nilpotent α satisfies $\alpha^2 = c_j$ for a unique $j \in \{1, 2, 3\}$, placing α and c_j in the same component. No rank-2 idempotent can be connected to any constant map: since all are idempotents and distinct idempotents are non-adjacent (by the theorem on the edgeless idempotent subgraph), no edge exists between them. Inspection of Table 1 (edges 1, 2, 7, 11, 15, 17) confirms that only non-constant nilpotents are connected to constant maps.

Rank-2 pair components (ii). Each proper quasi-idempotent α satisfies $\alpha^2 = e$ for a unique rank-2 idempotent e (edges 4, 5, 10, 12, 18, 19 of Table 1). Since e is idempotent, $e^k = e$ for all $k \geq 1$, so e cannot be adjacent to any other idempotent or constant map. No further edges are incident to these vertices in the table. Hence $\{e, \alpha\}$ is an isolated 2-vertex component. *Permutation component (iii).* Every permutation $\pi \in \mathcal{T}_3$ satisfies $\pi^{|\pi|} = \varepsilon$, so all six permutations are connected to ε . No power of a bijection is a non-bijection, so this component is disjoint from all others.

Distinctness and count. The 27 elements are distributed as: $3 \times 3 = 9$ in nilpotent components, $6 \times 2 = 12$ in rank-2 pair components, and 6 in the permutation component, totalling $9 + 12 + 6 = 27 = |\mathcal{T}_3|$. The $3 + 6 + 1 = 10$ components are pairwise disjoint. $\square$

TABLE 2. Element-wise verification for $\mathcal{P}(\mathcal{T}_3)$. “Steps to idem” is the minimum $k \geq 1$ such that α^k is idempotent. Elements that are themselves idempotent have $k = 1$. Components C_1, C_2, C_3 are the three nilpotent components (rooted at c_1, c_2, c_3 respectively); $D_1, \dots, D_6$ are the six rank-2 pair components; C_{perm} is the permutation component.

Element	Type	Idempotent reached	Steps	Component
$(1, 1, 1) = c_1$	Idempotent (rank 1)	c_1	1	C_1
$(2, 2, 2) = c_2$	Idempotent (rank 1)	c_2	1	C_2
$(3, 3, 3) = c_3$	Idempotent (rank 1)	c_3	1	C_3
$(1, 1, 2)$	Nilpotent	c_1	2	C_1
$(1, 3, 1)$	Nilpotent	c_1	2	C_1
$(2, 2, 1)$	Nilpotent	c_2	2	C_2
$(3, 2, 2)$	Nilpotent	c_2	2	C_2
$(2, 3, 3)$	Nilpotent	c_3	2	C_3
$(3, 1, 3)$	Nilpotent	c_3	2	C_3
$(1, 1, 3)$	Idempotent (rank 2)	$(1, 1, 3)$	1	D_1
$(1, 2, 1)$	Idempotent (rank 2)	$(1, 2, 1)$	1	D_2
$(1, 2, 2)$	Idempotent (rank 2)	$(1, 2, 2)$	1	D_3
$(1, 3, 3)$	Idempotent (rank 2)	$(1, 3, 3)$	1	D_4
$(2, 2, 3)$	Idempotent (rank 2)	$(2, 2, 3)$	1	D_5
$(3, 2, 3)$	Idempotent (rank 2)	$(3, 2, 3)$	1	D_6
$(3, 3, 1)$	Prop. quasi-idem.	$(1, 1, 3)$	2	D_1
$(2, 1, 1)$	Prop. quasi-idem.	$(1, 2, 2)$	2	D_3
$(2, 1, 2)$	Prop. quasi-idem.	$(1, 2, 1)$	2	D_2
$(2, 3, 2)$	Prop. quasi-idem.	$(3, 2, 3)$	2	D_6
$(3, 3, 2)$	Prop. quasi-idem.	$(2, 2, 3)$	2	D_5
$(3, 1, 1)$	Prop. quasi-idem.	$(1, 3, 3)$	2	D_4
ε	Idempotent (rank 3)	ε	1	C_{perm}
$(1, 3, 2)$	Permutation (ord. 2)	ε	2	C_{perm}
$(2, 1, 3)$	Permutation (ord. 2)	ε	2	C_{perm}
$(3, 2, 1)$	Permutation (ord. 2)	ε	2	C_{perm}
$(2, 3, 1)$	Permutation (ord. 3)	ε	3	C_{perm}
$(3, 1, 2)$	Permutation (ord. 3)	ε	3	C_{perm}

Remark 4.5. *Table 2 confirms that every element of $\mathcal{T}_3$ reaches an idempotent in at most 3 steps. The maximum distance to the nearest idempotent is 2 for the two 3-cycles (via the permutation component), and the nilpotent and permutation components each have diameter 2.*

Remark 4.6. *The partition of $\mathcal{T}_3$ in Table 2 accounts for all 27 elements:*

$$\underbrace{3}_{\text{const. maps}} + \underbrace{6}_{\text{rank-2 idem}} + \underbrace{1}_{\varepsilon} + \underbrace{6}_{\text{nilpotents}} + \underbrace{6}_{\text{quasi-idem}} + \underbrace{5}_{\text{perm.}} = 27 = 3^3 = |\mathcal{T}_3|,$$

providing a complete and consistent combinatorial decomposition of the full transformation semigroup on three elements, as certified by the GAP computation.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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