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**ON SOLVABILITY OF GENERALIZED VARIATIONAL INEQUALITIES**

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**Abstract.** In this paper, we investigate the solvability of a class of nonlinear variational inequalities involving a combination of relaxed monotone operators with nonexpansive mappings in a Hilbert space.

**Keywords**: variational inequality; Hilbert space; nonexpansive mapping; solution.

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## 1. Introduction-Preliminaries

Let $H$ be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $K$ be a closed convex subset of $H$ and let $A, T_1, T_2, g : H \to H$ be a nonlinear mappings. Let $K(\cdot) : H \to P(H)$ be the multi-valued mapping. Then the following problems is said to be a nonlinear variational inequality (NVI) problem: determine an element $u \in K$ such that:

$$\begin{aligned} &\text{(i) } g(u) \in K(u), \\ &\text{(ii) } \langle Ag(u), v - g(u) \rangle \geq \langle Au, v - g(u) \rangle - \rho \langle (T_1 + T_2)u, v - g(u) \rangle, \quad \forall v \in K(u), \end{aligned} \tag{1.1}$$

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where $\rho > 0$ is a constant which was introduced by Verma [1]. Next, we shall denote the solution of (1.1) by $\Gamma$.

Next, we consider some special cases of (NVI) problem (1.1) as following.

For $K(u) = K$ for all $u \in H$, the (1.1) reduces to: find an element $u \in K$ such that $g(u) \in K$ and

$$\langle Ag(u), v - g(u)\rangle \geq \langle Au, v - g(u)\rangle - \rho\langle (T_1 + T_2)u, v - g(u)\rangle, \quad \forall v \in K, \tag{1.2}$$

where $\rho > 0$ is a constant.

Replacing $g(u)$ by $u_0$ for some $u \in K$ in (1.2), the (1.2) reduces to Noor [2]: find $u_0 \in K$ such that

$$\langle Au_0, v - u_0\rangle \geq \langle Au, v - u_0\rangle - \rho\langle (T_1 + T_2)u, v - u_0\rangle, \quad \forall v \in K, \tag{1.3}$$

where $\rho > 0$ is a constant.

For $T_1 = I$, the identity mapping on $H$, the (1.1) collapses to: find an element $u \in K$ such that $g(u) \in K(u)$ and

$$\langle Ag(u), v - g(u)\rangle \geq \langle Au, v - g(u)\rangle - \rho\langle (I + T_2)u, v - g(u)\rangle, \quad \forall v \in K(u), \tag{1.4}$$

where $\rho > 0$ is a constant.

Variational inequalities introduced by Stampacchia [3] in the early sixties have had a great impact and influence in the development of almost all branches of pure and applied sciences and have witnessed an explosive growth in theoretical advances, algorithmic development, see [4-20] and references therein. It is well known that the variational inequalities are equivalent to the fixed point problems. This alternative equivalent formulation is very important from the numerical analysis point of view and has played a significant part in several numerical methods for solving variational inequalities and complementarity. In particular, the solution of the variational inequalities can be computed using the iterative methods. Recently, variational inequalities have been generalized and extended in several directions using innovative and novel techniques. General variational inequalities represent an important and extremely useful class of nonlinear problems arising from applied mathematics, optimization and control theory, economics, mechanics, engineering sciences, physics and others.

Next, we recall the following:

(1) A mapping $T$ of $H$ into $H$ is called monotone if

$$\langle Tu - Tv, u - v\rangle \geq 0, \quad \forall u, v \in H.$$

(2) $T$ is called $r$-strongly monotone if there exists a constant $v > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq r\|x - y\|^2, \quad \forall x, y \in H.$$

This implies that

$$\|Tx - Ty\| \geq r\|x - y\|, \quad \forall x, y \in K,$$

that is, $A$ is $r$-expansive and, when $r = 1$, it is expansive.

(3) $A$ is said to be $\mu$-cocoercive if there exists a constant $\mu > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \mu\|Tx - Ty\|^2, \quad \forall x, y \in H.$$

Clearly, every $\mu$-cocoercive mapping $A$ is $\frac{1}{\mu}$-Lipschitz continuous.

(4) $A$ is called relaxed $\gamma$-cocoerceive if there exists a constant $u > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-\gamma)\|Tx - Ty\|^2, \quad \forall x, y \in H.$$

(5)$A : H \to H$ is said to be a relaxed monotone operator if there exists a constant $g > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq -r\|x - y\|^2, \quad \forall x, y \in H.$$

Relaxed operators were applied to the study of constrained problems in reflexive Banach spaces, where the set of all admissible elements is nonconvex but star-shaped. As a result, corresponding variational formulations are no longer variational inequalities but, instead, become hemi-variational inequalities [11].

(6) $A : H \to H$ is called Lipschitz continuous if there is a constant $L > 0$ such that for all $x, y \in H$, we have

$$\|Tx - Ty\| \leq L\|x - y\|.$$

(7) $A$ is said to be relaxed $(\gamma, r)$-cocoercive if there exist two constants $u, v > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-\gamma)\|Tx - Ty\|^2 + r\|x - y\|^2, \quad \forall x, y \in H.$$

For $u=0$, $A$ is $v$-strongly monotone. This class of mappings is more general than the class of strongly monotone mappings. It is easy to see that we have the following implication:

$r$-strongly monotonicity $\Rightarrow$ relaxed $(\gamma, r)$-cocoercivity.

(8) $S : H \to H$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in H.$$

$S$ is said to be strictly pseudocontractive if there exists a constant $\kappa \in [0,1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I-S)x - (I-S)y\|^2, \quad \forall x, y \in H.$$

It is clear that strictly pseudocontractive mappings include nonexpansive mapping as a special case. Next, we shall denote the fixed point set of $S$ by $F(S)$.

We now recall some well-known concepts and results:

**Lemma 1.1.** *For any $z \in H$, $u \in K$ satisfies the inequality:*

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in K,$$

*if and only if $u = P_K z$.*

**Lemma 1.2.** *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\lambda_n)a_n + b_n, \quad \forall n \geq n_0,$$

*where $n_0$ is some nonnegative integer, $\{\lambda_n\}$ is a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$, then $\lim_{n\to\infty} a_n = 0$.*

**Lemma 1.3** [1] *An element $u \in K$ is a solution of the (NVI) problem (1.1) if and only if $u \in K$ such that $g(u) \in K(u)$ and*

$$\langle A(g(u)) - Au + \rho(T_1 + T_2)u, v - g(u)\rangle \geq 0, \quad \forall u \in K(u). \tag{1.5}$$

**Lemma 1.4** [1] *An element $u \in K$ is a solution of the (NVI) problem (1.1) if and only if $u \in K$ such that $g(u) \in K(u)$ and*

$$g(u) = m(u) + P_K[g(u) + Au - Ag(u) - \rho(T_1 + T_2)u - m(u)], \tag{1.6}$$

*where $P_K$ is the projection of $H$ onto $K$.*

It follows from (1.6) that

$$u = u - g(u) + m(u) + P_K[g(u) + Au - Ag(u) - \rho(T_1 + T_2)u - m(u)].$$

If $u$ is a common element of the set of fixed points of some nonexpansive mapping $S$ and set of solutions of the (NVI) problem (1.1), we have

$$\begin{aligned} u &= u - g(u) + m(u) + P_K[g(u) + Au - Ag(u) - \rho(T_1 + T_2)u - m(u)] \\ &= S\{u - g(u) + m(u) + P_K[g(u) + Au - Ag(u) - \rho(T_1 + T_2)u - m(u)]\} \end{aligned} \tag{1.7}$$

## 2. Algorithms

**Algorithm 2.1.** For $u_0 \in H$, $u_{n+1}$ is defined by an iterative scheme

$$\begin{aligned} u_{n+1} = (1-\alpha_n)u_n + \alpha_n\big(\alpha I + (1-\alpha)S\big)\{u_n - g(u_n) + m(u_n) \\ + P_K[g(u_n) + Au_n - Ag(u_n) - \rho(T_1 + T_2)u_n - m(u_n)]\}, \end{aligned} \tag{2.1}$$

for all $n \geq 0$, where $\rho > 0$ is a constant and $S$ is a $\kappa$-strictly pseudocontractive mapping.

If $S = I$, the identity mapping on $H$, the algorithm (2.1) reduces to the following.

**Algorithm 2.2.** For $u_0 \in H$, $u_{n+1}$ is defined by an iterative scheme

$$\begin{aligned} u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{u_n - g(u_n) + m(u_n) \\ + P_K[g(u_n) + Au_n - Ag(u_n) - \rho(T_1 + T_2)u_n - m(u_n)]\}, \end{aligned} \tag{2.2}$$

for all $n \geq 0$, where $\rho > 0$ is a constant.

If $S = I$, the identity mapping on $H$ and $\{\alpha_n\} \equiv 1$, the algorithm (2.1) reduces to the following algorithm which was mainly considered by Verma [9].

**Algorithm 2.3.** For $u_0 \in H$, $u_{n+1}$ is defined by an iterative scheme

$$u_{n+1} = u_n - g(u_n) + m(u_n) + P_K[g(u_n) + Au_n - Ag(u_n) - \rho(T_1 + T_2)u_n - m(u_n)], \tag{2.3}$$

for all $n \geq 0$, where $\rho > 0$ is a constant.

If $S = g = I$, the identity mapping on $H$ and $\{\alpha_n\} \equiv 1$, the algorithm (2.1) reduces to the following.

**Algorithm 2.4.** For $u_0 \in H$, $u_{n+1}$ is defined by an iterative scheme

$$u_{n+1} = m(u_n) + P_K[u_n - \rho(T_1 + T_2)u_n - m(u_n)], \tag{2.4}$$

for all $n \geq 0$, where $\rho > 0$ is a constant.

## 3. Main results

**Theorem 3.1.** *Let $H$ be a real Hilbert space, $K$ be a finite closed convex subset of $H$ and the set $K(u) = m(u) + K$ be defined by a multi-valued mapping $K(u) : H \to P(H)$, where $m : H \to H$ is Lipschitz continuous with the Lipschitz continuity constant $\lambda > 0$. Let $T_1 : H \to H$ be a relaxed $(\gamma_1, r_1)$-cocoerceive and $\mu_1$-Lipschitz continuous mapping, $T_2 : H \to H$ be Lipschitzian and relaxed monotone with the Lipschitz continuity constant $p > 0$ and the relaxed monotonicity constant $q > 0$, $A : H \to H$ be Lipschitz continuous with the Lipschitz continuity constant $s > 0$ and $g : H \to H$ be a relaxed $(\gamma_2, r_2)$-cocoerceive and $\mu_2$-Lipschitz continuous mapping. Let $S$ be a kappa-strictly pseudocontractive mapping such that $F(S) \cap \Gamma \neq \emptyset$. Assume that $\alpha \in [\kappa, 1)$, $\{\alpha_n\} \subset [0,1]$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $2\theta_2 + \theta_1 + 2\lambda + s + s\mu_2 < 1$, where $\theta_1 = \sqrt{1 + 2\rho\gamma_1\mu_1 - 2\rho r_1 + 2\rho q + \rho^2(\mu_1 + p)^2}$ and $\theta_2 = \sqrt{1 + \mu_2^2 - 2r_2 + 2\gamma_2\mu_2^2}$. Then the sequence $\{u_n\}$ generated by (2.1) converges strongly to $u^* \in F(S) \cap \Gamma$.*

**Proof.**

Fix $u^* \in F(S) \cap \Gamma$ and Let $T : \alpha I + (1 - \alpha)S$. In view of the restriction, we find that $T$ is nonexpansive with $F(T) = F(S)$; see Zhou [12]. we have

$$\begin{aligned} u^* = {} & (1 - \alpha_n)u^* + \alpha_n T\{u^* - g(u^*) + m(u^*) \\ & + P_K[g(u^*) + Au^* - Ag(u^*) - \rho(T_1 + T_2)u^* - m(u^*)]\}. \end{aligned}$$

It follows that

$$\begin{aligned} \|u_n - u^*\| \leq {} & (1 - \alpha_n)\|u_n - u^*\| + \alpha_n\|u_n - g(u_n) + m(u_n) \\ & + P_K[g(u_n) + Au_n - Ag(u_n) - \rho(T_1 + T_2)u_n - m(u_n)] \\ & - \{u^* - g(u^*) + m(u^*) \\ & + P_K[g(u^*) + Au^* - Ag(u^*) - \rho(T_1 + T_2)u^* - m(u^*)]\}\| \end{aligned} \tag{3.1}$$

Now, we consider the rightside of second term of (3.1)

$$\begin{aligned}
&\|u_n - g(u_n) + m(u_n) + P_K[g(u_n) + Au_n - Ag(u_n) - \rho(T_1+T_2)u_n - m(u_n)] \\
&\quad - \{u^* - g(u^*) + m(u^*) + P_K[g(u^*) + Au^* - Ag(u^*) - \rho(T_1+T_2)u^* - m(u^*)]\}\| \\
&\le 2\|u_n - u^* - [g(u_n) - g(x^*)]\| + 2\|m(u_n) - m(u^*)\| + \|A(u_n) - A(u^*)\| \\
&\quad + \|Ag(u_n) - Ag(u^*)\| + \|u_n - u^* - \rho[(T_1+T_2)u_n - (T_1+T_2)u^*]\| \\
&\le 2\|u_n - u^* - [g(u_n) - g(x^*)]\| + \|u_n - u^* - \rho[(T_1+T_2)u_n - (T_1+T_2)u^*]\| \\
&\quad + (2\lambda + s + s\mu_2)\|u_n - u^*\|.
\end{aligned} \tag{3.2}$$

Next, we will consider the second term of the rightside of (3.2).

$$\begin{aligned}
&\|u_n - u^* - \rho[(T_1+T_2)u_n - (T_1+T_2)u^*]\|^2 \\
&\le \|u_n - u^*\|^2 - 2\rho\langle u_n - u^*, T_1u_n - T_1u^*\rangle - 2\rho\langle u_n - u^*, T_2u_n - T_2u^*\rangle \\
&\quad + \rho^2(\|T_1u_n - T_1u^*\| + \|T_2u_n - T_2u^*\|)^2 \\
&\le \|u_n - u^*\|^2 - 2\rho(-\gamma_1\|T_1u_n - T_1u^*\|^2 + r_1\|u_n - u^*\|^2) + 2\rho q\|u_n - u^*\| \\
&\quad + \rho^2(\mu_1 + p)^2\|u_n - u^*\|^2 \\
&\le \theta_1^2\|u_n - u^*\|^2,
\end{aligned} \tag{3.3}$$

where $\theta_1 = \sqrt{1 + 2\rho\gamma_1\mu_1 - 2\rho r_1 + 2\rho q + \rho^2(\mu_1 + p)^2}$. In view of the assumption that $g$ is relaxed $(\gamma_2, r_2)$-cocoercive and $\mu_2$-Lipschitz continuous, we obtain

$$\begin{aligned}
&\|u_n - u^* - g(u_n) - g(u^*)\|^2 \\
&\le \|u_n - u^*\|^2 - 2[-\gamma_2\|g(u_n) - g(u^*)\|^2 + r_2\|u_n - u^*\|^2] + \|g(u_n) - g(u^*)\|^2 \\
&\le \|u_n - u^*\|^2 + 2\mu_2^2\gamma_2\|u_n - u^*\|^2 - 2r_2\|u_n - u^*\|^2 + \mu_2^2\|u_n - u^*\|^2 \\
&= (1 + 2\mu_2^2\gamma_2 - 2r_2 + \mu_2^2)\|u_n - u^*\|^2 \\
&= \theta_2^2\|u_n - u^*\|^2,
\end{aligned} \tag{3.4}$$

where $\theta_2 = \sqrt{1+\mu_2^2 - 2r_2 + 2\gamma_2\mu_2^2}$. Substitute (3.3) and (3.4) into (3.2) yields that

$$\begin{aligned} &\|u_n - g(u_n) + m(u_n) + P_K[g(u_n) + Au_n - Ag(u_n) - \rho(T_1+T_2)u_n - m(u_n)] \\ &\quad - \{u^* - g(u^*) + m(u^*) + P_K[g(u^*) + Au^* - Ag(u^*) - \rho(T_1+T_2)u^* - m(u^*)]\}\| \\ &= (2\theta_2 + \theta_1 + 2\lambda + s + s\mu_2)\|u_n - u^*\|. \end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.1), we find that

$$\begin{aligned} \|u_n - u^*\| &\le (1-\alpha_n)\|u_n - u^*\| + \alpha_n(2\theta_2 + \theta_1 + 2\lambda + s + s\mu_2)\|u_n - u^*\| \\ &= [1 - \alpha_n(1 - 2\theta_2 - \theta_1 - 2\lambda - s - s\mu_2)]\|u_n - u^*\|. \end{aligned} \tag{3.6}$$

In view of Lemma 1.2, we can get the desired conclusion easily. This completes the proof.

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